This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

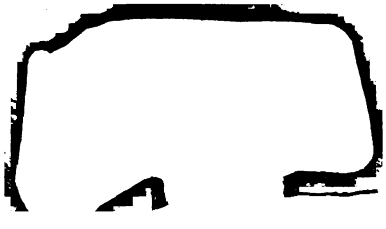
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`



APPLIED MECHANICS.

APPLIED MECHANICS:

AN ELEMENTARY GENERAL INTRODUCTION TO

THE THEORY OF

STRUCTURES AND MACHINES.

WITH DIAGRAMS, ILLUSTRATIONS, AND EXAMPLES.

BY

JAMES H. COTTERILL, F.R.S.,

ASSOCIATE MEMBER OF COUNCIL OF THE INSTITUTION OF NAVAL ARCHITECTS; ASSOCIATE
MEMBER OF THE INSTITUTION OF CIVIL ENGINEERS; PROFESSOR OF APPLIED
MECHANICS IN THE ROYAL NAVAL COLLEGE, GREENWICH.

SECOND EDITION REVISED.

London:
MACMILLAN AND CO.
AND NEW YORK.
1890.

All rights reserved.

WHAT are the conditions of a science? and when may any subject be said to enter the scientific stage? When the facts of it begin to resolve themselves into groups when phenomena are no longer isolated experiences, but appear in connection and order; when, after certain antecedents, certain consequences are uniformly seen to follow; when facts enough have been collected to furnish a basis for conjectural explanation; and when conjectures have so far ceased to be utterly vague that it is possible in some degree to foresee the future by the help of them.—FROUDE.

22086
SD
.C82

PREFACE.

ON the author's appointment to lecture on Mechanics in the Royal Naval College, a course of elementary lessons was commenced, based on RANKINE'S well-known treatise, with such assistance as could be obtained from other sources. After some years this course assumed a tolerably permanent form, and it was thought desirable to print it, partly from the inconvenience to students of being exclusively dependent on oral instruction, and partly from an idea that it might be useful to others besides those who were immediately addressed. The place which these lectures occupy in the programme of the College will be found explained in an Appendix.

The preparation of the work for the press has extended over a considerable period, and has been subject to many interruptions. There is therefore not always the unity desirable in a scientific treatise; nor is it by any means complete, even when due account is taken of the stringent limitations explained in the Introduction. It is, however, hoped that these deficiencies may be partly compensated for by the fact that the book is the product of a great deal of experience in teaching the subject, and a great deal of consideration as to the matter which ought to find a place in a general elementary treatise. Nearly the whole has been delivered in the form of lectures, and some part has actually been printed from notes taken throughout one session by a member of the junior class (Mr. H. J. Oram, R.N.) at that time, which were afterwards transcribed for the press by the author's assistant. Everything, however, of any importance has been re-written, with alterations and additions, to make it better fit for publication. Throughout, the object has been to give reasons, not rules, and details of application are consequently subordinated to the principles on which the theory is based. Especially has the author endeavoured to distinguish as clearly as possible between those parts of the subject which are universally and necessarily true, and those parts which rest on hypotheses more or less questionable. The book is intended to give that general knowledge of the mechanics of structures and machines which should accompany the detailed study either of naval architecture or of any special branch of engineering to which a student proposes to devote himself. Much, therefore, is excluded which might naturally be expected to form part of the work, simply because, however important, it is required only by a special class of students.

The introduction of descriptive details is not necessary to the plan of this work, except in certain parts of the theory of mechanism, nor, indeed, in a general treatise would it be possible to include them systematically within any reasonable compass. In the chapters on mechanism, however, they are required, and elsewhere it has been thought advisable to introduce them occasionally. Care has been taken to select working examples almost exclusively, the plates representing which have mostly been drawn by Mr. T. A. Hearson, to whom the author is indebted for many suggestions and portions of the descriptive matter, together with some assistance in revising proof sheets and transcribing lecture notes for the press. The proofs have been read by Professor W. C. Unwin, M.I.C.E., to whose great technical knowledge some corrections are due. In a general elementary work there is not room for much that is new : in the references at the end of each chapter and in the Appendix the various sources of information have been stated fully.

GREENWICH, *May*, 1884.

PREFACE TO THE SECOND EDITION.

In this new edition a number of errors have been corrected and some verbal emendations introduced. For many of these corrections the author is indebted to the care of his assistant, Mr. J. H. Slade; and for others to the kindness of friends who have used the book. In two or three instances these errors are of some importance: in such cases they have been noticed in the Appendix. Some additional matter rendered necessary by the progress of the subject or for other reasons has been placed in a separate Appendix. Should the book continue to be found useful, the author would be encouraged, in a future edition, to dwell at greater length on some of the important subjects which (especially in Part V.) have suffered from too great condensation. An abridged edition is now in active preparation, in which those parts of the whole subject which are of most general importance will be treated in fuller detail.

GREENWICH, *December*, 1889.

CONTENTS.

GENERAL INTRODUCTION.

PART I.—STATICS OF STRUCTURES.

CHAPTER I.—FRAMEWORK LOADED AT THE JOINTS.

a

ARTICLE PAGE
1. Preliminary Explanations and Definitions, ... 1

SECTION I.—TRIANGULAR FRAMES.

2. Diagram of Forces for a Simple Triangular Frame, ... 3
3. Triangular Trusses, ... 5
4. Cranes and Derricks, ... 6
5. Sheer Legs and Tripods, ... 8
6. Effect of the Tension of the Chain in Cranes, ... 9
Examples, ... 10

SECTION II.—INCOMPLETE FRAMES.

7. Preliminary Remarks, ... 11
8. Simple Trapezoidal or Queen Truss, ... 12
9. General Case of a Funicular Polygon under a Vertical Load, . . 13
10. Suspension Chains, Arches, Bowstring Girders, ... 15
11. Suspension Chains (*continued*), Bowstring Suspension Girder, . . 19
Examples, ... 21

SECTION III.—COMPOUND FRAMES.

12. Compound Triangular Frames for Bridge Trusses, ... 22
13. Roof Trusses in Timber, ... 25
14. Queen Truss for Large Iron Roofs, ... 27
15. Concluding Remarks, Diagrams of Forces in General, ... 29
Examples and References, ... 31

CHAPTER II.—STRAINING ACTIONS ON A LOADED STRUCTURE.

16. Preliminary Explanations, ... 33

SECTION I.—BEAMS.

ARTICLE PAGE
17. Straining Actions on a Beam, ... 34
18. Example of a Balanced Lever, General Rules, ... 35
19. Beam supported at the Ends and Loaded at an Intermediate Point, . . 36
20. Beam supported at the Ends and Loaded Uniformly, ... 38
21. Beam Loaded at the Ends and supported at Intermediate Points, . . 39
22. Application of the Method of Superposition, ... 40
Examples, ... 41

SECTION II.—FRAMEWORK GIRDERS.

23. Preliminary Explanations, ... 42
24. Warren Girders under Various Loads, ... 44
25. N Trusses, ... 48
Examples, ... 48

SECTION III.—GIRDERS WITH REDUNDANT BARS.

26. Preliminary Explanations, ... 49
27. Lattice Girders, Flanged Beams, ... 51
Examples and References, ... 53

CHAPTER III.—STRAINING ACTIONS DUE TO ANY VERTICAL LOAD.

28. Preliminary Remarks, ... 54
29. Relation between the Shearing Force and Bending Moment, . . . 54
30. Application to the Case of a Loaded Beam, ... 59
31. Application to a Vessel Floating in Water, ... 60
32. Maximum Straining Actions, ... 64
Examples, ... 64
33. Travelling Loads, ... 65
34. Counter-Bracing of Girders, ... 68
Examples, ... 69
35. Method of Sections applied to Incomplete Frames, Culmann's Theorem, . 70
36. Method of Sections in General, Ritter's Method, ... 71
Examples and References, ... 73

CHAPTER IV.—FRAMEWORK IN GENERAL.

37. Straining Actions on the Bars of a Frame. General Method of Reduction, . 74
38. Hinged Girders, Virtual Joints, ... 76
39. Hinged Arches, ... 78
40. Structures of Uniform Strength, ... 79
41. Stress due to the Weight of a Structure, ... 81
42. Straining Actions on a Loaded Structure in General, ... 83
43. Framework with Redundant Parts, ... 84
44. Concluding Remarks, ... 85
Examples, ... 86

PART II.—KINEMATICS OF MACHINES.

45. Introductory Remarks, ... 91

CHAPTER V.—LOWER PAIRING.

ARTICLE PAGE
46. Definition of Lower Pairs, ... 93
47. Definition of a Kinematic Chain, ... 95
48. Mechanism of a Direct-Acting Engine, ... 97
49. Velocity of Piston, ... 99
Examples, ... 102
50. Mechanisms derived from the Slider Crank Chain, ... 103
51. Double Slider-Crank Chains, ... 109
52. Crank Chains in General, ... 111
53. Diagram of Velocities in Linkwork, ... 112
54. Screw Chains, ... 114
55. Parallel Motions derived from Crank Chains, ... 115
56. Closure of Kinematic Chains, ... 119
Examples and References, ... 120

CHAPTER VI.—CONNECTION OF TWO LOWER PAIRS BY HIGHER PAIRING.

SECTION I.—TENSION AND PRESSURE ELEMENTS.

57. Preliminary Remarks, Tension Elements, ... 122
58. Simple Pulley Chain, Blocks and Tackle, ... 123
59. Wheel and Axle, ... 125
60. Pulley Chains with Friction Closure, Belts, ... 127
61. Shifting of Belts, Fusee Chain, ... 129
62. Simple Hydraulic Chain, ... 129
Examples, ... 130

SECTION II.—WHEELS IN GENERAL.

63. Higher Pairing of Rigid Elements, ... 131
64. Rolling Contact, ... 132
65. Augmentation of a Chain, Trains of Wheels, ... 134
66. Wheel Chains involving Screw Pairs, ... 136
Examples, ... 137

SECTION III.—TEETH OF WHEELS.

67. Preliminary Explanations, ... 138
68. Involute Teeth, ... 139
69. Path of Contact, the Pitch Circle, ... 141
70. Path of Contact, any Circle, ... 144
71. Addendum and Clearance of Teeth, ... 145
72. Endless Screw and Worm Wheel, ... 146
Examples, ... 147

SECTION IV.—CAMS AND RATCHETS.

73. Reduction of a Crank Chain by omission of the Coupling Link, . . 147
74. Cams with Continuous Action, ... 149
75. Mechanisms with Intermittent Action, ... 151
76. Ratchets, ... 152
77. Other Forms of Ratchet Mechanisms, ... 153
78. Screw Cams, ... 154
Examples, ... 154

CHAPTER VII.—MECHANISM IN GENERAL.

ARTICLE PAGE
79. Plane Motion in General, Centrodes, ... 156
80. Axoids, Elementary Examples of Centrodes, ... 158
81. Profiles for given Centrodes, ... 160
82. Centrodes for a Higher Pair connecting Lower Pairs, ... 162
83. Construction of Centres of Curvature of Profiles, Willis's Method, . . 163
84. Sphere Motion, ... 164
85. Screw Motion, ... 165
86. Classification of Simple Kinematic Chains, ... 165
87. Compound Kinematic Chains, ... 167

PART III.—DYNAMICS OF MACHINES.

CHAPTER VIII.—PRINCIPLE OF WORK.

SECTION I.—BALANCED FORCES (STATICS).

88. Preliminary Explanations, Definition of Work, ... 175
89. Oblique Resistance, ... 176
90. Variable Resistance, ... 177
91. Resistance to Rotation, Stability of a Vessel, ... 179
92. Internal and External Work, ... 181
93. Energy, Principle of Work, ... 181
94. Machines, ... 184
95. Verification of the Principle of Work in Special Examples, ... 185
96. Periodic Motion of Machines, ... 187
97. Power, Sources of Energy, ... 188
98. Reversibility, Conservation and Storage of Energy, ... 188
Examples, ... 189

SECTION II.—UNBALANCED FORCES (KINETICS).

99. Kinetic Energy of a Particle, ... 190
100. Partially Unbalanced Forces, Principle of Work, ... 191
101. Kinetic Energy of the Moving Parts of a Machine, ... 193
102. Conservation of Energy, ... 195
103. Examples of the Principle of Work, ... 196
Examples and References, ... 197

CHAPTER IX.—DYNAMICS OF THE STEAM ENGINE.

104. Construction of Polar Curves of Crank Effort, ... 199
105. Construction of Linear Curves of Crank Effort, ... 201
106. Ratio of Maximum and Minimum Crank Effort to Mean, ... 202
107. Fluctuation of Energy, ... 203
108. Fluctuation of Speed, Fly Wheels, ... 208
109. Correction of Indicator Diagram for Inertia of Reciprocating Parts, . . 211
110. Construction of Curves of Crank Effort for any given Indicator Diagram, . 216
111. Periodic Motion of Machines in General, ... 218
Examples, ... 218

CHAPTER X.—FRICTIONAL RESISTANCES.

ARTICLE PAGE
112. Preliminary Remarks, ... 221

SECTION I.—EFFICIENCY OF LOWER PAIRING.

113. Ordinary Laws of Sliding Friction, ... 221
114. Friction of Bearings, ... 222
115. Friction of Pivots, ... 224
116. Friction and Efficiency of Screws, ... 226
117. Efficiency of Mechanism by Exact Method, ... 228
118. Efficiency of Mechanism by Approximate Method, ... 229
119. Experiments on Sliding Friction (Morin), ... 230
120. Limitations to the Ordinary Laws in Plane Surfaces, ... 231
121. Experiments on Axle Friction, ... 233

SECTION II.—EFFICIENCY OF HIGHER PAIRING.

122. Rolling Friction, ... 234
123. Friction of Ropes and Belts, ... 235
124. Driving Belts, ... 237
125. Slip of Belts, ... 238
126. Stiffness of Ropes, ... 239
127. Friction of Toothed Wheels and Cams, ... 240

SECTION III.—FRICTIONAL RESISTANCES IN GENERAL.

128. Efficiency of Mechanism in General, ... 241
129. Friction Brakes, ... 243
Examples and References, ... 244

CHAPTER XI.—INCOMPLETE CONSTRAINT. STRAINING ACTIONS ON MACHINES.

130. Preliminary Remarks, ... 246

SECTION I.—ELEMENTARY PRINCIPLES OF DYNAMICS.

131. Impulse and Momentum, ... 246
132. Centrifugal Force, ... 248
133. Principle of Momentum, ... 249
134. Internal and External Kinetic Energy, ... 250
135. Examples of Incomplete Constraint, ... 251
Case I. Recoil of a Gun, ... 251
Case II. Collision of Vessels, ... 252

SECTION II.—CENTRIFUGAL REGULATORS.

136. Preliminary Remarks, ... 253
137. Simple Revolving Pendulum, ... 253
138. Speed of a Governor to overcome given Frictional Resistances, Loaded Governors, ... 254
139. Variation of Height of a Governor by a Change of Position of the Arms, . 256
140. Parabolic Governors, ... 257
141. Stability of Governors, ... 258

SECTION III.—STRAINING ACTION ON THE PARTS OF A MACHINE.

ARTICLE PAGE
142. Transmission of Stress in Machines, ... 259
143. Dynamometers, ... 261
144. Stability of Machines, Balancing, ... 262
145. Straining Actions on the Parts of a Machine due to their Inertia, . . 264
146. Virtual Machines, ... 265
Examples, ... 266

PART IV.—STIFFNESS AND STRENGTH OF MATERIALS.

147. Introductory Remarks, ... 271

CHAPTER XII.—SIMPLE TENSION, COMPRESSION, AND BENDING OF PERFECTLY ELASTIC MATERIALS.

SECTION I.—TENSION AND COMPRESSION.

148. Simple Tension, ... 274
149. Work done in Stretching a Rod, ... 276
150. Thin Pipes and Spheres under Internal Fluid Pressure, . . . 277
151. Remarks on Tension, ... 279
152. Simple Compression, ... 279
Examples, ... 280

SECTION II.—SIMPLE BENDING.

153. Proof that the Stress at each Point varies as its Distance from the Neutral Axis, ... , 281
154. Determination of the Position of the Neutral Axis, ... 283
155. Determination of the Moment of Resistance, ... 284
156. Remarks on the Theory of Bending, ... 286
157. Calculation of Moments of Inertia, ... 286
158. Beams of I Section with Equal Flanges, ... 287
159. Ratio of Depth to Span in I Beams, ... 288
160. Proportion of I Beams for Equal Strength, ... 289
161. Beams of Uniform Strength, ... 292
162. Unsymmetrical Bending, ... 292
Examples, ... 294

CHAPTER XIII.—DEFLECTION AND SLOPE OF BEAMS.

163. Deflection due to the Maximum Bending Moment, ... 296
164. General Equation of Deflection Curve, ... 297
165. Elementary Cases of Deflection and Slope, ... 298
166. Beam propped in the Middle, ... 299
167. Beam fixed at the Ends, ... 300
168. Stiffness of a Beam, ... 302
169. General Graphical Method, ... 303
170. Examples of Graphical Method, ... 304
171. Resilience of a Bent Beam, ... 306
Examples, ... 307

CHAPTER XIV.—TENSION OR COMPRESSION COMPOUNDED WITH BENDING. CRUSHING BY BENDING.

ARTICLE PAGE
172. General Formula for the Stress due to a Thrust or a Pull in Combination with a Bending Moment, ... 309
173. Strut or Tie under the action of a Force parallel to its Axis in cases where Lateral Flexure may be neglected, ... 309
174. Effect of a Thrust on a Loaded Beam, ... 311
175. Remarks on the Application of the General Formula, ... 311
176. Straining Actions due to Forces normal to the Section, ... 312
177. Maximum Crushing Load of a Pillar, ... 312
178. Manner in which a Pillar crushes. Formula for Lateral Deviation, . . 315
179. Actual Crushing Load, ... 316
180. Gordon's Formula, ... 318
181. Collapse of Flues, ... 320
Examples, ... 321

CHAPTER XV.—SHEARING AND TORSION OF PERFECTLY ELASTIC MATERIAL.

182. Distinction between Tangential Stress and Normal Stress. Equality of Tangential Stress on Planes at Right Angles, ... 323
183. Tangential Stress equivalent to a Pair of Equal and Opposite Normal Stresses. Simple Distorting Stress, ... 324
184. Torsion of a Shaft, ... 327
185. Diameter of a Shaft to transmit a given Power, ... 329
186. Distance apart of Bearings, ... 330
187. Web of a Beam of I Section, ... 331
188. Distribution of Shearing Stress on the Section of a Beam, ... 333
189. Deflection due to Shearing, ... 334
190. Weakening of Beams by Insufficient Resistance to Longitudinal Shearing of the Web, ... 335
191. Joints and Fastenings, ... 335
Examples, ... 338

CHAPTER XVI.—IMPACT.

192. Preliminary Remarks. General Equation of Impact, ... 340
193. Augmentation of Stress by Impact in Perfectly Elastic Material, . . 341
194. Sudden Application of a Load, ... 343
195. Action of a Gust of Wind on a Vessel, ... 344
196. Impact at High Velocities. Effect of Inertia, ... 345
197. Vibrating Loads. Synchronism, ... 347
198. Impact when the Limits of Elasticity are not exceeded. Resilience, . 349
Examples,. 350

CHAPTER XVII.—STRESS, STRAIN, AND ELASTICITY.

SECTION I.—STRESS.

199. Ellipse of Stress, ... 352
200. Principal Stresses. Axes of Stress, ... 354
201. Varying Stress. Lines of Stress. Bending and Twisting of a Shaft, . 355

ARTICLE PAGE
202. Straining Actions on the Web of an I Beam, ... 356
203. Remarks on Stress in General, ... 357
Examples, ... 358

SECTION II.—STRAIN.

204. Simple Longitudinal Strain. Two Strains at Right Angles, ... 359
205. Comparison between Stress and Strain, ... 361

SECTION III.—CONNECTION BETWEEN STRESS AND STRAIN.

206. Equations connecting Stress and Strain in Isotropic Matter, . . . 361
207. Elasticity of Form and Volume, ... 362
208. Modulus of Elasticity under Various Circumstances. Elasticity of Flexion, 363
209. Thick Hollow Cylinder under Internal Pressure, ... 365
210. Strengthening of Cylinder by Rings. Effect of great Pressures, . . 369
Examples, ... 370

CHAPTER XVIII.—MATERIALS STRAINED BEYOND THE ELASTIC LIMIT. STRENGTH OF MATERIALS.

211. Plastic Materials, ... 371
212. Flow of Solids, ... 373
213. Preliminary Remarks on Materials. Stretching of Wrought Iron and Steel, 373
214. Real and Apparent Strength of Ductile Metals, ... 377
215. Increase of Hardness by Stress beyond the Elastic Limit, ... 378
216. Compression of Ductile Materials, ... 379
217. Bending beyond the Elastic Limit, ... 380
218. Stretching of Cast Iron, ... 381
219. Crushing of Rigid Materials, ... 383
220. Breaking of Cast Iron Beams, ... 384
221. Shearing and Torsion, ... 384
222. Theories of Compound Strength, ... 386
223. Connection between the Co-efficients of Strength for Shearing and Tension, 387
224. Wöhler's Experiments on Alternate Bending and Twisting, ... 388
225. Influence of Fluctuation of Stress, ... 390
226. Impact, ... 391
227. Factors of Safety and Co-efficients of Working Strength, ... 392
228. Values of Co-efficients, ... 393
229. Fibrous Materials. Ropes, ... 394
230. Tables of Strength, ... 395
Examples, ... 398
Description of Plate VIII., ... 399

PART V.—TRANSMISSION AND CONVERSION OF ENERGY BY FLUIDS.

231. Introductory Remarks, ... 403

CHAPTER XIX.—ELEMENTARY PRINCIPLES OF HYDRAULICS.

232. Velocity due to a given Head, ... 405
233. Hydraulic Resistances in General, ... 407
234. Discharge from Small Orifices, ... 408

ARTICLE PAGE
235. Incomplete Contraction, ... 410
236. Discharge from Large Orifices in a Vertical Plane, ... 410
237. Head relatively to Moving Orifices, ... 412
238. Steady Flow through Pipes. Conservation of Energy, ... 412
239. Distribution of Energy in an Undisturbed Stream. Vortex Motion, . 414
240. Viscosity, ... 416
241. Surface Friction in General, ... 417
242. Surface Friction in Pipes, ... 419
243. Discharge of Pipes, ... 420
244. Open Channels, ... 421
245. Virtual Slope of a Pipe, ... 421
246. Loss of Energy by Eddies and Broken Water, ... 422
247. Bends in a Pipe. Surface Friction, ... 427
248. Summation of Losses of Head, ... 427
249. Resistance of Deeply Immersed Bodies. Ships at Low Speeds, . . 428
250. Direct Impulse and Reaction, ... 432
251, Oblique Action. Curved Surfaces, ... 433
252. Impulse and Reaction of Water in a Closed Passage, ... 435
Examples, ... 436

CHAPTER XX.—HYDRAULIC MACHINES.

253. Hydraulic Motors in General, ... 439
254. Weight Machines, ... 439
255. Hydraulic Pressure Machines in Steady Motion, ... 440
256. Hydraulic Pressure Machines in Unsteady Motion, ... 442
257. Examples of Hydraulic Pressure Machines, ... 445
258. Hydraulic Brakes, ... 446
259. Transmission of Energy by Hydraulic Pressure, ... 449
260. Pumps, ... 450
Examples, ... 453
261. Impulse and Reaction Machines in General, ... 454
262. Angular Impulse and Momentum, ... 457
263. Reaction Wheels, ... 458
264. Turbine Motors, ... 460
265. Turbine Pumps, ... 464
265A. Approximate Investigation of the Efficiency of a Centrifugal Pump, . 466
266. Limitation of Diameter of Wheel, ... 468
267. Impulse Wheels,. 468
268. Propellers, ... 469
Examples, ... 472

CHAPTER XXI.—PNEUMATICS AND THERMODYNAMICS.

269. Preliminary Remarks, ... 475
270. Expansive Energy, ... 475
271. Transmitted Energy, ... 477
272. Cycle of Operations in a Pneumatic Motor. Mechanical Efficiency, . . 479
273. Pneumatic Pumps, ... 480
274. Indicator Diagrams, ... 481
275. Cycle of Thermal Operations in a Heat Engine, ... 483
276. Mechanical Equivalent of Heat, ... 484
277. Mechanical Value of Heat, ... 484
278. Thermal Efficiency, ... 486

ARTICLE PAGE
279. Compound Engines, ... 487
280. Internal Energy. Internal Work, ... 488
281. Steady Flow through a Pipe. Conservation of Energy, ... 491
282. Velocity of Efflux of a Gas from an Orifice, ... 492
283. Discharge from an Orifice, ... 493
283A. Flow of Gases through Pipes, ... 496
284. Flow of Gases under Small Differences of Pressure, ... 498
285. Varying Temperature. Chimney Draught, ... 500
Examples, ... 501

APPENDIX.

A. NOTES AND ADDENDA.

I. Statics of Structures, ... 507
II. Kinematics of Machines, ... 510
III. Dynamics of Machines, ... 511
IV. Stiffness and Strength of Materials, ... 514
V. Hydraulics and Pneumatics, ... 522

B. ORGANIZATION OF THE CLASSES IN ENGINEERING AND NAVAL ARCHITECTURE IN THE ROYAL NAVAL COLLEGE, ... 527

C. ADDITIONAL NOTES (*added to the Second Edition*), ... 530

ERRATA.

PAGE 20, line 5 from bottom—*for* $PC = Wx$, *read* $PC = wx$.
PAGE 262, line 2 from bottom—*for* c, *read* $2c$.
PAGE 263, line 3—*for* $\sqrt{2c}$, *read* $c.\sqrt{2}$.
PAGE 498, line 12—*for* per 29 ft., *read* per sq. ft.

INTRODUCTION.

THE province of the Engineer, as described by TREDGOLD in a well-known passage in the Charter of the Institution of Civil Engineers, is to "direct the forces of nature" and apply them to useful purposes, such forces being either destructive or useful according to the manner in which they are directed. The protection of life and property from destructive forces is accomplished by means of pieces rigidly connected with one another which transmit their action to bodies to which they are not injurious; while the utilization of such forces in moving weights, changing the form of bodies, and other similar operations, is effected by a set of moving pieces which transmit their action to the required place and modify it in some given way. In the first case the pieces are called, collectively, a STRUCTURE, in the second, a MACHINE. The object of the present work is to give an outline of the principles on which structures and machines are designed.

The actual form of a mechanical construction of any kind is almost always the final result of a process of evolution by which it has been gradually perfected by adaptation from some previously existing construction. To meet new wants the engineer selects some arrangement, suggested by experience of some nearly similar case, which appears likely to answer the purpose by its simplicity, facility of construction, and adaptation to the forces which it is proposed to control and utilize. If the new arrangement is merely a copy of the old this may be sufficient and the construction may be at once proceeded with, but if there be any important difference it is necessary, before incurring the expense and risk of actual construction, to ascertain that the design is in conformity with certain general principles which reason and experience alike show to be necessarily true in all cases. To a certain extent this has already been considered by the designer, whose knowledge and experience enable him to avoid at once arrangements which are obviously inadmissible, but complete conformity can only be secured by comparison with results deduced by reasoning from those principles.

In any branch of knowledge the explanation of a set of facts by a general principle, from which new results can be obtained, is properly described as a Theory of the phenomena to which they relate. When its principles are well established it enables us to predict the results of experiment; when they are not, it is even more necessary, to direct the course which experiment should take for more perfect knowledge. The systematic study of structures and machines with a view to discover the theoretical principles on which their construction is based, and the deduction from those principles of results which may be useful to the designer, forms a branch of science which, following **Rankine**, we may describe as Applied Mechanics. In some cases, the subject may have been so exhaustively studied and may be in its nature so limited, that all the arrangements which can be employed for a given purpose may be foreseen and the best determined by *à priori* considerations. The process of invention itself then becomes a problem in science. This, however, is the rare exception; in general, the use of theory is limited to the answering of certain questions relating to an arrangement which has already been proposed. Among the most important of these are—

(1.) What should be the dimensions of the parts of the construction that they may be strong enough to resist the action of the forces to which it is exposed?

(2.) Will the construction be sufficiently stable and rigid?

(3.) Are the natural forces, which it is proposed to utilize, sufficient for the proposed purpose and are they under proper control?

It is only in the very simplest cases that these and similar questions can be answered completely, without reference to the direct results of experience in order to interpret theoretical reasoning and render it applicable. Even, therefore, after the general plan of a construction is decided on, the work of the practical designer includes much which cannot be reduced to a mere process of deduction from given data. Nevertheless the part of theory in controlling and directing inventive power is of great and constantly-increasing importance, by furnishing principles of universal application, in conformity with which every mechanical construction must be designed, and by which the researches of the experimentalist must be guided.

The mechanics of structures and machines is based on the properties of materials, and on the general laws connecting matter and motion, the study of which is the object of Abstract Mechanics, but the special nature of the subject matter occasions a certain difference in the methods employed. In the elementary branches of purely abstract mechanics the number of bodies considered seldom exceeds two; if more are introduced the questions to be considered become impracticably complex considered as abstract mathematical problems. In applied mechanics a number of pieces are connected with comparatively little freedom so as to form an organic whole, and the results of experience or of mathematical investigations too complex for ordinary use are admitted freely for the purpose of simplification. Hence the calculations employed are of a coarser type, and, in particular, graphical methods are everywhere employed when possible, not only to exhibit, but also to obtain, numerical results. On the other hand, no investigation is considered as complete until it has been checked by reference to experience, and unless its errors are approximately known. The elementary principles of abstract statics, dynamics, and hydrostatics must be supposed already known, and some practical knowledge of machines and structures is presupposed.

The classification of mechanical constructions depends in great measure on the number of pieces connected and on the mode of connection. We have first the broad distinction between structures, in which the pieces have no movements except such as may be due to changes in their form and dimensions consequent on the forces to which they are exposed, and machines in which the object is attained by means of such movements. This distinction is so fundamental that there is no word in common use which includes both.

Structures may be ranged in order of simplicity according to the degree of constraint with which their parts are connected as follows:—

(1.) Structures with pin joints without redundant parts.
(2.) Structures with pin joints which include redundant parts.
(3.) Blockwork and earthwork structures.
(4.) Structures with rivetted or other forms of fastened joints.

A pin joint, such as is shown in a simple form in Figs. 1 and 2, Plate VIII., page 399, is one in which the pieces connected are united by a single pin fitting into holes in the pieces, and, in consequence, neglecting friction, the mutual action between the pieces connected necessarily passes through the axis of the pin. A redundant part is one which may be removed without destroying the structure if the remaining parts be sufficiently strong. The first class of structures therefore possess a peculiar characteristic which renders their theory much more simple than that of any other, namely, that the forces acting on each piece depend only on the external forces acting on the whole and not on the material or the dimensions of the pieces. In the theory of structures, then, this class is first considered, and the answer to the first of the general questions propounded above consists in the solution of two general problems.

(1.) Being given the load on the structure, it is required to find the forces acting on each part.

(2.) Being given the forces acting on a piece of material, it is required to find its dimensions that it may be sufficiently strong and stiff.

The first forms a part of the subject which may be properly described as the "Statics of Structures"; while the second, which depends on the properties of the materials of construction, is known as the "Strength and Stiffness of Materials." The results obtained are in continual requisition in the theory of the more complex structures, but require to be supplemented by further investigations and by results derived from direct experience, peculiar to each class. The present treatise, being simply introductory, refers to the more complex structures only incidentally.

A Machine is a structure the parts of which are in motion. The motion introduces new forces, often of great magnitude and importance, which must be taken into account in its design; but we have, in addition, to consider the third general question mentioned above, namely, the adaptation of the natural forces available to the work which the machine has to do. The simplest machines consist chiefly of a number of rigid pieces, and their theory is divided into two parts—one concerned with the motion of the machine, the other with the work it does. In many of the most important machines fluids are used, and their theory forms a distinct branch of the subject not less important than the rest, some account of which is indispensable. Thus the whole subject is divided into five parts.

Since the parts of structures as well as machines possess, though to a very limited extent, freedom to move, and since such movements often have to be supposed for the purposes of an investigation, the most natural arrangement perhaps would be to commence with the first part of the theory of machines, and then pass on to the statics of structures. It has, however, been found convenient to invert this order, and we now, therefore, commence with structures.

PART I.

STATICS OF STRUCTURES.

PART I.—STATICS OF STRUCTURES.

CHAPTER I.

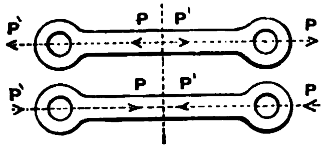
FRAMEWORK LOADED AT THE JOINTS.

1. *Preliminary Explanations and Definitions.*—A frame is a structure composed of bars, united at their extremities by joints, which offer no resistance to rotation. In the first instance we may suppose the centre lines of the bars all in one plane, and in that case the joints may consist simply of smooth pins passing through holes at the ends of the bars, which are to be imagined forked, if necessary, so as to allow the centre lines to meet in a point. A large and important class of structures, known to engineers as "trusses," approach so closely to frames that calculations respecting them may be conducted by treating them as if they were frames. The differences between a truss and a frame will appear as we proceed.

The frame may be acted on by forces applied at points in one or more of its bars, or at the joints which unite the bars together. An important simplification, however, is effected by supposing, in the first instance, that the joints only are loaded, an assumption which will be made throughout this chapter, except in a few simple examples. It will be shown hereafter (p. 75), that all other cases may be derived from this by means of a preliminary reduction.

Assuming, then, that the frame is acted on by forces at the joints, due either to weights or other external causes, or to the reaction of supports on which the frame rests, the problem to be solved is to find the forces called into play on each of the bars of which it is constructed. These forces are caused by the pressure of the pins on the sides of the holes through which they pass, and it at once follows, since no other forces act on the bar, that for each bar these pressures must be equal and opposite, their common line of action being the line joining the centres of the holes. There are two possible cases shown in Figs. 1a, 1b; in the first the bar is acted on by a pair of equal and opposite forces tending to lengthen it, and in the second to shorten it. The pairs of forces are called a Pull and a Thrust respectively, while the bars subjected to their action are called Ties or Struts respectively. Between a pull and a thrust there is no statical difference but that of sign; the constructive difference, however, between a tie and a strut is great. The first may theoretically be a rope or chain, and the second may be made up of pieces simply butting against one another without fastening, while a rigid bar will serve either purpose, though its powers of resistance are generally entirely different in the two cases.

Figs. 1a, 1b.



It often happens that it is unknown whether a bar be a strut or a tie, and the pair of forces are then called a STRESS on the bar. This word "stress" was introduced by Rankine to denote the mutual action between any two bodies, or parts of a body, and here means, in the first instance, the mutual action between the parts of the frame united by the bar we are considering. If, however, we imagine the bar cut into two parts, *A* and *B*, by any transverse section, as shown in Figs. 1a, 1b, those parts are held together in the case of a pull, or thrust away from each other in the case of a thrust, by internal molecular forces called into play at each point of the transverse section, and acting one way on *A* and the other way on *B*. As *A* and *B* must both be in equilibrium, it is obvious that these internal forces must be exactly equal to the original forces, and thus it appears that the stress on the bar may also be regarded as the internal molecular action between any two parts into which it may be imagined to be divided. Stress, regarded in this way, will be fully considered in a subsequent division of this work; it will be here sufficient to say that its intensity is measured by dividing the total amount by the sectional area of the bar, and is limited to a certain amount, depending on the nature of the material of which the bar is constructed.

It is further manifest from what has been said, that the stress on a bar may likewise be regarded as a mutual action between the bar and either of the pins at its ends which are pulled towards the middle of the bar in the case of a pull, or thrust away from it in the case of a thrust; each pin is therefore acted on, in addition to any load which may be suspended from it, by forces, the directions of which are the lines joining the centres of the pins, from which it follows at once that *every joint may be regarded as a point kept in equilibrium by the load at that joint and by forces of which the bars of the frame are the lines of application.* This principle enables us to find the stress on each bar of a frame loaded at the joints whenever such stress can be determined by statical considerations alone, without reference to the material or mode of construction, that is to say, in all cases which properly belong to the present division of our work.

Forces are measured in pounds-weight or, when large, in tons of 2240 lbs. They are often distributed over an area or along a line, and are then reckoned per square foot or per "running" foot, the last expression being commonly abbreviated to "foot-run."

The bars need not be connected by simple pin joints as has been supposed for clearness, provided that their centre lines if prolonged meet in a point through which passes the line of action of the load on the joint. This point may be called the centre of the joint, and we may replace the actual joint by a simple pin, or, if the bars are not in one plane, by a ball and socket which has the same centre. We shall return to this hereafter, but now pass on to consider various kinds of frames, commencing with the simplest.

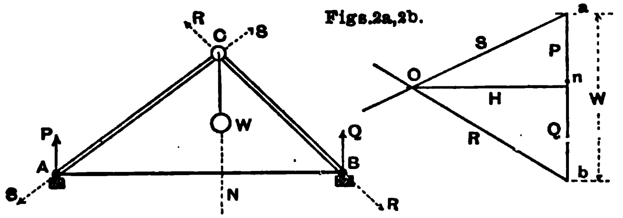
SECTION I.—TRIANGULAR FRAMES.

2. *Diagram of Forces for a Simple Triangular Frame.*—The simplest kind of frame is a triangle.

In Fig. 2a, ACB is such a triangle; it is supported at AB so that AB is horizontal, and loaded at C with a weight W. Then evidently the effect of the weight is to compress AC, BC, and to stretch AB, which is conveniently indicated by drawing AC, BC in double lines, and AB in a single line. Also the weight produces certain vertical pressures on the supports A, B, which will be balanced by corresponding reactions P and Q.

To find the magnitude of the thrust on AC, BC, the pull on AB, and the reactions, the diagram of forces Fig. 2b is drawn: ab is a vertical line representing W on any convenient scale, while aO, bO are lines drawn through a, b respectfully, parallel to AC, BC, to meet in O, and finally On is drawn parallel to AB, or, what is the same thing, perpendicular to ab. Now, applying the fundamental principle laid down above, we observe that C is a point kept in equilibrium by three forces, the load at C, namely W, the thrust of AC which we will call S, and the thrust of BC which we will call R. In the second figure the triangle

Oab has its sides parallel to these forces, and hence it follows that *Oa*, *Ob* represent *S*, *R* on the same scale that *ab* represents *W*. Again *A* is a point kept in equilibrium by three forces, the thrust of *AC*, the pull of the tie *AB*, which we will call *H*, and the upward reaction *P* of the support *A*. But referring to the figure 2*b*, *On*, *an*, are respectively parallel to the two last forces, so that, by the triangle of forces, they represent *H*, *P* on the same scale that *Oa* represents *S*. The same reasoning applies to the point *B*, and therefore *bn* represents the other supporting force *Q*, as is also obvious from the consideration that $P + Q = W$. We thus see that all the forces acting upon and within the triangular frame *ACB* are represented by corresponding lines in Fig. 2*b*,


Figs. 2a, 2b.

which is thence called the "diagram of forces" for the triangular frame. Such a diagram can be drawn for any frame, however complicated, and its construction to scale is the best method of actually determining the stresses on the several parts of the frame.

The force *H* requires special notice: it is called the "*thrust*" of the frame. In the present case the thrust is taken by the tension of the third side of the triangle, but this may be omitted, and the supports *A* and *B* must then be solid and stable abutments capable of resisting a horizontal force *H*. In many structures such a horizontal thrust exists; and its amount and the mode of providing against it are among the first things to be considered in designing the structure. Besides the graphical representation just given, which enables us to obtain the thrust of a triangular frame by constructing a simple diagram, it may also be calculated by a formula which is often convenient. Let *AC* be denoted by *b* and *BC* by *a*, as is usual in works on trigonometry, and let *AN*, *BN* their projections on *AB* be called *b'*, *a'*, and let the height of the triangle be *h* and its span *l*, then by similar triangles,

$$\frac{P}{H}=\frac{an}{On}=\frac{CN}{AN}=\frac{h}{b'},$$

$$\frac{Q}{H}=\frac{bn}{On}=\frac{CN}{BN}=\frac{h}{a'}.$$

Therefore, by addition,

$$\frac{W}{H}=\frac{ab}{On}=h\left(\frac{1}{b'}+\frac{1}{a'}\right)$$

$$\text{or}\quad H=W\frac{a'b'}{lh}.$$

In practical questions it often happens that a', b', h are known by the nature of the question, whence H is readily determined. The case when the load bisects the span may be specially noticed; then $a'=b'=\frac{1}{2}l$ and

$$H=\frac{Wl}{4h}.$$

When the height of the frame is small compared with the span, this calculation is to be preferred to the diagram, which cannot then be constructed with sufficient accuracy.

The simple frame here considered may be inverted, in which case the diagram of forces and the numerical results are unaltered, the only change being that the two struts have become ties and the tie a strut.

3. *Triangular Trusses.*—Triangular frames are common in practice, and the rest of this section will be devoted to some of the commonest forms in which they appear.

Fig. 3a shows a simple triangular truss consisting of a beam, AB, supported by a strut at the centre, the lower extremity of which is carried by tie rods, AC, BC, attached to the ends of the beams. If now a weight, W, be placed at the centre, immediately over the strut, it does not bend the beam (sensibly) as it would do if there were no strut, but is transmitted by the strut to the joint C, so that the truss is equivalent to the simple triangular frame of the last article. This, however, supposes that the strut has exactly the proper length to prevent any bending of the beam; if it be too short or too long the load on the frame will be less or greater than W, a point which will be further considered presently. It should be noticed that D is not necessarily at the centre.

Fig.3a.

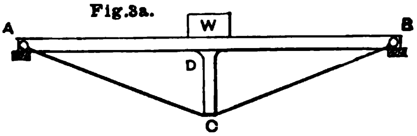


Fig.3b.

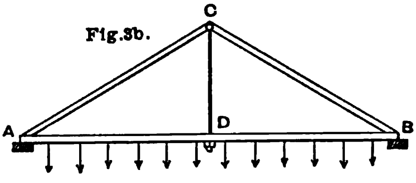


Fig. 3b shows the same construction inverted. CD is a tie by which D is suspended from C; we will suppose this rod to pass through AB and a nut applied below, by means of which D may be raised or lowered.

Let *AB* now be uniformly loaded with a given weight, then the bending of *AB* is resisted by *CD*, which supports it and carries a part of the load, which may be made greater or less by turning the nut. If, however, we imagine *AB*, instead of being continuous through *D*, to be jointed at *D*, then the tie *CD* necessarily carries half the weight of *AD* and half the weight of *BD*, that is to say, half the whole load, whatever be its exact length. This simple example illustrates very well the most important difference between a truss and a mathematical frame; namely, that in the truss one or more of the bars is very often continuous through a joint. Such cases can only be dealt with on the principles of the present division of our work, by making the supposition that the bar in question, instead of being continuous, is jointed like the rest. The error of such a supposition will be considered hereafter; it is sufficient now to say that in order that it may be exact in the particular case we are considering, the nut must be somewhat slackened out so that *D* may be below the straight line *AB*, and that being dependent on accuracy of construction, temperature, and other varying circumstances, such errors cannot be precisely stated, but must be allowed for in designing the structure by the use of a factor of safety. The supposition is one which is usual in practical calculations, and will be made throughout this division of our work.

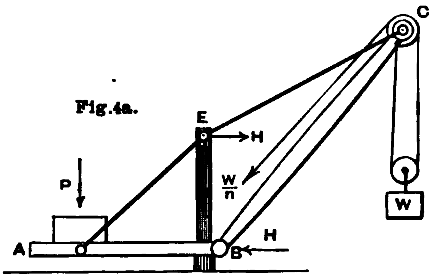
The foregoing is one of the simplest cases where, as is very common in practice, the bars of the frame are loaded and not the joints alone. When such bars are horizontal and uniformly loaded, the effect is evidently the same as if half the load on each division of the loaded bar were carried at each of the joints through which it passes. This is also true if the loaded bars be not horizontal, but the question then requires a much more full discussion, which is reserved for a later chapter (see Ch. IV.).

When one of the joints of the loaded bar is a point of support, like *A* in Fig. 3, the supporting force is due partly to the half weight of one or more divisions of the loaded bar, and partly to the downward pull or thrust of other bars meeting there: the first of these causes does not affect the stress on the different parts of the truss, and the calculations are therefore made without any regard to it. The explanations given in this article should be carefully considered, as they apply to many of the examples subsequently given.

The triangular truss in both the forms given in this article is frequently employed in roofs and bridges of small span, as well as for other purposes.

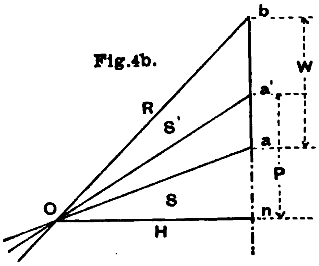
4. *Cranes.*—The arrangements adopted for raising and moving weights furnish many interesting examples of triangular frames. Fig. 4a shows one of the forms of the common crane, a machine the essential members of which are the jib, *BC*, supported by a stay, *CE*, attached to the crane-post, *BE*, which is vertical. In cranes proper this third member rotates, carrying *BC* and *CE* with it, but in the sailors' derrick a fixed mast plays the part of a crane-post and the stay, *CE*, is a lashing of rope frequently capable of being lengthened and shortened by suitable tackle, so as to raise and lower the jib, a motion very common in cranes and hence called a derrick motion. The weight is generally also capable of being raised and lowered directly by blocks and tackle, but for the present will be supposed directly suspended from *C*.

Fig.4a.



The diagram of forces now assumes the form shown in Fig. 4b, in which the lettering is the same as in Fig. 2b, page 4, the only difference in the diagrams being that in the present case *AC*, which is now a tie, is divided into two parts, *AE* and *EC*, inclined at an angle. The stress on *AE* is therefore not the same as on *EC*, but is got by drawing a third line, *Oa'*, parallel to *AE*. The perpendicular *On* gives us in this instance not only the stress on *AB* and the horizontal thrust of *CB* at *B*, but also the horizontal pull of *CE* at *E*—we may call this *H* as before. There is an upsetting moment on the structure as a whole which is equal to the product of the weight *W* by its horizontal distance from *B* (often called the radius of the crane) and also to the force *H*, multiplied by the length of the crane-post, *BE*. One principal difference between different types of cranes lies in the way in which this upsetting moment is provided against.

Fig.4b.

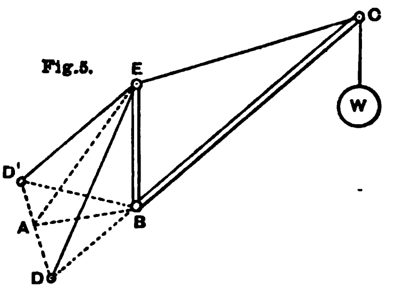


(*a.*) In portable cranes, such as shown in Fig. 4a, there is a horizontal platform, *AB*, supported by a stay, *AE*, and carrying a counterbalance weight, *P*, sometimes capable of being moved in and out so as to provide for different loads. The right magnitude of counterbalance weight and the pull on the stay *AE* are shown by the diagram *P* corresponding to the supporting force at *A* in the previous case.

(*β.*) In the pit crane, the post is prolonged below into a well and the lower end revolves in a footstep, the upper bearing being immediately below *B*. In this instance the post has to be made strong enough to resist a bending action at *B*, equal to the upsetting moment, and the bearings have to resist a horizontal force equal to *H* multiplied by the ratio of the length of the crane-post, *BE*, to that of its prolongation below the ground.

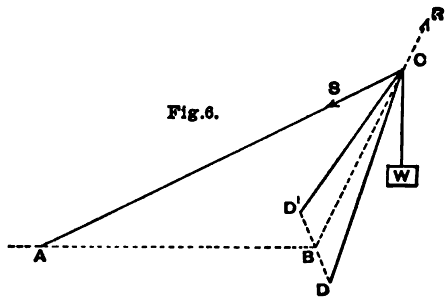
(γ.) The upper end of the crane-post may revolve in a headpiece, which is supported by a pair of stays anchored to fixed points in the ground. The upright mast of a derrick frequently requiring support in the same way, this arrangement is known as a derrick-crane. It is shown in Fig. 5, *ED*, *ED′* being the stays. To find the stress on the stays it is necessary to prolong the vertical plane through *EC*, to intersect the line *DD′*, joining the feet of the stays in the point *A*, and imagine the two stays, *ED*, *ED′*, replaced by a single stay *EA* : then a diagram of forces, drawn as in the previous case, determines *S′*, the pull on this stay. But it is clear that *S′* must be the resultant pull on the two original stays, and may be considered as a force applied at *E* in the direction of *AE* to the simple triangular frame *DED′*. A second diagram of forces therefore will determine the pull on each stay, just as in the next following case.

Fig. 5.



5. *Sheer Legs and Tripods.*—Instead of employing an upright post to give the necessary lateral stability to the triangle, one of its members may be separated into two. Thus in moving very heavy weights sheer legs are used, the name being said to be derived from their resemblance to a gigantic pair of scissors (shears) partly opened and standing on their points. In Fig. 6, *CD*, *CD′* are spars, or tubular struts, often of great length, resting on the ground at *DD′* and united at *C*, so as to be capable of turning together about *DD′* as an axis. The load is carried at *C* and the legs are supported by a stay, *CA*, which is sometimes replaced by a rope and tackle, capable of being lengthened or shortened

Fig. 6.



so as to raise or lower the sheers. Drawing AB to the middle point of DD', the pair of legs are to be imagined replaced by a single one, CB, then the diagram of forces may be constructed just as in Fig. 4b, and we shall obtain the tension of the rope S and the resultant thrust on the pair of legs R. Now draw the triangle CDD', as in Fig. 7a, and imagine it loaded at C with a weight, R, then drawing the diagram of forces, Fig. 7*b*, we get R' the thrust on each leg. The horizontal force, H', in this second diagram represents the tendency of the feet of the legs to spread outwards laterally, while the force, H, of the original diagram represents their tendency to move inwards perpendicular to DD'. In some cases the guy rope and tackle CA are replaced by a third leg called the back leg, and the sheers are then raised and lowered by moving A by a large screw; the force H is then also the force to be overcome in turning the screw.

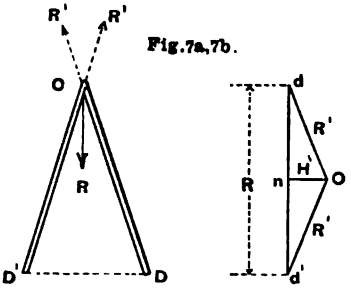


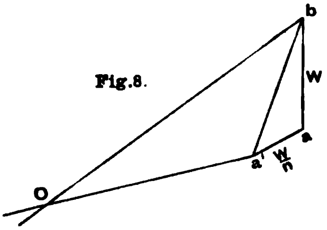
Fig. 7a, 7b.

Instead of having only two legs, as in sheers, we may have three forming a tripod. This arrangement is frequently used to obtain a fixed point of attachment for the tackle required to raise a weight, and is sometimes called a "gin," or as military engineers prefer to spell the word, a "gyn." The thrust on each leg and the tendency of the legs to move outwards can be obtained by a process so similar to that in the preceding examples that we need not further consider it.

6. *Effect of the Tension of the Chain in Cranes.*—In most cases the load is not simply suspended from C as has been hitherto supposed, but is carried by a chain passing over pulleys and led to a chain barrel, generally placed somewhere on the crane-post. The tension of the chain in this case is W/n, where n is a number depending on the nature of the tackle, and this tension is to be considered as an additional force applied at C to be compounded with the load W, the effect of which has been previously considered. Fig. 8 shows the form the diagram of forces assumes in this case. Drawing ba as before to represent W, and aa' parallel to the direction in which the chain is led off from the pulley at C and equal to the tension W/n, the third side of the triangle, ba' must be the resultant force at C due to both forces, whence drawing $a'O$ parallel to the stay and bO parallel to the jib, and reasoning as before as to the equilibrium of the forces at C, we see that these lines must be the tension of the stay and the thrust on the jib. The

effect of the tension of the chain is generally to diminish the pull on the stay and increase the thrust on the jib, sometimes very considerably, as for example in certain older types of crane still used for light loads under the name of "whip" cranes. In these cranes the chain passes over a single fixed pulley at the end of the jib, and is attached directly to the weight, so that the tension of the chain is equal to the weight. The other end of the chain is led off along a horizontal stay to a wheel and axle at the top of the crane post, a chain from the wheel of which passes to a windlass below. This arrangement, the double windlass of which facilitates changes in the lifting power corresponding to the load to be raised, is a development of the primitive machine in which the wheel was a tread wheel worked by men or animal power. In this case the pull on the stay is diminished by the whole weight lifted, and is thus reduced very much. Where a crane has to be constructed of timber only, this is a considerable advantage, from the difficulty of making a strong tension joint in this material.

Fig. 8.



EXAMPLES.

1. The slopes of a simple triangular roof truss are each 30°. Find the thrust of the roof and the stress on each rafter when loaded with 250 lbs. at the apex.

Thrust of roof = 216·5 lbs.
Stress on rafters = 250 „

2. A beam 15 feet long is trussed with iron tension rods, forming a simple triangular truss 2 feet deep. Find the stress on each part of the frame when loaded with 2 tons in the middle.

Thrust on strut = 2 tons.
Pull of tension rods = 3·88 „
Thrust on beam = 3·75 „

3. The platform of a foot bridge is 20 feet span, and 6 feet broad, and carries a load of 100 lbs. per sq. ft. of platform. It is supported by a pair of triangular trusses each 3 feet deep, one on each side of the bridge. Find the stress on each part of one of the trusses.

The whole load of 12,000 lbs. rests equally on the two trusses, there is therefore 6,000 lbs. distributed uniformly along the horizontal beam of each truss.

Thrust on strut = 3,000 lbs.
Tension of tie rods = 5,220 „
Thrust on horizontal beam = 5,000 „

4. The slopes of a simple triangular roof truss are 30° and 45° and span 10 feet. The rafters are spaced $2\frac{1}{2}$ feet apart along the length of the wall, and the weight of the roofing material is 20 lbs. per sq. ft. Find by graphical construction the thrust of the roof.

Each rafter carries a strip of roof $2\frac{1}{2}$ feet wide, the load on rafter = 50 lbs. per foot length of rafter. Find the lengths by construction or otherwise. The virtual load at apex = $\frac{1}{2}$ weight on the two rafters = 311 lbs.

Thrust of roof = 198 lbs.

5. The jib AC of a ten-ton crane is inclined at 45° to the vertical, and the tension rod BC at an angle of 60°. Find the thrust of the jib, and the pull of the tie rod when fully loaded, the tension of the chain being neglected. If a back stay BD be added inclined at 45°, and attached to the end of a horizontal strut AD, find the counterbalance weight required at D to balance the load on the crane, and find also the tension of the back stay.

Thrust on jib AC = 33·5 tons.
Tension of tie rod = 27·5 ,,
Counterbalance weight = 23·5 ,,
Tension of back stay = 33·5 ,,

6. A pair of sheer legs are 40 feet high when standing upright, the lower extremities rest on the ground 20 feet apart, the legs stand 12 feet out of the perpendicular, and are supported by a guy rope attached to a point 60 feet distant from the middle point of the feet. Find the thrust on each leg, and the tension of the guy rope under a load of 30 tons.

Thrust on each leg = 19·5 tons.
Tension of guy rope = 12·8 ,,

7. In example 5 the tension of the chain is half the load, and the chain barrel is so placed that the chain bisects the crane post AB. Find the stress on the jib and tie rod.

Thrust of jib = 36 tons.
Pull of tie rod = 25 ,,

8. In a derrick crane the projections of the stays on the ground form a right-angled triangle, each of the equal sides of which is equal to the crane post. The jib is inclined at 45° and the stay at 60° to the vertical. Find the stress on all the parts (1) when the plane of the jib bisects the angle between the stays; (2) when it is moved through 90° from its first position. Load 3 tons.

Answer.—Case 1. Pull on each stay = 7·1 tons.
Case 2. Pull on one = thrust on other = 7·1 ,,

9. A load of 7 tons is suspended from a tripod, the legs of which are of equal length and inclined at 60° to the horizontal. Find the thrust on each leg. If a horizontal force of 5 tons be applied at the summit of the tripod in such a way as to produce the greatest possible thrust on one leg, find that thrust and determine the stress on the other two legs.

Answer.—Case 1. Thrust on each leg = 2·7 tons.
Case 2. Thrust on one leg = 6·7 ,,
Pull on each of the others = 3½ ,,

SECTION II.—INCOMPLETE FRAMES.

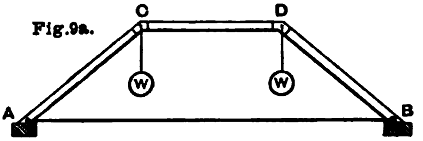
7. *Preliminary Remarks.*—A frame may have just enough bars and no more to enable it to preserve its shape under all circumstances, or the number of bars may be insufficient or there may be redundant bars. The distinction between these three classes of frames is very important: in the first the structure will support any load consistent with strength, and the stress on each bar bears a certain definite relation to the load, so that it can be calculated without any reference to the material or mode of construction; in the second, the frame assumes different forms according to the distribution of the load, but the stress on each bar can still be calculated by reference to statical considerations alone; in the third, where the frame has redundant bars, the stress on some or all of the bars depends on the relative yielding of the several bars of the

frame. It is to the second class, which may be called "incomplete" frames, that the present section will be devoted.

In incomplete frames the structure changes its form for every distribution of the load, and, strictly speaking, therefore, such constructions cannot be employed in practice, because the distribution of the load is always variable to a greater or less extent. But when the greater part of the load is distributed in some definite way the principal part of the structure may consist of an incomplete frame, designed for the particular distribution in question, and subsequent moderate variations of distribution may be provided for either by stiffening the joints or by subsidiary bracing. Such cases are common in practice, and investigations relating to incomplete frames are therefore of much importance.

8. *Simple Trapezoidal or Queen Truss.*—We will first consider a frame which is composed of four bars. The most common case is that in which two of the bars are horizontal and the other two equal to one another, thus forming a trapezoid. The structure is called a *trapezoidal frame.*

Fig.9a.



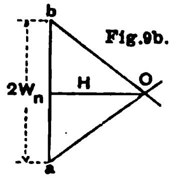


Fig.9b.

It is suitable for carrying weights applied at the joints *CD*, either directly or by transmission through vertical suspending rods from the beam *AB*. From the symmetry of the figure it is evidently necessary for stability that the loads at *C* and *D* should be equal. This fact will also appear from the investigation. Consider first the joint *C*, and draw the triangle of forces, *Oan*, for that point; *an* being taken to represent *W*, *aO* will represent the thrust on *AC* and *On* that along *CD*. The triangle *Obn* will represent the forces at the joint *D*, *Ob* representing the thrust of *BD*; *bn* will represent the load at *D*, and from the symmetry of the figure must equal *an*, and hence weight at *D* must for equilibrium equal that at *C*. Now let us proceed to joint *A*, where there are also three forces acting, one along *AC* is now known and represented by *aO*, thus *On* will represent the tension of *AB*, and *an* will be the necessary supporting force at *A* equal to *W*, as might be expected. The tension of *AB* is equal to the thrust on *CD*. We observe that the diagram of forces is the same as that of a triangular frame, carrying 2*W* at the vertex and of span equal to the difference between *AB* and *CD*.

Trapezoidal frames are employed in practice for various purposes.

(α.) A beam, AB (Fig. 10a), loaded throughout its length may be strengthened by suspending pieces, CN, DM, transmitting a part of the weight to the arch of bars AC, CD, BD, an arrangement common in small bridges.

Fig. 10a.

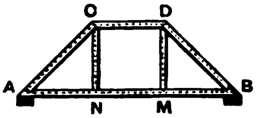
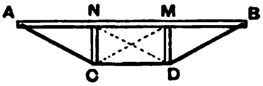


Fig. 10b.



(β.) As a truss for roofs, in which case there will be a direct load at C and D due to the weight of the roofing material, while vertical members serve partly as suspending rods by which part of the weight of tie beam and ceiling (if any) is transmitted to CD, and partly to enable the structure to resist distortion under an unequal load. When made of wood, this is the old form of roof called by carpenters a "Queen Truss," CN, DM, being the "queen posts" (see Section III. of this chapter). This name is constantly used for all forms of trapezoidal truss erect or inverted which include the vertical "queens."

(γ.) Not less common is the inverted form, Fig. 10b, applied to the beams carrying a traversing crane, the cross girders which rest on the main girders of a railway bridge and carry the roadway, and many other purposes. The bars AC, CD, BD are now iron tie rods. In this case also if the two halves of the beam are unequally loaded there will be a tendency to distortion, to resist which completely, diagonal braces, CM, DN, must be provided, as shown in the figure by dotted lines. Such diagonal bars occur continually in framework, and their function will be fully considered in the next chapter. But in the present case they are quite as often omitted, the heavy half of the beam then bends downwards and the light half bends upwards (see Ex. 4, p. 86), but the resistance of the beam to bending is found to give sufficient stiffness.

9. *General case of a Funicular Polygon under a Vertical Load. Example of Mansard Roof.*—We next take a general case. In Fig. 11a 0 1 2 3 ...6 is a rope or chain attached to fixed points at its ends and loaded with weights W_1 W_2...suspended from the points 1, 2, etc. The figure shows 5 weights, but there may be any number. The rope hangs in a polygon the form of which depends on the proportions between the weights. It is often called a "funicular polygon" and possesses very important properties. We shall find it convenient to distinguish the sides of this polygon by letters a, b, c, etc. We are about to determine the proportions between the weights when the rope hangs in a given form, and, conversely, the form of the rope when the weights are given.

In Fig. 11b draw *ab* vertical to represent W_1, the load suspended at the angle of the polygon where the sides *a* and *b* meet, then draw *aO*, *bO* parallel to *a*, *b* respectively to meet in *O*, thus forming a triangle *Oab*, which we distinguish by the number 1, which represents the forces acting on the point 1, so that the tensions of the sides *a*, *b* are thus determined. Now draw *Oc* parallel to the side *c* to meet the vertical in *c*; we thus obtain a triangle distinguished by the number 2, which represents the forces acting at that point, and as *Ob* is already known to be the tension of *b* it follows that *bc* must be the weight W_2, and *Oc* the tension of the side *c*. Proceeding in this way we get as many triangles as there are weights, and the sides of these triangles must represent the weights and the tensions of the parts of the rope to which they are respectively parallel. Thus, if the form of the rope is known and one of the weights, all the rest can be determined. Conversely, to find the form of the funicular polygon when the weights are given in magnitude and line of action, we have only to set downwards on a vertical line the weights in succession and join the points *a b*..., which will now be known, to any given point *O*, then the funicular polygon must have its angles on the lines of action of the weights and its sides parallel to the radiating lines *Oa*, *Ob*, *Oc*, etc., so that the sides can be drawn in succession, starting from any point we please.

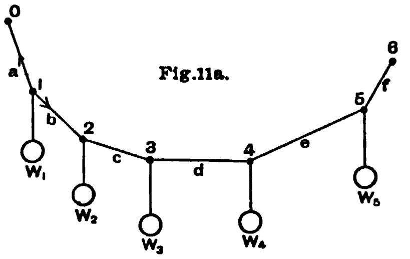


Fig. 11a.

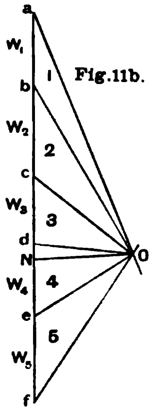


Fig. 11b.

In the diagram of forces, Fig. 11b, if *ON* be drawn horizontal to meet the vertical *a*, *b*, *c*...in *N*, this line must represent the horizontal tension of the rope.

The rope may be replaced by a chain of bars which may be inverted, thus forming an arch resting on fixed points of support, the diagram of forces will be unaltered, and *ON* will represent the thrust of the arch. As an elementary example of an arch of bars we will consider a truss used for supporting a roof of double slope called a Mansard roof. We will take the usual case in which the truss is symmetrical about the centre. Suppose it is loaded at the joints. There is one proportion of load which the truss is able to carry without any bracing bars being added.

From symmetry the weights at 2 and 2′ (See Fig. 12a) must be equal. To find the proportion between the weights at 1, and at 2, 2′, together with the stresses on the bars of the frame, in Fig. 12b set down aa' to represent W at 1, and draw aO and $a'O$ parallel to a and a', the thrusts along these bars will be determined. Then, considering the equilibrium of either 2 or 2′, say 2, one of the three forces acting at the joint, namely aO, along the bar a being known, the other two forces may be determined by drawing ab and Ob parallel to them, ba parallel to W_2, and Ob to the bar b. If ON be drawn horizontally it will give the amount of the horizontal thrust of the roof or the tension of a tie bar 3 3′, if there is such a bar. If the proportion of W_2 to W_1 is greater than ab to aa^1 the structure will give way by collapsing, 2 and 2′ coming together; and if the proportion is less, the structure will give way by 2 and 2′ moving outwards and 1 falling down between. In practice it is impossible to secure the necessary proportion of loads, on account of variation of wind pressure and other forces, and therefore stiffening or some kind is always needed. If bracing bars be placed as shown by the dotted lines 2 3′, 2′3, 2 2′, the structure will stand whatever be the proportion between the loads. The truss may be partially braced by the horizontal bar 2 2′ only. Then the proportion between the loads W_1 and W_2 may be anything we please, but the loads at 2 and 2′ must be equal, at least theoretically, but in practice the stiffness of the joints will generally be sufficient for stability, especially if vertical pieces be added connecting these points to the tie beam as in a queen truss.

Fig.12a.

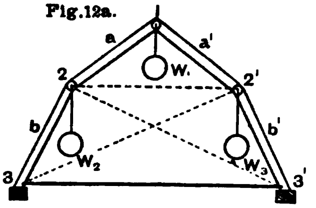
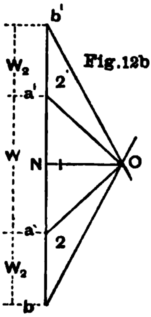


Fig.12b.



10. *Suspension Chains. Arches. Bowstring Girders.*—We now go on to consider another important example, in which the number of bars composing the frame is very much increased, as found in the common suspension bridge.

Let AB (Fig. 13a) be the platform of a bridge of some considerable span, which has little strength to resist bending. Suppose it divided into a number of equal parts, an odd number for convenience, say nine, and each point suspended by a vertical rod from a chain of bars secured at the ends to fixed points, D and E, in a horizontal line. In the figure only half the structure is shown. Suppose the platform loaded with a uniformly distributed weight; we require to know the stress on each bar and the form on which the chain will hang. Equal weights on each division of the platform will produce equal tensions in the vertical suspending rods, and if we neglect the differences of weight of the rods and bars themselves, the load at each joint of the chain of bars will be the same. (Comp. Art. 11.) Let W = load at each joint. Now the centre link, KK', since there is an odd number and the chain is symmetrical, will be horizontal. Let us consider the equilibrium of the half chain between C and D. The four weights, W, hanging at K, L, M, N, are sustained in equilibrium by the tensions of the bars KK' and ND.

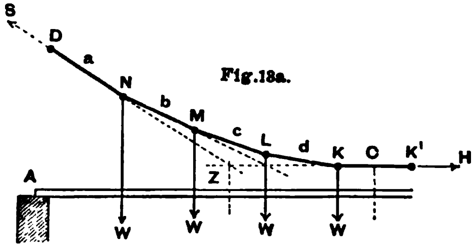


Fig.13a.

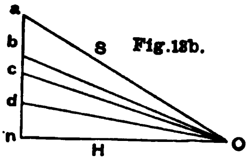


Fig.13b.

The resultant of the four W's will act at the middle of the third division from the left end, and since this resultant load together with the tensions of the middle and extreme links maintain the half chain in equilibrium, the three forces must meet in a point, the point Z shown in the figure. Thus the direction of the extreme link DN may be drawn. The direction and position of the other links may be found also. Considering the portion of the chain NC carrying three weights, the resultant of which is in the line through L, the link NM must be in such a direction as to pass through the point where this resultant cuts KK' produced. Having drawn NM, ML may be drawn in a similar way, and then LK. Returning to the consideration of the half chain, the three forces which keep it in equilibrium may be represented by the three sides of a triangle. Set down an (Fig. 13b) to represent

4 *W*, and draw *aO* and *nO* parallel to *DZ* and *ZC*; *aO* will be the tension of *DN* and *nO* of *KK'*. If *an* be divided into 4 equal parts, and the points *b*, *c*, *d*, joined to *O*, these lines will represent the tensions of links *NM*, *ML*, and *LK*. It may be easily shown that they will be parallel to those links. We see that the tension increases as we pass from link to link, from the centre to the ends.

In many cases in practice, the number of vertical suspending rods and links in the chain is very great. We may then, in what follows, without sensible error, regard the chain as forming a continuous curve.

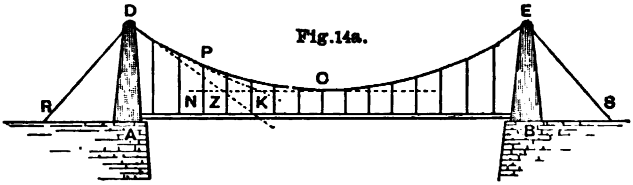


Fig. 14a.

In such a case, *C*, the lowest point of the chain (Fig. 14*a*), is over the middle of the platform. The tangent at *C*, which is horizontal, will meet the tangent to the chain at *D*, in a point *Z*, which will be over the middle of the half platform, for that will be a point in the line of action of the resultant load on the half chain. We can now draw a triangle of forces *anO*, for the half chain as before; *On* will represent the tension of the chain at the lowest point, or the horizontal component of the tension of the chain at any point. We can easily obtain a convenient expression for this horizontal tension thus:—Let l = span of the bridge, and w = load per foot run. Then $\frac{1}{2}wl$ = weight on the half chain represented by *an*. Let H = horizontal tension, then

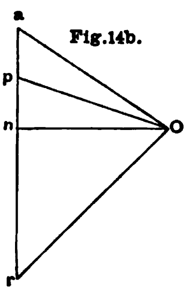


Fig. 14b.

$$\frac{H}{\frac{1}{2}wl}=\frac{On}{an}.$$

But if we drop a perpendicular from *D* to cut the horizontal tangent in a point *V* (not shown in the figure), *DV* will be the dip of the chain *d*, and comparing the triangles *DVZ*, *aOn*,

$$\frac{On}{an}=\frac{VZ}{DV}=\frac{\frac{1}{4}l}{d}=\frac{H}{\frac{1}{2}wl}$$

$$\therefore\ H=\frac{1}{8}wl\frac{l}{d}$$

which, since wl = total load on chain, may be written

$$H=\frac{1}{8}\text{ load on chain }\frac{\text{span}}{\text{dip}}.$$

This is the same as the horizontal thrust of a triangular frame of the same height which carries a uniformly distributed load of the same intensity.

Having found the magnitude of the horizontal tension of the chain we can calculate the tension at D, the highest point of the chain. Let S be this greatest tension, represented in the diagram of forces by aO, then since $\overline{aO}^2 = \overline{an}^2 + \overline{nO}^2$

$$S^2 = \left(\frac{W}{2}\right)^2 + H^2.$$

The tension at any point P of the chain may be found by drawing from O a line op parallel to the tangent to the chain at P. It will cut an in a point p such that $np : na ::$ length of platform below $PC : \frac{1}{2}$ span.

$$\text{Since } \overline{Op}^2 = \overline{np}^2 + \overline{On}^2$$

$$\text{Tension at } P = \sqrt{\left(\frac{np}{na}\frac{W}{2}\right)^2 + H^2}.$$

The loaded platform, instead of being suspended from the chain of bars, may rest by means of struts on an arch of bars as in the figure.

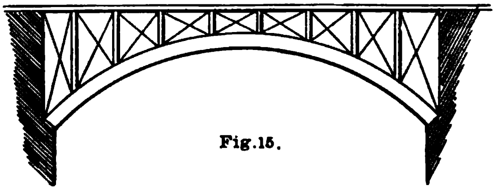


Fig. 15.

In this case all the bars will be in compression instead of tension, as in the previous case. If the form of the arch is similar to that in which the chain hung, it will have no tendency to change its form under the load. There will be simple thrust of varying amount at different parts of the arch. The horizontal thrust at the top of the arch is given by the same expression as for the horizontal tension of the chain, and the thrust of any bar of the arch may be determined in a manner similar to that for finding the tension of any link of a chain. We shall show presently that the proper form of the arch and chain under a uniform load is a parabola. Hence, the structure just described is called a Parabolic Arch. In iron bridges the platform is not unfrequently carried by a number of ribs placed side by side. Each rib is approximately parabolic in form, usually of I. section, of depth from $\frac{1}{40}$th to $\frac{1}{60}$th the span at the crown, increasing somewhat towards the abutments. The roadway is supported sometimes by simple vertical struts, as in the ideal case just considered, sometimes by spandrils of more complex form, chiefly for the sake of appearance. When uniformly loaded, the stress on the ribs is nearly as found above: for resistance to variation in the load reliance is placed on the resistance to bending of the ribs and platform. The case of a stone or brick arch is far more complex, and is not considered here.

There is yet another very common structure designed on the same principles. In this the platform, instead of resting on an arch below it, is suspended from an arch above it. In this case the thrust of the arch is taken by the platform, which serves as a tie, just as the string ties together the ends of a bow. Hence it is called a *Bowstring Girder*. In this, like the others, the loading proper to the parabolic form is a uniformly distributed one, and any variation of the loading will tend to distort the bow. The structure may, however, be enabled to sustain a varying load by the addition of bracing bars as shown by the diagonal lines. When the bridge is heavily loaded it will almost always happen that the greater part of the weight is uniformly distributed, and is sustained by simple thrust of the arch, so that the bracing is only a subsidiary part of the structure.

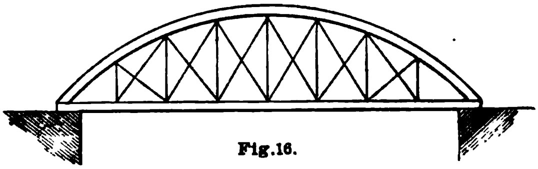


Fig. 16.

11. *Suspension Chains (continued). Bowstring Suspension Girder.*—In describing the suspension bridge we spoke of the chain as being secured at the ends to fixed points. In practice the securing of the ends is effected thus. The chain is led to the top of a pier of cast-iron or masonry, and instead of being simply attached to the top of the pier, and thus producing an enormous tendency to overturn the pier, the chain is secured to a saddle which rests on rollers on the top of the pier, and on the other side the chain is prolonged to the ground, passes through a tunnel for some little distance, and is finally secured by means of anchors to a heavy block of masonry. By this arrangement the only force acting on the pier is a purely vertical one, and a comparatively slender pier will be sufficient to sustain it. It is not necessary that the tension of the chain should be the same on each side of the pier, or that it should be inclined at the same angle. What is necessary is that the horizontal component of the tension on each side should be the same. If an (Fig. 14b, page 17) = half weight on chain as before, and $On = H$, the horizontal tension (which may either be calculated from the formula just obtained, or found by construction), then aO will be the pull of the chain S at the top of the pier. Then considering the equilibrium of the saddle, the pull of the chain Q on the short side and the upward reaction of the pier may be found by completing the triangle of forces aOr; Or will be the pull on the anchor, and ar the total vertical pressure on the pier.

In connection with this description of the method of securing the ends of the suspension chain, we may mention a form of structure in which the arch and chain are combined, a good example of which occurs in the railway bridge at Saltash. The horizontal pull of the chain is here balanced by the thrust of an arch, so that the combined effect is to produce simply a vertical pressure on the piers. The suspending rods are secured to the chains and prolonged to the arch above, so that a portion of the load is carried by the arch, producing a thrust, and a portion by the chain, causing a pull. To prevent any tendency to overturn the piers (this is insured by means of saddles resting on rollers) the horizontal component of the thrust of the arch must equal the horizontal component of the pull of the chain. The proportion between the loads on arch and chain will depend on the proportion between the rise of the arch and dip of the chain.

If W_1 = load on arch, and W_2 = load on chain,
d_1 = rise of arch, and d_2 = dip of chain,

then
$$H = \frac{W_1 l}{8d_1} = \frac{W_2 l}{8d_2}; \quad \therefore \frac{W_1}{W_2} = \frac{d_1}{d_2};$$

also
$$W_1 + W_2 = \text{total load on bridge}:$$

from which the stresses on the structure may be determined. It is known as a Bowstring Suspension Girder (pp. 42, 70).

We shall next show that the form of the curve of a chain carrying a uniformly loaded platform is a parabola. Referring to Fig. 14*a*, let P be any point in the chain, drop a perpendicular PN to meet the tangent at C, and bisect CN in K. Then KP must be the direction of the pull of the chain at P in order that the portion PC may be kept in equilibrium. The triangle PNK has its sides parallel to the three forces which act on PC, and the sides are therefore proportional to the forces Let $CN = x$ so that the load hanging on $PC = Wx$, also let $PN = y$.

Then
$$\frac{H}{wx} = \frac{NK}{PN} = \frac{\frac{1}{2}x}{y}.$$

$$\therefore x^2 = \frac{2H}{w}y; \text{ or, since } H = \frac{wl^2}{8d},$$

$$x^2 = \frac{l^2}{4d}y;$$

therefore x^2 is proportional o y.

Now the curve whose co-ordinates have this relation one to another is called a *parabola.*

If the load, instead of being uniformly distributed on a horizontal platform, were simply due to the weight of the chain itself, then the curve in which the chain would hang would deviate somewhat from the parabola; for in that case, since the slope increases as we approach the piers, the load also, per horizontal foot, would increase as we approach the piers, causing the chain near the piers to sink and become more rounded, and at the centre to rise and become more flattened. The curve in which the chain hangs by its own weight is called the *catenary.* In the catenary, as in the parabola, the tension increases as we approach the piers. This may be taken account of by proportioning the section of the chain to the tension at the various points; this would tend still more to make the weight of chain, per horizontal foot, increase as we approach the piers, and cause the chain to deviate still further from the parabolic form. Such a curve is called the catenary of uniform strength.

In an actual suspension bridge, where there is a uniformly loaded platform, as well as a heavy chain, the true curve in which it hangs will lie somewhere between the parabola and the catenary; but since in most cases the deviation from uniformity of the weight of chain is small compared with the load it carries, the deviation from the parabola is not great. The error involved in assuming the curve to be parabolic is generally greatest in bridges of large span; in such cases a preliminary calculation of approximate weights may be necessary so as to be able to apply the general process of article 9.

EXAMPLES.

1. A trapezoidal truss is 16 feet span and 4 feet deep, the length of the upper bar is 6 feet. Find the stress on each part when loaded with two tons at each joint.

Stress on sloping bars = 3·2 tons,
„ horizontal „ = 2·5 „

2. The platform of a bridge, 8 feet broad and 27 feet span, is loaded with 150 pounds per square foot. It is supported on each side by an inverted queen truss placed below, the queen posts, each 3 feet deep, dividing the span into three equal portions. Find the stress on each part.

Load on each truss = half whole load on platform = 16,200.
⅓ 16,200 = 5,400 is the load at each of the two joints of one of the queen trusses.
Tension of sloping bars = 17,074 lbs.
Tension and thrust of horizontal bars = 16,200.

3. The height of a mansard roof without bracing is 10 feet and span 14 feet. The height of the triangular upper portion is 4 feet and span 8 feet. The load being 1 ton at the ridge, find the necessary load at each intermediate joint and the thrust of the roof.

By the construction described in the text, load at each intermediate joint = ½ ton, and the thrust of the roof = ½ ton.

4. If the roof in the last question be partly braced by a bar joining the intermediate joints, find the stress on the bar when the load at each intermediate joint is 1 ton.

Thrust on bar = ¼ ton.

5. The load on the platform of a suspension bridge, 600 feet span, is ½ ton per foot run, inclusive of chains and suspending rods. The dip is $\frac{1}{13}$th the span. Find the greatest and least tensions of one of the chains.

Least tension = horizontal tension = 243¾ tons.
Greatest tension = 255 tons.

6. The load on a simple parabolic arch, 200 feet span and 20 feet rise, is 360 tons, determine the thrust and greatest stress on the arch.

Thrust = 450 tons; greatest stress = 484 tons.

7. The rise of a bowstring bridge is 15 feet and span 120 feet, find the thrust when loaded with 2,000 lbs. per foot run.

Thrust 240,000 lbs. = 107$\frac{1}{7}$ tons.

8. In example 5 the ends of the chain are attached to saddles resting on rollers on the tops of piers 50 feet high, and prolonged to reach the ground at points 50 feet distant from the bottoms of the piers, where they are anchored. Find the load on the piers and the pull on the anchors.

Load on the pier = 637½ tons;
Pull on each anchor = 344·6 tons.

9. A light suspension bridge is to be constructed to carry a path 8 feet broad over a channel 63 feet wide by means of 6 equidistant suspending rods, the dip to be 7 feet. Find the lengths of the successive links of the chain. Supposing a load of 1 cwt. per square foot of platform, find the sectional areas of the links of the chain, allowing a stress of 4 tons per square inch.

$\frac{6}{7}$ of the whole load is carried by the chains and the remaining portion by the piers directly. Tension of each suspending rod = 36 cwt.

Links.	Tensions.	Areas.	Lengths.
Centre	277·7	3·47	9·
2nd	280·	3·5	9·08
3rd	287·	3·6	9·3
4th	298·	3·72	9·66

10. Construct a parabolic arch, the thrust of which is half the total load.

Span = four times the rise.

11. If the weight of a uniformly loaded platform be suspended from a chain by vertical rods, show that the corners of the funicular polygon lie on a parabola.

SECTION III.—COMPOUND FRAMES.

12. *Compound Triangular Frames for Bridge Trusses.*—By a compound frame is meant a frame formed from two or more simple frames by uniting two or more bars. Many frames of common occurrence in practice may conveniently be considered as combinations of the simpler

examples already described. They are generally dealt with by use of what we may call the principle of superposition, which may be thus stated :—*The stress on any bar due to any total load is the algebraical sum of the stresses due to the several parts of the load.*

We will now consider some examples of compound frames, which are used in bridge trusses. In these structures the object is to carry a distributed load by means of a comparatively slender beam. A prop in the centre may still leave the halves too weak to carry the weight on them, and the beam may be strengthened by supporting it in more than one point.

(1) Suppose the beam supported by a number of equidistant struts, the lower ends of which are carried by tension rods attached to the ends of the beam, we then have a structure called a *Bollman truss.* There may be any number of struts—2, 3, 4, or more; the structure has been used for bridges of comparatively large span. If the actual load is distributed in some manner over the beam, we must first reduce the case to that of a structure loaded at the joints only. The loads on the struts are due to the weights resting on the adjacent divisions of the beam, and may be determined by supposing the beam broken or jointed at the points where the struts are applied.

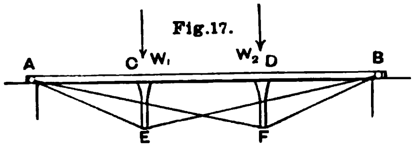


Fig.17.

Let us suppose the beam has three divisions, and that the load on the two struts are W_1 and W_2. These loads will be transmitted down the struts to the apices (Fig. 17) E and F, and will be independently supported, each by its own pair of tension rods. We may thus separately determine the stress on each part of either of the elementary triangular frames AEB or AFB. AB will be in compression an account both of the load at E and also at F. On account of W_1, using the formula previously obtained, the horizontal thrust $H_E = W_1\dfrac{a'b'}{lh}$, and on account of W_2 at F, $H_F = W_2\dfrac{a'b'}{lh}$.

Tension of AE, $T_{AE} = H_E \sec EAB$, $\quad T_{FB} = H_F \sec FBA$;

„ EB, $T_{EB} = H_E \sec EBC$, $\quad T_{AF} = H_F \sec FAD$.

The actual tensions of the sloping rods are simply as written, but since AB is a part of both triangular frames, the total thrust along it is found by summing the thrusts due to each : so

$$H = H_E + H_F.$$

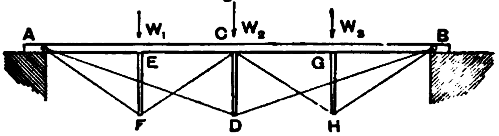
This is an example of the *principle of superposition* stated above.

(2) Suppose the beam which carries the distributed load to be

supported by a central strut forming a simple triangular truss, and further let the halves of the beam, not being strong enough to carry the load on them, each be subdivided and trussed by a simple triangular truss, the tension rods from the bottom of the subdividing struts proceeding only to the ends of each half beam. If the quarter spans are still too great, they may each of them be trussed in a similar way, and so on. Such a structure is called a *Finck truss.*

Suppose, for example, we have three struts. (Fig. 18.) We must first determine the load at the joints—that is, in this case the load on the struts due to the distributed load on the beam. Suppose that on account of the weights on the *adjacent* subdivisions those loads are W_1, W_2, W_3. If the load is uniformly distributed over the beam the W's are each of them equal to $\frac{1}{4}$ total weight on beam.

Fig. 18



We may now separately consider the triangular frame AFC carrying the load W_1. On account of it there will be a thrust on AC

$$H_F = W_1\frac{AC}{4h} = W_1\frac{l}{8h}.$$

The tensions of AF and FC are each $= H_F \sec FAE$. We get similar results from the triangle CHB. Just in the same way we may consider the principal triangular frame ADB, but in this case the thrust down the strut CD, which is the load at D, is not simply W_2, but greater by the amount of the downward pull of the two tension rods CF and CH. The vertical components of these tensions are $\frac{1}{2}W_1$ and $\frac{1}{2}W_3$, so that the total thrust down the strut $= W_2 + \frac{1}{2}(W_1 + W_3)$. This is the load which must be taken to act at D in determining the stresses on the members of ADB.

Thus $H_D = (W_2 + \frac{1}{2}W_1 + \frac{1}{2}W_3)\frac{l}{4h}$, and the tensions of AD and DB are each $= H_D \sec DAB$.

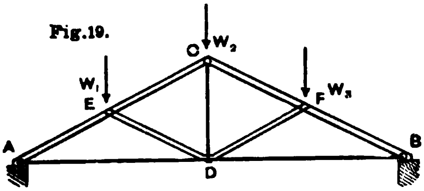
It will be seen that the thrust on the central strut and tensions of the longer rods are the same as if the secondary trusses had not been introduced. For example, if the W's each $= \frac{1}{4}$ whole load on beam, then the virtual load at $D = \frac{1}{2}$ weight on beam. The mere strengthening of each half the beam by trussing it can no more relieve the central strut of the load it has to carry, than the fact of strengthening a structure of any kind can relieve the two points of support from the duty each must have of bearing its own proper share of the weight. In stating the thrust on the beam we must divide it into two portions AC and CB. The portion AC is subjected to the thrust of the triangles AFC and ADB; $\therefore H_{AC} = H_F + H_D$, and CB being a portion of the triangles CHB and ADB, $H_{CB} = H_H + H_D$. When W_3 is not equal to W_1, the thrusts on the two portions will be different. This is quite possible although the beam AB may be a continuous one.

Both these simple forms of truss have been used for bridges of considerable span. As an example of the first may be mentioned the bridge at Harper's Ferry, U.S., destroyed during the war. It was 124 feet span in 7 divisions. The great length of the tension rods and their inequality appears objectionable. The second in 8 or 16 divisions has been much used in America; but in England other forms mentioned in a later chapter are much more common.

13. *Roof Trusses in Timber.*—In roofs of small span, 10 or 12 feet only, the roofing material, slates or tiles, rests on a number of laths set lengthways to the roof, and these laths rest on sloping rafters spaced 1 or 2 feet apart, with their feet resting on the walls of the building; the stability of the walls being depended on for taking the thrust.

When we come to larger and more important roofs we find additional members added for strength and security. The closely spaced rafters just mentioned are called common rafters. These being too long and slender to carry the weight of the roofing material and transmit it to the walls, are supported, not only at the ends by the walls and ridge piece, but also at the middle by a longitudinal beam of wood called a *purlin*, and the purlin is supported at intervals of its length by principal rafters. The principal rafters again are supported by struts at their central points, immediately below the purlins. To carry the lower ends of the struts, a vertical tension piece is introduced, by which they are suspended from the apex of the principals, while the thrust is taken by a tie beam connecting the feet of the rafters. In such a roof, a ceiling or floor may frequently be required to be supported by the tie beam, and to prevent it from sagging under the weight an additional tension will come on the vertical suspending rod. This rod is then a very important member of the structure, and is called the king post, and the whole structure, consisting of the principal rafters, king post, &c., is called a *King post truss*. This truss is often constructed entirely of wood. The sloping struts then for constructive reasons (Ch. XV.) butt on an enlarged part at the bottom of the king post above the point where the horizontal tie beam is attached, but for calculation purposes may be regarded as meeting at that point as shown in Fig. 19.

Fig. 19.



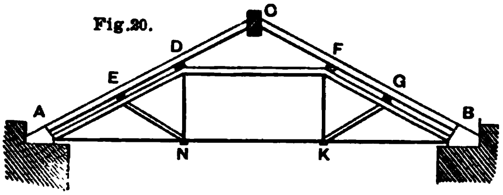
By means of the purlins and the ridge piece the weight of the roofing material will produce loads at the joints $ECF = W_1 W_2 W_3$ suppose.

Now treat the structure as made up of three simple triangular frames *AED*, *DFB*, and *ACB*. First consider *AED* with the load W_1 at vertex *E*. The horizontal thrust of this frame $H_E = W_1 \frac{AD}{4h}$ where h is the height of point *E* above *AD*. Also the thrust along *AE* and *ED* due to the load at $E = H_E \sec EAD$. In an exactly similar manner we may consider the triangle *DFB*; the results for this will be to those for *AED* in the proportion of W_3 to W_1. Next as to the primary triangle *ACB*. There is at *C* a direct load of W_2 due to the weight between *E* and *C*, and *F* and *C*. But beside this, the king post pulls the point *C* downwards, so that the total load at $C = W_2 +$ tension of king post. In addition to a portion of the weight of the ceiling (if any) the post has to support *D* against the downward thrust of the two struts *ED* and *FD*. The vertical components of these thrusts are $\frac{1}{2} W_1$ and $\frac{1}{2} W_3$, therefore, neglecting the weight of ceiling, the virtual load at $C = W_2 + \frac{1}{2}(W_1 + W_3)$. Let us call this total load W, then H_C the horizontal thrust of $ACB = W \frac{AB}{4CD}$ and the thrusts along *AC* and *CB* due to load at $C = H_C \sec A$.

Now in the complete structure, since *AD* is a member both of the triangular frame *AED* and *ACB*, the total tension of $AD = H_E + H_C$.
For the same reason tension of $DB = H_F + H_C$,
and thrust of $AE = (H_E + H_C) \sec A$,
" " $FB = (H_F + H_C) \sec A$.
The other members of the structure are portions of one elementary frame only, and the stress is due only to the load at the apex of that frame.

The king post truss serves for roofs of spans under 30 feet, but for spans greater than this trusses of more complicated construction are required. If the span is from 30 to 50 feet, then instead of supporting the common rafters by a purlin at the centre of its length only, as in the king post truss, two supporting purlins may be used, dividing the length of the rafter into three equal portions. These purlins may be carried by a queen truss, the sloping members of which are supported in the middle by struts, as shown in the figure (Fig. 20).

Fig. 20.



The vertical queen posts DN and FK serve to sustain the downward thrust of the struts EN and GK, and also to support the weight of a ceiling, if there is one. Supposing the weight of the ceiling omitted, let W be the weight of roofing material on one side for a length of roof equal to the spacing of the trusses, then $\frac{1}{3}W$ will, through the common rafters and purlins, act at E, and $\frac{1}{3}$ at D; and similarly for the other side. At the ridge C there will also be $\frac{1}{3}W$ acting; but this will be distributed equally amongst the common rafters which are carried by the truss, and will produce compression in those rafters without directly affecting the truss. The part of the thrust of the roof arising from this will, however, generally, like the rest, ultimately come on the principal tie beams.

To find the stresses on the different members of the truss. Consider first the small triangles AEN and BGK, each carrying $\frac{1}{3}W$ at the vertex. We then consider the trapezoidal truss $ADFB$. The loads at D and F will be $\frac{1}{3}W$ + tension of queen post. Since the tension of the queen post DN = the vertical component of the thrust along EN it will equal $\frac{1}{2}.\frac{1}{3}W = \frac{1}{6}W$, and the total load at each joint of the trapezoidal truss will be $\frac{1}{3}W + \frac{1}{6}W = \frac{1}{2}W$, the same as would have acted if there had been no purlin at E and no strut EN. After having determined the respective stresses due to the triangles and trapezoid separately, we must add the results for any bar which is a part of both. Were it not for the friction at the joints and the power of resistance of the continuons rafters AC, CB to bending, this structure would be stable only under a symmetrical load. In practice, however, it is able to sustain an unsymmetrical load, such as roofs are frequently subjected to.

14. *Queen Truss for large Iron Roofs.*—As the span of the roof is still further increased we find other kinds of trusses employed to support them. A common form in iron roofs is constructed, as shown in Fig.

21. It is in reality a further development of the wooden queen truss, and is known by the same name. *AC* and *CB* are divided into a number of equal parts, and sloping struts and vertical suspending rods are applied as shown. Suppose the load the same at each joint on one side of the roof, the load on the right, however, not being necessarily equal to that on the left. Let the upward supporting force at $A = P$. P will be $\frac{1}{2}$ total weight if the loading is symmetrical, but in any other case it may be found by taking moments of the loads about *B*. We might solve the problem of finding the stress on each member of the structure by treating separately each elementary triangle into which the structure may be divided, and summing the stresses for any bar which may form a part of two or more triangular frames. But we will describe another method.

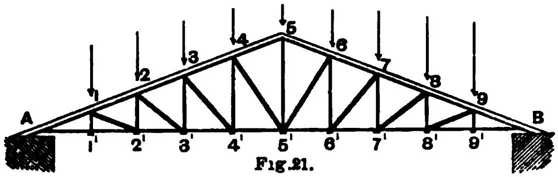


Fig. 21.

First, to find the tension of the vertical suspending rods consider $A12'$ as an independent triangle, carrying a load W at its vertex. The slope of $12'$ being the same as that of $A1$, the tension rod $22'$ must supply a supporting force to the joint $2' = \frac{1}{2}W$. Considering next the triangle $A23'$ and its equilibrium about the point *A*. The forces along 23 and $3'4'$ have no moment about *A*, so that the moment of the two weights W at 1 and 2 about *A* must be balanced by the upward pull of the tension rod $33'$. $\therefore$ tension of $33' = W$.

In a similar way we can see that the tension of $44' = \frac{3}{2}W$. However many divisions of the roof there may be, the tensions of the vertical suspending rods will increase in arithmetical progression, with the same difference between each. The rod $11'$, except so far as may be due to the weight of the rod $A2'$, will have no tension on it. Calling this the 1st tension rod, the tension of the $n^{\text{th}} = \frac{n-1}{2}W$. We must notice that the rod $55'$ is common to both sides of the roof, and we must add the two tensions to get the total. Now consider any joint, say $4'$ in the tie bar *AB*, and resolve vertically and horizontally. If $R =$ thrust of $34'$, θ its inclination to the horizontal, and T the pull on that division of *AB* which is indicated by the numerical suffix placed below it,

$$R \sin \theta = \tfrac{3}{2}W,$$

$$R \cos \theta = T_{3'4'} - T_{4'5'};$$

$$\therefore\ T_{3'4'} - T_{4'5'} = \tfrac{3}{2}W \cot \theta.$$

But from figure $\cot \theta = \frac{1}{3} \cot A$;

$$\therefore\ T_{3'4'} - T_{4'5'} = \tfrac{1}{2} W \cot A.$$

Whichever joint we select we should find the same result—namely, that the difference between the tensions of two consecutive portions of the tie rod is a constant quantity $= \frac{1}{2} W \cot A$. So that these tensions are in arithmetical progression diminishing towards the centre.

If we call $A2'$ the 1st division of tie rod, then for the joint between the $n-1$th and nth we have

$$R \sin \theta = \frac{n-1}{2} W.$$

$$R \cos \theta = T_{n-1} - T_n, \text{ and } \cot \theta = \frac{1}{n-1} \cot A;$$

$$\therefore\ T_{n-1} - T_n = \tfrac{1}{2} W \cot A.$$

If $A1$ is the 1st division of the rafter, then the thrust on the nth division $= T_n \sec A$.

Now, the tension of the tie rod in the

$$\begin{aligned} &1^{\text{st}} \text{ division} &&= P \cot A, \\ &2^{\text{nd}} \quad ,, &&= (P - \tfrac{1}{2} W) \cot A, \\ &n^{\text{th}} \quad ,, &&= \left(P - \frac{n-1}{2} W\right) \cot A. \end{aligned}$$

The thrust on the nth division of rafter $= \left(P - \frac{n-1}{2} W\right) \operatorname{cosec} A$.

The thrust on any strut may best be found by squaring and adding the two equations of equilibrium of the lower joint of it. We get

$$\text{Thrust of } n^{\text{th}} \text{ strut} = \frac{W}{1} \sqrt{n^2 + \cot^2 A}.$$

15. *Concluding Remarks.—General Method of Constructing Diagrams of Forces.*—Cases of framework often occur which are much more complicated than those which we have hitherto considered, but if there are no redundant bars the stress on each part depends on statical principles only, without reference to the relative yielding of the several parts of the structure. Such cases may always be treated by use of the general principle stated in Art. 1, and we shall conclude this chapter by explaining briefly a graphical method of applying that principle invented by the late Professor Clerk Maxwell. The forces will be supposed all in one plane, and each of them will be supposed known, that is to say, if there be any unknown reactions at points of support they will be supposed previously found by a graphical or other process, from the consideration that the whole must form a set of forces in equilibrium. In Fig. 22*a* a frame is shown acted on by known forces *PQR*..., an ideal example is chosen which is better suited for the purpose of ex-

plaining the method than any case of common occurrence in practice. First seek out a joint where only two bars meet: there will usually be two such joints if there be no redundant bars in the frame, and in the present instance we will choose the joint where P acts. Distinguish all the triangles, making up the frame by letters A, B, C, &c., and place numbers or letters outside the frame, one for each bar. In Fig. 22*b* draw 18 parallel to the force P and representing it in magnitude, 8*a* parallel to 8, 1*a* parallel to 1, to intersect in the point *a*; then, as in previous examples, 8*a*, 1*a* represent the stress on the two bars to which they are parallel. Pass now to the joint where Q acts: this joint is chosen because only three bars meet there, on one of which we have just determined the stress; draw 12 parallel to Q and representing it, then *ab* parallel to the bar lying between the triangles A and B, and 2*b* parallel to the bar 2; we thus get a polygon 12*ba*, the sides of which are parallel to the four forces acting at the joint where Q acts, while two of them represent two forces already known, the other two, therefore, will represent the remaining two forces. Proceed now to the joint where W acts and complete in the same way the polygon 8*abc*7, then to the joint where R acts, and so on. We at length arrive at the triangle 4*f*5, the third side of which, if we have performed the construction accurately, and if the forces be really in equilibrium, must be parallel to the last force T. On examination of the diagram of forces (Fig. 22*b*) it will be seen that to every joint of the frame corresponds a polygon representing the forces at that joint, while each line, such as *ab* or 7*c*, gives the stress on the bars separating those letters or numbers in the frame-diagram. The polygon 12...8 is the polygon of external forces, each side representing the force to which it is parallel.

Fig.22a.

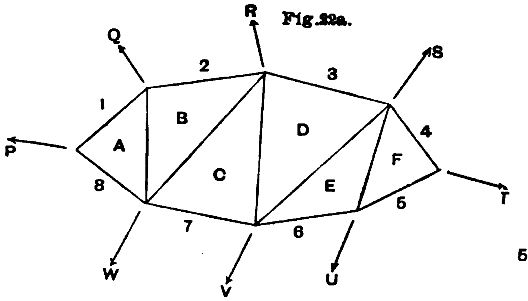
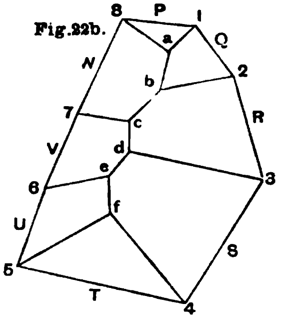


Fig.22b.



The method here described is easy to understand in the general case we have considered, and with a little practice the transformations the diagram of forces undergoes will offer no difficulty. Some joints are usually unloaded, and the corresponding lines in the polygon of external forces vanish; the forces may be parallel, in which case the polygon becomes a straight line, while not unfrequently the sides of two of the polygons representing the forces at the joints coincide. The figure, however, always possesses the same properties.

In Mr. Bow's excellent work referred to at the end of this chapter over 200 examples will be found of the application of this method, including almost all known forms of bridge and roof trusses.

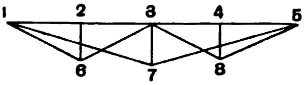
EXAMPLES.

1. A Bollman truss of three divisions is 21 feet span, and is loaded uniformly with 1 ton per foot. The depth of the truss is 3½ feet. Find the stress on each part.

Load on each strut = 7 tons.
Tension of short rods = 10·4 ,,
,, longer ,, = 9·6 ,,
Total thrust on beam = 18⅔ ,,

being 9⅓ due to each triangle.

2. A Finck truss of 4 divisions, 20 feet span and 3 feet deep, is loaded with 1 ton per foot, find the stress on each part.



Thrust on 26 and 48 = 5 tons.
,, 37 = 10 ,,
Tensions of 16, 63, 38, and 85 = 4·86 ,,
,, 17 and 75 = 17·4 ,,
Thrust on 13 and 35 = 4⅙ + 16⅔ = 20⅚ tons.

3. In the last question suppose one half the truss loaded with an additional 1 ton per foot. Find the stress on each part.

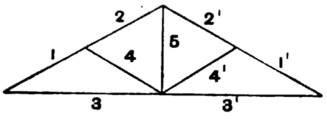
Suppose the additional load on the right-hand side.

Thrusts.	Tensions.
On 26 = 5 tons.	On 16 and 63 = 4·86 tons.
,, 37 = 15 ,,	,, 38 ,, 85 = 9·72 ,,
,, 48 = 10 ,,	,, 17 ,, 75 = 26·1 ,,
,, 13 = 4⅙ + 25 = 29⅙.	
,, 35 = 8⅓ + 25 = 33⅓.	

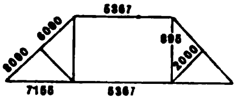
4. A roof 28 feet span, height 7 feet, rests on king-post trusses spaced 10 feet apart. The weight of roof is 20 lbs. per square foot. Find the stress on each part. Also obtain results when an additional load of 40 lbs. per square foot rests on one side.

Load at each joint. 1st case = 1566·6 lbs.

Bars.	Stress in lbs.		Bars.	Stress.	
	Equal Load.	Additional Load.		Equal Load.	Additional Load.
1	5254	8756	1′	5254	12261
2	3503	7006	2′	3503	7006
3	4700	7833	3′	4700	10966
4	1752	1752	4′	1752	5255
5	1566·6	3113			



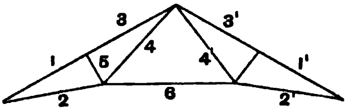
5. A roof 48 feet span, 12 feet high, rests on queen trusses 8 feet high, spaced 10 feet apart. Find the stresses for a load of 20 lbs. per square foot.



6. An A roof, braced as in the figure, is 40 feet span, and 10 feet high; the horizontal

tie bar is 8 feet below the vertex. Find the stresses on each part when loaded with 2 tons at each joint by constructing a diagram of forces or otherwise.

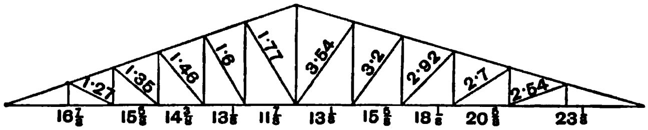
Bars.	Stress.
1	10·4
2	9·4
3	9·4
4	4·7
5	1·8
6	5·



7. In the last question suppose an accumulation of snow on one side equivalent to an additional load of 2 tons at the middle of the rafter, and 1 ton at the ridge. Find the stress on each part.

Bars.	Stress.	Bars.	Stress.
1	13·9	1'	17·3
2	12·5	2'	15·7
3	12·8	3'	15·4
4	5·5	4'	8·6
5	1·8	5'	3·6
6	7·5		

8. Suppose there are 11 suspending rods in iron roof shown in the figure, the height of which is $\frac{1}{10}$th the span. Find the stress on each part—1st, when loaded with $\frac{1}{2}$ ton at each joint on both sides, and, 2nd, when loaded with an additional $\frac{1}{2}$ ton at each joint on one side, not including the ridge.



Additional load is on right-hand side, and the figures on the diagram refer to case 2.

9. The roadway of a bridge, 80 feet span, is carried by a pair of compound trapezoidal trusses, each consisting of three simple trapezoids of the same height, the six "queens" of which are equidistant, forming seven divisions of length four thirds the height of the truss. Find the stress on all the bars due to $\frac{1}{4}$ ton per foot run on the bridge.

10. Find the stress on each part of a "straight-link suspension" bridge formed by inverting the truss of the last question, assuming the pull at the centre of the platform zero.

REFERENCES.

For further information on the subjects treated of in the present chapter the reader may refer amongst other works to

GLYNN—*Construction of Cranes.* Weale's series.
HURST—*Carpentry.* Spon, 1871.
BOW—*Economics of Construction.* Spon, 1873.

CHAPTER II.

STRAINING ACTIONS ON A LOADED STRUCTURE.

16. *Preliminary Explanations.*—In the preceding chapter we have considered only those structures in which the parts are subject to compression and tension alone, except by way of anticipation in a few special cases. But the parts of a structure are generally subject to much more complex forces, and besides, although the forces acting on each bar have been determined, we should, if we stopped here, have a most imperfect idea of the way in which the load affects the structure as a whole.

If we imagine a structure to be made up of any two parts, A and B, united by joints, or distinguished by an ideal surface cutting through the structure in any direction, the whole of the forces acting on the structure may be separated into two sets, one of which acts on A, the other on B. Since the structure is in equilibrium as a whole, the two sets of forces must balance one another, and must therefore produce equal and opposite effects on A and B, effects which are counteracted by the union existing between the parts. The two sets of forces taken together constitute a STRAINING ACTION of which each set is an element, and the object of this and the next two chapters is to consider the straining actions to which loaded structures and parts of structures are subject.

Straining actions differ in kind, according to the nature of the effects which they tend to produce. Four simple cases may be distinguished:—

(1) The parts A and B may tend to move towards each other or away from each other perpendicular to a given plane. This effect is called Compression or Extension, and the corresponding straining action is a thrust or a pull.

(2) A and B may tend to slide past each other parallel to a given plane. This effect is called Shearing.

(3) A and B may tend to rotate relatively to each other about an axis lying in a given plane. This is called Bending.

(4) *A* and *B* may tend to rotate relatively to each other about an axis perpendicular to a given plane. This is called Twisting.

In the first two cases the straining action reduces to two equal and opposite forces, and in the second two to two equal and opposite couples. In general, straining actions are compound, consisting of two or more simple straining actions combined. The given plane with reference to which the straining actions are reckoned may always be considered as an ideal section separating *A* and *B* even when the actual dividing surface is different. We shall commence by considering the straining actions on a beam of small transverse section.

SECTION I.—BEAMS.

17. *Straining Actions on a Beam.*—The action of a simple thrust or pull on a bar has already been sufficiently considered in Chapter I. They are usually considered as separate cases, and the simple straining actions on a bar are therefore reckoned as five in number. The other three are (1) shearing, (2) bending, and (3) twisting, of which the last rarely occurs, except in machines, and will, therefore, be considered in a later division of this work, under that head.

Shearing and bending are due to the action of forces, the directions of which are at right angles to the bar: in structures, the forces usually lie in one plane passing through the axis of the bar. A bar loaded in this way is called a beam.

Simple shearing is due to a pair of equal and opposite forces, *F* (Fig. 23), applied to points very near together, tending to cause the two parts *A* and *B* to slide past one another, as shown in the figure (Figs. 23*a*, 23*b*). Either element is called the shearing force, and is a measure of the magnitude of the shearing action, but in considering the sign we must consider both together. In this work, if the right-hand portion, *A*, tends to move upwards, and *B* downwards, as in Fig. 23*b*, the shearing action will usually be reckoned positive, while in the converse case (Fig. 23*a*) it will be reckoned negative.

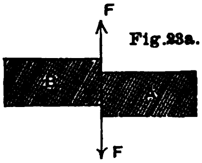


Fig. 23a.

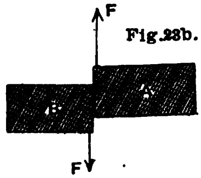


Fig. 23b.

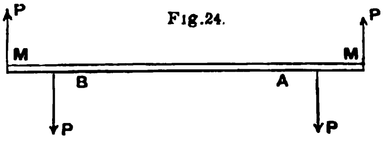


Fig. 24.

Simple bending is due to a pair of equal and opposite couples applied to the bar, one acting on *A*, the other on *B*, as in Fig. 24, tending to make A and B rotate in opposite directions. The magnitude of the bending is measured by the moment of either couple which is called the bending moment. In this work bending moments will usually be reckoned positive when the left-hand half, B, rotates with the hands of a watch, and the right-hand half in the opposite direction. That is to say, when the beam tends to become convex downwards, as in the ordinary case of a loaded beam supported at the ends. In loaded beams shearing and bending generally exist together, and vary from point to point of the beam. We shall now consider various special cases.

18. *Example of a Balanced Lever. General Rules for calculating S.F. and B.M.*—First take the case of a beam, AB, supported at C (Fig. 25), and loaded with weights, PQ, at its ends.

If the weights are such that $P.AC = Q.BC$ the beam will be in equilibrium, but the two parts, AC, BC, tend to turn about C in opposite directions, there is therefore a bending action at C, of which the equal and opposite moments $P.AC$, $Q.BC$ are the elements. Either of these is the bending moment usually denoted by M, so that we write

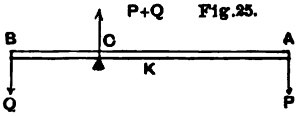


Fig. 25.

$$M_C = P.AC = Q.BC.$$

Not only is there a bending action at C, but if we take any point, K, and consider the forces acting on AK, BK separately, we see that AK tends to turn about K under the action of the force P, while BK tends to turn about K under the action of the forces $P + Q$ at C and Q at B. The first tendency is immediately seen to be simply the moment $P.AK$, while the second is $Q.BK - (P + Q)CK$. The last quantity reduces to $Q.BC - P.CK$, or, remembering that $Q.BC = P.AC$ to $P.AK$. The two moments then, as before, are equal and opposite, and constitute a bending action at K, measured by the bending moment

$$M_K = P.AK.$$

This example will sufficiently explain the general rule for calculating the bending moment at any point, K, of a beam. *Divide the forces into two sets, one acting to the right and the other to the left of K, and estimate the moment of either set about K, then the result will be the bending moment at K.* The example shows that the calculation of one of the two moments will generally be more simple than that of the other, and cases constantly occur, as where a beam is fixed at one end in a wall, where nothing is known about one set of forces except that they balance the other set. In each case the simplest calculation is of course to be preferred.

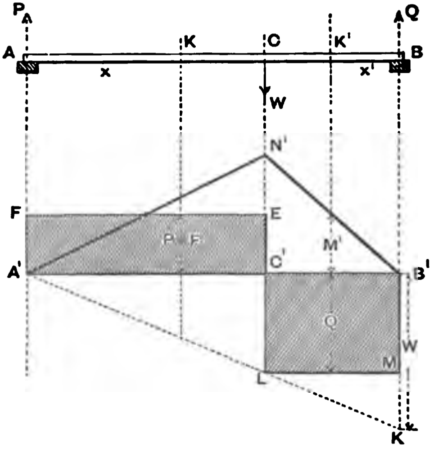
Moments are measured numerically by unit weight acting at unit leverage, as, for example, 1 ton acting at a leverage of 1 foot, for which the expression "foot-ton" is commonly employed. This phrase, however, is used also for a wholly different quantity, namely, the unit of mechanical work, and for this reason it would be preferable to call the unit of moment a ton-foot for the sake of distinction.

The peculiar action called shearing will be better understood when we come to consider the action of forces on a framework girder in the next section; it will here be sufficient to say that if the sum of the forces acting on AK, BK are not separately equal to zero, they must tend to cause AK, BK to move past each other in the vertical direction, thus constituting a shearing action measured by the magnitude of the shearing force, which may be thus calculated for any point K. *Divide the forces into two sets, one acting to the right of K and the other to the left of K, the algebraical sum of either set is the shearing force at K.* As before either set may be chosen, whichever gives the result most simply. In the example just given the shearing force at any point of AC is P; and at any point of BC, Q.

19. *Beam Supported at the Ends and Loaded at an Intermediate Point.*—We will next consider the case of a beam supported at the ends and loaded at some intermediate point. Before we can apply the rules previously enunciated, to find the shearing force and bending moment at any point, we must first determine the supporting forces at the two ends. We find the force P acting at A, Fig. 26, by taking moments about B, thus,

$$P(a+b)=Wb; \quad \therefore P=\frac{Wb}{a+b},$$

Fig.26.



and similarly $$Q = \frac{Wa}{a+b}.$$

First as to the shearing force. Taking any point K in AC, and considering the forces acting on AK, of which there is only one,

$$F_K = P = \frac{Wb}{a+b}.$$

At any point K' between C and B we have

$$F_{K'} = Q = \frac{Wa}{a+b}.$$

It will be noticed that at K the tendency is for the left-hand portion to slide upwards relatively to the right, whereas at K' the tendency is for the right-hand portion to slide upwards relatively to the left. It is advantageous to distinguish between these two tendencies, as previously stated, by calling the one positive and the other negative.

We may draw a diagram to represent the shearing force at any point thus. Let $A'B'$ be drawn parallel to and below AB to represent the length of the beam, and let $CC'L$ be the line of action of the weight. If we set up an ordinate $A'F = P$, and downwards an ordinate $B'M = Q$, and draw FE and ML parallel to $A'B'$ to meet the vertical $EC'L$; the shearing force at any point will be represented by the ordinates of the shaded figure $A'FELMB'$, measured from the base line $A'B'$. Not only will the magnitude of the shearing force be represented, but also the direction of the sliding tendency. This is why on one side of C' the ordinate was set downwards.

In this example the supporting forces may be found by construction, and thus the whole operation of determining and representing the shearing force performed graphically. For, set down $B'K = W$, join $A'K$, and where the vertical through C' cuts $A'K$, draw LM horizontal, then $B'M = Q$ and $MK = P$. Then set up $A'F = MK$; and draw FE horizontal.

Next as to the bending moment at any point. Take any point K in AC distant x from A, then

$$M_K = Px = \frac{Wb}{a+b}x,$$

and similarly at K' in CB distant x' from B,

$$M_{K'} = Qx' = \frac{Wa}{a+b}x';$$

so for either side of C, the bending moment is greater the greater the distance of the point from the end of the beam. Thus the greatest bending moment is at C.

If in the value of M_K we put $x = a$,
or „ $M_{K'}$ „ $x' = b$,

we get the same result, viz., that

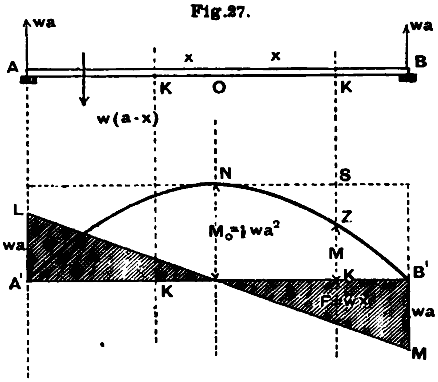
$$M_C = \frac{Wab}{a+b} = \text{greatest bending moment.}$$

The graphical representation of the bending moment at any point is very useful and instructive. We may construct the diagram thus:—$A'B'$ representing the length of the beam set up from C', $C'N$ the bending moment at $C' = \frac{Wab}{a+b}$ on some convenient scale, on such a scale for instance as 1 inch = 20 ft.-lbs. Then joining $A'N$ and $B'N$, the ordinate of the figure $A'NB'$, measured from the base line $A'B'$, will express on the scale chosen the bending moment at any point of the beam. If $a = b = \frac{1}{2}$ span, so that the load is applied at the centre of the beam, then

$$M_C = \tfrac{1}{4}W \times \text{span} = \text{greatest bending moment.}$$

20. *Beam Supported at the End and Loaded Uniformly.*—The next example for consideration is that of a beam supported at the ends and loaded uniformly throughout its length with w lbs. per foot. (Fig. 27.)

Fig. 27.



Let the span $= 2a$. Take any point, K, distant x from the centre O. The load on AK is wAK, and therefore the shearing force at K, reckoning the forces on the left-hand side must be

$$F_K = wa - wAK = wa - w(a-x) = wx.$$

That is, the shearing force is proportional to the distance of the point from the centre of the beam. At the end A where $x = a$,

$$F_A = wa,$$

and at B where $x = -a$,

$$F_B = -wa.$$

If from $A'B'$, below AB in the diagram, we set up and down ordinates at A' and $B' = wa$ on some scale, and join LM, the ordinates of the sloping line will represent the shearing force at any point. The shearing force at the centre of the beam is zero.

In finding the bending moment at K, reckoning still from the left-hand side, we must clearly take account not only of the supporting force at A, but also of the effect of the load which rests on the portion of the beam AK. The moment of this load about K is the same as if it were all collected at its centre of gravity, namely, at the centre of AK. Thus

$$M_K = wa\,.\,AK - wAK.\frac{AK}{2}$$

$$= \frac{w}{2}AK(2a - AK) = \frac{w}{2}AK.KB.$$

That is to say, the bending moment at any point is proportional to the product of the segments into which the beam is divided by the point.

Putting $AK = a - x$ and $BK = a + x$,

$$M_K = \tfrac{1}{2}w(a^2 - x^2),$$

which is greater the less x is. At the centre $x = 0$, and we have the maximum bending moment

$$M_0 = \tfrac{1}{2}wa^2.$$

If we put $2wa = W$, the total load on the beam

$$M_0 = \tfrac{1}{8}W \times \text{span}.$$

This is only one half the bending moment due to the same load when concentrated at the centre of the beam.

If ordinates be set up from $A'B' = \frac{1}{2}w\ (a^2 - x^2)$, at all points, the extremities of the ordinates will lie on a curve which may easily be seen to be a parabola with its axis vertical and vertex above the middle point of the beam. For

$$SZ = SK - KZ = \tfrac{1}{2}wa^2 - \tfrac{1}{2}w(a^2 - x^2) = \tfrac{1}{2}wx^2.$$

So that SZ is proportional to SN^2, showing that the curve is a parabola.

21. *Beam Loaded at the Ends and Supported at Intermediate Points.*—Next, suppose a beam (Fig. 28) supported at A, B, and loaded with weights P, Q, at the ends C, D, which overhang the supports. If AC, AB, BD are denoted by a, l, b respectively, the supporting force S at A (by taking moments about B) is given by

$$Sl = P(a + l) - Qb.$$

Similarly R, the supporting force at B, is given by

$$Rl = Q(b + l) - Pa.$$

Take now a point K distant x from A; then

$$F_K = S - P = \frac{Pa - Qb}{l} = \frac{M_A - M_B}{l},$$

where M_A, M_B are the bending moments at A, B.

Also for the bending moment at K,

$$M_K = -Sx + P(a+x) = -\frac{M_A - M_B}{l}.x + M_A,$$

or, as we may write it,

$$M_K = M_A\frac{l-x}{l} + M_B\frac{x}{l}.$$

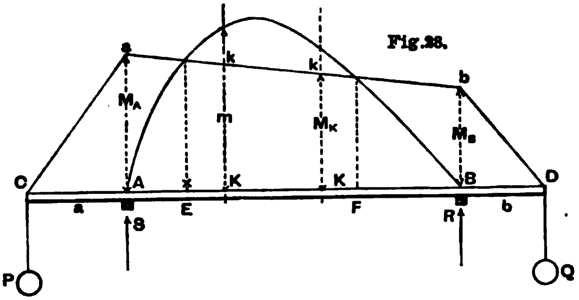


Fig. 28.

These formulae show that the shearing force is constant while the bending moment varies uniformly. In the diagram this is indicated by setting up ordinates *Aa, Bb*, to represent the bending moments at *A*, *B*, and joining *a*, *b*; the ordinate *Kk* of this line corresponding to an intermediate point *K*, will represent the bending moment there. The moments are in this example reckoned positive for upward bending.

An important special case is when $M_A = M_B$; then the bending moment is constant, and the shearing force *zero*. We have then no shearing but only bending. Simple bending is unusual in practice, but an instance occurs in the axle of a carriage.

The ordinates of the straight lines *Ca, Db*, represent the bending moment at any point of the overhanging parts of the beam.

22. *Application of the Method of Superposition.*—When a beam is acted on by several loads, the principle of superposition already stated in Chap. I. is often very useful in drawing diagrams and writing down formulae for the straining action at any point. Thus, for example, in the preceding case, if there be many weights on the overhanging end of a beam, the bending moment and shearing force at each point must be the sum of that due to each taken separately; and hence it follows that, whatever be the forces acting on a beam, if there be a part *AB* under the action of no load, and the bending moments at the ends of that part be M_A, M_B, the straining actions at any intermediate point *K* will always be given by the formulae just written down. And, further, if there be a load of any kind on *AB*, and *m* be the bending moment,

on the supposition that the beam simply rests on supports at *A*, *B*, then the actual bending moment must always be given by

$$M = M_A.\frac{l-x}{l} + M_B.\frac{x}{l} + m,$$

a general formula of great importance. The result is shown graphically in the diagram, where the curve represents the bending moment *m*, and the straight line *ab* the effect of the bending moments at the ends, supposed, as is frequently the case, to be in the opposite direction to *m*; then the intercept between the curve and the straight line represents the actual bending moment.

If several weights act on a beam, triangles may readily be constructed showing the bending moment due to each weight; then adding the ordinates of all the triangles at the points of application of the weights, and joining the extremities by straight lines, a polygon is obtained which is the polygon of bending moments for the whole load. This process may also be applied to shearing forces. It is simple, but somewhat tedious when there are many weights, and other methods of construction will be explained hereafter.

EXAMPLES.

1. A beam, *AB*, 10 feet long is fixed horizontally at *A*, and loaded with 10 tons distributed uniformly, and also with 1 ton at *B*. Find the bending moment in inch tons at *A*, and also at the middle of the beam.

M = 720 inch tons at *A*.
= 210 ,, at the centre.

2. In the last question find the shearing force at the two points mentioned.

F = 11 tons at *A*.
= 6 ,, at the centre.

3. A beam, *AB*, 10 feet long is supported at *A* and *B*, and loaded with 5 tons at a point distant 2 feet from *A*. Find the shearing force in tons, and the bending moment in inch-tons at the centre of the beam. Find also the greatest bending moment.

F at the centre = 1 ton.
M at the centre = 60 inch tons.
Maximum bending moment = 96 ,,

4. In the last question suppose an additional load of 5 tons to be uniformly distributed. Find the shearing force and bending moment at the centre of the beam.

F at centre = 1 ton as before.
M at centre = $11\frac{1}{4}$ foot tons = 135 inch-tons.

5. A beam, *AB*, 20 feet long is supported at *C* and *D*, two points distant 5 feet from *A* and 6 feet from *B* respectively. A load of 5 tons is placed at each extremity. Find the bending moment at the middle of *CD* in inch-tons.

Moment = 330 inch-tons.

6. In the example just given draw the diagrams of shearing force and bending moment at each point of the beam.

7. A foundry crane has a horizontal jib, *AC*, 21 feet long attached to the top of a crane post 14 feet high, which turns on pivots at *A* and *B*. The crane carries 15 tons, which may be considered as suspended at the extremity of the jib. The jib is supported by a strut attached to a point in it 7 feet from *A*, and resting on the crane post at *B*. Find

the stress on crane post and strut, and the shearing force and bending moment at any point of the jib.

Tension of crane post = 30 tons.
Thrust on strut = 50 ,,

8. A rectangular block of wood 20 feet long floats in water; it is required to draw the curves of shearing force and bending moment when loaded (1) with 1 cwt. in the middle; (2) with ½ cwt. at each end, and (3) ½ cwt. placed at two points equidistant from the middle and each end.

9. A beam, *AB*, 20 feet long is supported at the ends, and loaded at two points distant 6 feet and 11 feet respectively from one end with weights of 8 tons and 12 tons: employ the method of superposition to construct the polygons of shearing force and bending moment. Find the maximum bending moment in inch-tons.

Maximum moment = 972 inch-tons.

10. A beam is supported at the ends and loaded uniformly throughout a part of its length: show that the diagram of moments for the part of the beam outside the load is the same as if the load had been concentrated at the centre of the loaded part, and for the remainder is a parabolic arc. Construct this arc.

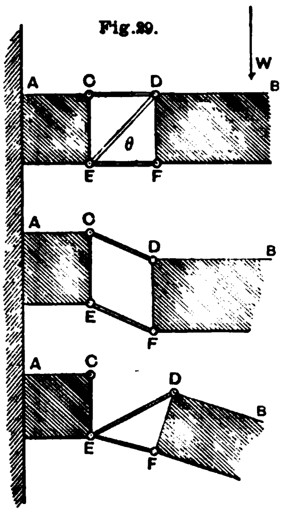
SECTION II.—FRAMEWORK GIRDERS WITH BOOMS PARALLEL, AND WEB A SINGLE TRIANGULATION.

23. *Preliminary Explanations.*—Hitherto we have only considered beams of small transverse section, but the part of a beam may be played by a framework or other structure under the action of transverse forces. Such a structure, when employed as a beam, is called a Girder, and consists essentially of an upper and a lower member called the Booms of the girder, connected together by a set of diagonally placed bars, called collectively the Web. The web consists sometimes of several triangulations of bars crossing each other, and may even be continuous. In the present section the booms will be supposed straight and parallel, and the web a single triangulation. The action of a load on such a girder furnishes the simplest and best illustration of the nature of the straining actions we have just been considering.

Suppose, in the first place, we have a rectangular beam of considerable transverse dimensions, which has one end fixed horizontally, and the other end loaded with a weight *W*. Now let a part of the length, *CD* (see Fig. 29), be cut away, and replaced by three bars, *CD*, *EF*, *DE*, jointed at their ends to the two parts of the beam—*CD*, *EF* forming a rectangle, of which *DE* is a diagonal. With this construction the load *W* will be sustained, as well as by the original beam, but the three bars will be subject to stresses which we shall now determine. To do this, suppose each of the three bars (in succession) removed, and examine the effect on the structure—an artifice which often enables us to see very clearly the nature of the stress on a given part of the structure.

In the first place, suppose *CD* removed; then the portion *EB* will turn about the joint *E*, as shown in the lower part of the diagram, so that the function of the bar *CD* must be to prevent this turning, which is exactly what we have previously described as bending. The tendency to turn round *E*—that is, the bending moment at *E*—is in this case simply $= W \times CB$. But if there is a system of loads, the bending moment at *E* may be found by methods previously described.

Fig. 29.



Now let $H =$ stress on *CD*. It may readily be seen to be a tensile stress, because, on the removal of the bar, the ends *C* and *D* separate from one another. Also, let $h =$ *CE* or *DF*, the depth of the beam. The power of *CD* to prevent *EB* from turning about *E* is measured by the moment about *E* of the force *H* which acts along it. Therefore

$$Hh = M_E.$$

And dividing the bending moment at *E* by the depth of the beam, we obtain the magnitude of the tension of *CD*.

Next, let the bar *EF* be removed. The structure will yield by turning round the joint *D*, the point *F* approaching *E*. Thus the bar *EF* is in compression, and by its thrust, $= H'$ say, towards *F*, it prevents *FB* from turning round *D*.

The tendency to turn round *D*, due to the action of the external forces $= M_D$, will be equal to the resisting moment $H'h$.

$$\therefore \quad H'h = M_D.$$

Therefore if we divide the bending moment for the joint *D* opposite to the bar, by the depth of the beam *h*, we obtain the magnitude of the compressive force H'.

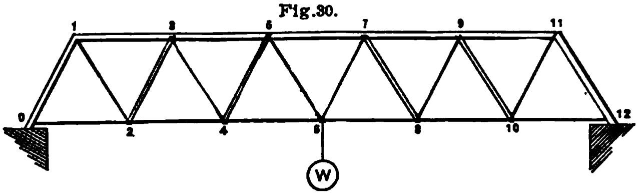
Lastly, let us suppose the diagonal bar *ED* to be removed, the effect is quite different from the two former cases; for instead of the overhanging portion of the beam turning about some point, it now gives way by sliding downwards (as shown in the centre of the diagram), remaining horizontal all the time. *CD* and *EF* turn about *C* and *E*, remaining parallel to one another. The rectangle *CDFE* becomes distorted by the shortening of the diagonal *ED* and the lengthening of *CF*. In the structure then the function of the diagonal bar *ED* is to prevent the sliding, by resisting the tendency to shorten. Thus the bar *ED* must be in compression, and by its thrust upon the point *D* it maintains FB from sliding downwards. Let S = thrust along ED and θ = angle it makes with the vertical. The force S may be resolved into two components, a horizontal one, $S \sin \theta$, and a vertical one, $S \cos \theta$. It is the vertical component alone which resists the sliding action, and maintains D in its proper position. Now the tendency to slide is no other thing than the shearing force on the structure, which we have previously been investigating. In this example the shearing force is simply W for all sections between A and B. But in other cases of loading the shearing force may be estimated by previously given methods. Since the downward tendency of the shearing force is balanced by the upward thrust of the vertical component of S along ED we have

$$S \cos \theta = F.$$

Instead of the points E and D being joined there might have been a bar CF, which, by the resistance to lengthening which it would offer, would have sustained the portion FB from sliding downwards. Such a bar would be in tension just as the bar ED is in compression, and in finding the stress on it we should use exactly the same equation. Now instead of having 3 bars only, the whole structure may be built up of horizontal and diagonal bars. The same principles will apply. On removing any one of the horizontal bars, we see that the structure yields by turning round a joint opposite: so we say the function of the horizontal bars is to resist bending. This is expressed by the equation $Hh = M$. On the other hand, the function of the diagonal bars being to resist the shearing tendency, we have always $S \cos \theta = F$.

24. *Warren Girders under various Loads.*—Fig. 30 shows a Warren Girder, so called from the name of the inventor, Captain Warren, a type of girder much used for bridges since its first introduction about the year 1850. It consists of a pair of straight parallel booms connected together by a triangulation of bars inclined to each other, generally at 60°, so that the triangles formed are equilateral. The booms in the actual structure are generally continuous through the junctions with the diagonal bars, but, if well constructed, there is no sensible error in regarding the structure as a true frame, in which the several divisions

Fig.30.



are all united by perfectly smooth joints. Any three bars forming a parallelogram and its diagonal may be considered as playing the same part as regards the rest of the structure as in the case just considered.

When a Warren girder is used, it is generally supported at the ends, and the loads are applied at one or more joints in the lower boom. We will examine some examples.

(1) Suppose there is a single load applied at a joint in the centre of the span.

First as to the diagonal bars. It was shown above that the duty of these bars was to prevent the structure yielding under the action of the shearing force; the vertical component of the stress on either of the diagonal bars being equal to the shearing force for the interval of the length of the girder within which the diagonal bar lies. This is expressed by the equation

$$S \cos \theta = F.$$

Now in the example which we are considering with the load in the centre, the shearing force will be the same at all sections to the right and left, namely $= \frac{1}{2} W$. Therefore the stress on all the diagonal bars is of the same magnitude,

$$S = \frac{W}{2 \cos 30^\circ} = \frac{W}{\sqrt{3}}.$$

If we consider the effect of removing either of the bars, we shall find that commencing from one end they prevent alternately the shortening and lengthening of the diagonals which they join, so that, commencing with one end, the bars are alternately in compression and tension. The compression bars are shown in double lines.

Next as to the several portions of the length of the top and bottom booms. As was shown above, the stress on any division of the horizontal bars has the effect of preventing a bending round the joint opposite; so that the moment of the stress about the joint is equal to the bending moment at the joint, due to the external forces. This is expressed by the equation

$$Hh = M.$$

Let a = length of a division.

Then, since the supporting force at the joint 0 is $\frac{1}{2}W$, the bending moments at the joints numbered 1, 2, 3, &c., are

$$M_1 = \frac{W}{2} \frac{a}{2} = \frac{Wa}{4},$$

$$M_2 = \frac{W}{2} a = \frac{Wa}{2},$$

$$M_3 = \frac{W}{2} \frac{3}{2} a = \frac{3Wa}{4},$$

and so on, the bending moments increasing in arithmetical progression.

Since the depth of the girder h is the same at all parts of the length; if we divide the M's each of them by h, we obtain the magnitude of the stress on the bars opposite the respective joints. Thus

$$H_{02} = \frac{Wa}{4h},\ H_{13} = \frac{Wa}{2h},\ H_{24} = \frac{3Wa}{4h},\ \text{and so on.}$$

We see, then, that the stress on the several divisions increases in arithmetical progression as we proceed from the ends towards the centre. By observing the effect of removing either of the bars, we see that all the divisions of the upper boom are in compression. This is expressed by drawing them with double lines in the figure. All the divisions of the lower boom are in tension.

(2) Next suppose the load is applied at some other joint not in the centre—the joint 4 for example. We must first calculate the supporting forces. Suppose they are P at 0 and Q at 12. For the portion of the girder to the left of 4 the shearing force will be the same at all sections and be equal to P. So the stress on all the diagonals between 0 and 4 will be equal to P sec 30°.

To the right of joint 4 the shearing force $= Q$, and the stress on all the diagonal bars from 4 to 12 will be Q sec 30°.

Proceeding from either end towards the joint where the load is applied, we observe that the diagonal bars are alternately in compression and tension—so that the bar 56 is now in compression, whilst the bar 54 is in tension. On these bars the nature of the stresses is just opposite to that to which they were exposed when the load was at the centre joint. Thus by varying the position of the load we not only vary the magnitude of the stress, but we may in some cases change the character of the stress, requiring a diagonal bar to act sometimes as a strut and sometimes as a tie.

For the divisions of the horizontal booms on the left of W the stresses are

$$\frac{Pa}{2h},\ \frac{2Pa}{2h},\ \frac{3Pa}{2h},\ \&c.,$$

in arithmetical progression up to the bar opposite the joint to which the load is applied; and to the right of W,

$$\frac{Qa}{2h},\ \frac{2Qa}{2h},\ \frac{3Qa}{2h},\ \&c.,$$

in arithmetical progression also up to the bar opposite the load. The upper bars are all in compression and the lower in tension as before.

When there are a number of loads placed arbitrarily at the different joints, the simplest way of determining the stresses is often to find the stress on the bars due to each load taken separately, and then apply the principle of superposition. In applying the principle due regard must be paid to the nature of the stress. A compressive stress must be considered as being of opposite sign to a tensile stress, and, in compounding, the algebraical sum of the stresses for each load will be the total stress on the bars.

(3) There is one particular case, that in which the girder is uniformly loaded, which it is advisable to examine separately.

In general, the load on the platform of the bridge is by means of transverse beams or girders transferred to the joints of the lower boom. The transverse beams may be the same number as the joints in the lower boom. In that case the girder will be loaded with equal weights at all the bottom joints. If the transverse beams are more numerous their ends will rest on the bottom booms, and tend to produce a local bending action in each division, in addition to the tensile stress which, as the bottom member of the girder, it will have to bear. In some cases, to lessen or get rid of this bending action, vertical suspending rods are introduced, by which means the middle points of the lower divisions are supported, and the loads transmitted to the upper joints of the girder. In such a case we may take all the joints both in the upper and lower booms to be uniformly loaded.

We will, however, suppose equal weights applied to the joints of the lower boom only. First as to the shearing forces. Between the end and the 1st weight the shearing force = the supporting force = half the total load = P say. In the next division the shearing force is less by the amount of the load at the 1st lower joint $= P - W$. In the third division of the lower boom from the end the shearing force $= P - 2W$, and so on. The stresses on the diagonals can now be found by multiplying the shearing force in the division within which any one diagonal lies by the secant of the angle which the diagonal makes with the vertical. The stresses will diminish in arithmetical progression as we pass inwards from the ends towards the centre. It will be observed that on the first and second diagonals from the end the stress is of the same magnitude. On the third and fourth it is alike also, and so on. The stresses are alternately compression and tension, commencing with compression on the first bar.

To find the stresses on the booms we must determine the bending moments at all the joints.

$$M_1 = \frac{P}{2}a. \qquad M_2 = \frac{P}{2}2a.$$

$$M_3 = \frac{P}{2}3a - W\tfrac{1}{2}a. \qquad M_4 = \frac{P}{2}4a - Wa.$$

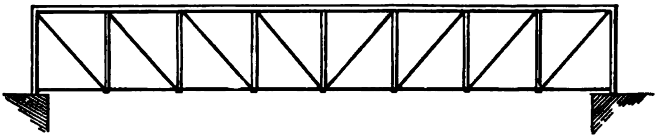
$$= \frac{a}{2}(3P - W). \qquad = \frac{a}{2}(4P - 2W).$$

$$M_5 = \frac{a}{2}(5P - 4W). \qquad M_6 = \frac{a}{2}(6P - 6W).$$

Divisions of the M's by h, the depth of the girder, will give the several horizontal stresses. They will be found to increase as we pass from the ends towards the centre.

25. *N Trusses.*—The web of the girder, instead of consisting of bars sloping both ways, forming a series of equilateral triangles, may be constructed of bars placed alternately vertical and sloping at an angle, so forming a series of right-angled triangles, looking like a succession of capital letters N. (See Fig. 31.) For this reason it is sometimes called an N girder. The ordinary practice is to divide the girder into a number of squares by means of the vertical bars, so that the diagonals slope at an angle of 45°. It is advantageous to place the diagonals so as to be in tension. For a load in the centre, or a uniformly distributed load, they should slope upwards from the centre towards the ends. The vertical bars will then be in compression. A short bar is better able to resist compression than a long one, whereas a tension bar is of the same strength whether short or long ; so it is manifestly economical of material, and a saving of weight, to place the long bars, that is the sloping bars, so as to be in tension. The same methods will apply to find the stresses on the bars, since as before the web resists the shearing action, and the booms the bending.

Fig. 31.



The simple queen truss, considered in Chapter I., Section II., is another example of a web consisting of alternate vertical and diagonal bars, but the diagonal is not usually inclined at 45° to the vertical.

EXAMPLES.

1. A trapezoidal truss is 24 feet span and 3 feet deep. The central part is 8 feet long and is braced by a diagonal stay so placed as to be in tension. Find the stress on each part when loaded with 4 tons at one joint and 5 tons at the other.

Stress on diagonal stay = ·95 tons.

2. A bridge is constructed of a pair of Warren girders, with the platform resting on the lower booms, each of which is in 6 divisions. The bridge is loaded with 20 tons in the middle. Find the stress on each part.

3. In example 2 obtain the result when the load is supported at either of the other joints.

4. From the results of examples 2 and 3 deduce the stress on each part of the girder when the bridge is loaded with 60 tons, divided equally between the three pairs of joints from one end to the centre.

Results for questions 2, 3, 4, the bars being numbered as in Fig. 30.

Bars.	Stress on Boom.				Bars.	Stress on Diagonals.			
	Load at 6.	at 4.	at 2.	at 6, 4, and 2.		Load at 6.	at 4.	at 2.	at 6, 4, and 2.
02	2·88	3·85	4·8	11·53	01	-5·76	-7·7	-9·6	-23·06
13	-5·76	-7·7	-9·6	-23·06	12	5·76	7·7	9·6	23·06
24	8·64	11·55	8·64	28·83	23	-5·76	-7·7	1·92	-11·54
35	-11·52	-15·36	-7·68	-34·56	34	5·76	7·7	-1·92	11·54
46	14·4	13·44	6·72	34·56	45	-5·76	3·85	1·92	0
57	-17·28	-11·52	-5·76	-34·56	56	5·76	-3·85	-1·92	0
68	14·4	9·6	4·8	28·8	67	5·76	3·85	1·92	11·54
79	-11·52	-7·68	-3·84	-23·04	78	-5·76	-3·85	-1·92	-11·54
8,10	8·64	5·76	2·88	17·28	89	5·76	3·85	1·92	11·54
9,11	-5·76	-3·84	-1·92	-11·52	9,10	-5·76	-3·85	-1·92	-11·54
10,12	2·88	1·96	·96	5·66	10,11	5·76	3·85	1·92	11·54
					11,12	-5·76	-3·85	-1·92	-11·54

5. A bridge 80 feet span is constructed of a pair of N girders in 10 divisions, the platform resting on the lower booms, and the diagonals so arranged as to be all in tension. A load of 80 tons is uniformly distributed over the platform. Find the stress on each bar.

SECTION III.—GIRDERS WITH REDUNDANT BARS.

26. *Preliminary Explanations.*—Again, returning to the (p. 43) beam out of which a portion has been cut and replaced by bars, let us suppose that instead of one diagonal bar only, there are two. We require to find the stresses on the bars. First, on the diagonal bars. In this case also the stress on these bars will be due to the shearing force. Together they prevent the structure yielding under the shearing action, but the amount each one bears is indeterminate until we know how the diagonals are constructed and attached to the rest of the structure. Suppose, for example, the diagonals are simple struts placed across the corners of the rectangle, but not secured at the ends. The struts will be incapable of taking tension; and the diagonal *ED*, which slopes in the direction, to be subject to compression will have to bear the whole shearing force. The other diagonal is ineffective. Secondly, suppose the diagonals to be simple ties, such as a chain or slender rod, and so incapable of withstanding compression. Then the bar *CF* will carry the whole shearing force. We may have any number of intermediate cases between these extreme ones according to the material of the diagonals and the method of attachment. In all cases one diagonal tends to lighten, and the other to shorten, and according to their powers of resistance to these tendencies they offer resistance to the shearing. If S_1 and S_2 be stresses on the two bars, then in all cases

$$(S_1 + S_2)\cos\theta = F.$$

If the diagonals are exactly similar rigid pieces similarly secured at the ends, equal changes of length will produce the same stress whether in compression or tension, so that each will bear an equal share of the shearing force. We shall then have

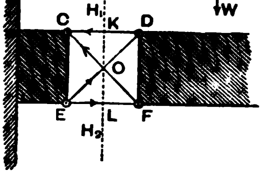
$$S_1 = S_2 = \tfrac{1}{2}F\sec\theta.$$

The foregoing is one of the simplest examples of a frame with redundant bars, and shows clearly why, in such cases, the stress on each bar cannot be determined by statical considerations alone, but depends upon the materials and mode of construction. In structures such as those considered in Chap. I., Sect. II., in which the principal part is an incomplete frame, stiffened by bracing or other means to provide against variations of the load, the bracing is usually redundant, and the stress on it cannot be calculated with certainty. Allowance has to be made for this in designing the structure by the use of a larger factor of safety. Redundant material is often no addition at all to the strength of the structure, and may even be a source of weakness, as will appear hereafter.

When framework girders were first introduced, it was objected by eminent engineers that failure of a single part would destroy the structure. Experience appears to have shown that risks of this kind are not serious, and the tendency of modern engineering design appears to be rather towards the employment of structures with as few parts as possible.

Next, as to the horizontal bars. These still sustain the bending moment, but not precisely in the same way as when there is only one diagonal. To find the magnitude of the forces, we employ a method similar to that used before, but instead of removing a bar we suppose the girder cut through one or more bars at any place convenient to our purpose; then the principle which we make use of is, that the action of each of the two halves on the other must be in equilibrium with the external forces which are applied to either half. In Figure 32 let us take a vertical section through the point of intersection of the diagonals, 4 bars are cut by the section, and through the medium of these 4 bars the structure to the left will act on the portion of the structure to the right of the section, and sustain it against the action of the external loads which rest on it.

Fig. 32.



First, there is the force H_1 pulling at K, and the force H_2 thrusting at L, and at O there are the two forces S_1 and S_2 on the two diagonals. Now, if we consider the tendency for the external forces to bend the right-hand portion round O, we see that the diagonal bars offer no resistance to this bending action, and must so far be left out of account. The whole resistance to bending is due to the bars CD and EF along which the forces H_1 and H_2 act, so that if M_0 be the bending moment at O due to the external forces,

$$(H_1 + H_2)\frac{h}{2} = M_0.$$

This will be true whatever be the proportion between S_1 and S_2, and H_1 and H_2. Instead, therefore, of taking the bending moment about a joint, as we did previously, we have in this case to take the moment about the point where the two diagonals cross.

But besides the balancing of the bending moment, there are other conditions to which the forces are subject, in order that the right-hand portion may be in equilibrium. One is, that all the forces which act on this portion must balance horizontally. There are no external forces which have any horizontal action, so that it is only the four internal forces which act along the bars cut, of which we have to take any account, and these must, on the whole, have no resultant horizontal action. The two thrusts must equal the two pulls; that is,

$$H_2 + S_2 \sin\theta = H_1 + S_1 \sin\theta.$$

$$\therefore \quad H_2 - H_1 = (S_1 - S_2)\sin\theta.$$

This also is true whatever be the distribution of the shearing force between the two diagonals.

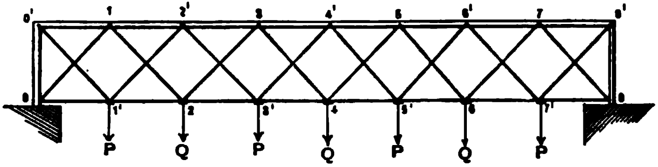
If, now, we suppose $S_2 = S_1$, then $H_2 = H_1 = H$, say. And the above formula becomes $Hh = M_0$, the same as we had before; but it must be applied a little differently, the moment now being taken about the point of intersection of the diagonals. If S_1 is not equal S_2, then H will be the mean of H_1 and H_2.

27. *Lattice Girders, Flanged Beams.*—Constructions with a double set of diagonals are common in practice. If, for example, in the N girder (Fig. 31) we place in each division two diagonals instead of one only, the construction is called a *lattice* or *trellis girder*. When employed for heavy loads, the diagonals are generally inclined at an angle of 45° to the vertical. In light structures, or when used for giving stiffness, they are often inclined at a much greater angle.

To determine the stresses, it will be necessary to make an assumption for the distribution of the shearing force between the two diagonals for each division of the girder, and it will generally be sufficiently correct to suppose each to carry half, and to write $S = \frac{1}{2}F \sec\theta$, and $Hh = M$ for the points where the diagonals intersect.

In lattice girders we more frequently find the double set of sloping bars introduced, but the vertical bars omitted. In this case it will not be true that the two diagonals in any one division are exposed to the same stress. We can determine the stresses otherwise. The structure may be divided into two elementary girders, each with its own system of diagonal bracing, and each with its own set of loads. Suppose, for simplicity, the number of divisions in the complete girder even, and each half girder loaded with equal weights applied to all the lower joints. Then if we make the simple, and in most cases safe, assumption that the thrusts on the two end vertical bars are equal, the forces on all the bars of the structure will be determinate. In the example shown in Fig. 33 the thrusts on the vertical end bars will be $2P$.

Fig. 33.



After we have calculated the stresses on each bar in each elementary girder, then, for any bar which is a portion of both, we must compound to obtain the total stress.

We may further increase the number of diagonal bars and obtain a girder, the web of which is a network of bars. In this case it will not be exactly, but will be very nearly, true that the horizontal bars take the bending, and the sloping bars the shearing action, the shearing force being regarded as equally distributed between all the diagonals cut by any one vertical section.

We may go on adding diagonal bracing bars until the space between the booms is practically filled up, and even then assume that the bending is taken by the horizontal bars and the shearing by the web. The numerous bracing bars may then be replaced by a vertical plate, which will form a continuous web to the girder. Such a construction is a very common one in practice, the horizontal members are called the top and bottom flanges of what is still a girder, and often called so, but more often a flanged or I beam. In the smallest class of these beams, they are rolled or cast in one piece; but for large spans they are built up of plates and angle irons rivetted together. For figures showing the transverse sections of such beams see Part IV. In taking the depth of such a girder, to make use of in the equation $Hh = M$, we ought to measure the vertical distance between the centres of gravity of the parts which we consider to be the flanges of the beam or girder. In the simple rolled or cast beam this will be the distance from centre to centre of depth of flanges. In the built-up beam account must be taken of the effect of the angle irons.

It must be remembered that this method of determining the strength of an I beam is only approximate. Its strength will be determined in a more exact way hereafter, when it will be found that the web itself assists in resisting the bending moment, but, area for area, to the extent only of about one-third that borne by the flange. On the other hand, the effective depth is less than the distance from centre to centre of the flanges. In rough preliminary calculations we may often neglect this, and employ the same formula as for lattice girders.

Girders are often of variable depths, so that the booms are not parallel; when this is the case the booms assist in resisting the shearing action of the load, as will be seen hereafter.

EXAMPLES.

1. A beam of I section is 24 feet span, and 16 inches deep; the weight of the beam is 1,380 lbs. It is loaded in the centre with 5 tons. Assuming the resistance to bending to be wholly due to the flanges, find the maximum total stress on each flange and the sectional area of each—the resistance to compression being taken to be 3 tons and to tension 4 tons per square inch.

Maximum total stress =53,505 lbs. = 23·88 tons.
Sectional area of upper flange = 8 square in.
,, ,, bottom ,, = 6 ,,

2. A trellis girder, 24 feet span and 3 feet deep, in three divisions, separated by vertical bars, with two diagonals in each division, is supported at the ends and loaded (1) with 20 tons symmetrically distributed over the middle division of the top flange, (2) with 20 tons placed over one of the vertical bars. Find the stress on each part of the girder, assuming each diagonal to carry half the corresponding shearing force.

Stress on diagonals—	Case 1.	14·2	0	14·2
	Case 2.	$18\frac{3}{5}$	$9\frac{4}{5}$	$9\frac{4}{5}$

Remark.—These results show the unsuitability of this construction for carrying a heavy load on account of the great inclination of the diagonals to the vertical.

3. A water tank, 20 feet square and 6 feet deep, is wholly supported on four beams, each carrying an equal share of the load. The beams are ordinary flanged ones, 2 feet deep. Find approximately the maximum stress on each flange, assuming that the weight of the tank is one-fourth the weight of water it contains.

$$\text{Distributed load on one beam} = \frac{187,500}{4} = 46,875 \text{ lbs.}$$

$$H_{\text{max.}} = 58,593 \text{ lbs.} = 26·1 \text{ tons.}$$

4. The Conway tubular bridge is 412 feet span. Each tube is 25 feet deep outside and 21 inside. The weight of tube is 1,150 tons, and the rolling load is estimated at $\frac{3}{4}$ ton per foot run. Find approximately the sectional areas of the upper and lower parts of the tube, the stress per square inch being limited to 4 tons.

$H_{\text{max.}}$ = 3,267 tons.
Area = 817 square in.

REFERENCES.

For details of construction of girders the reader is referred to
Girder Making . . . in Wrought Iron. E. HUTCHINSON. Spon, 1879.

CHAPTER III.

STRAINING ACTIONS DUE TO ANY VERTICAL LOAD.

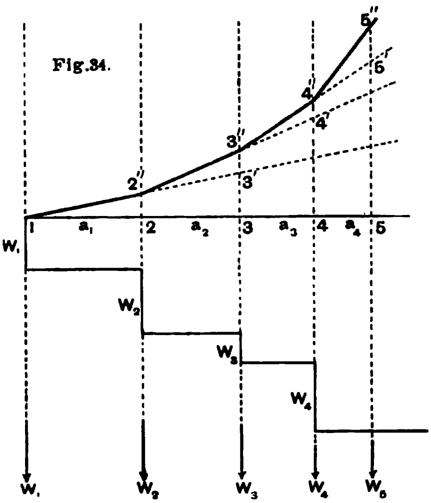
28. *Preliminary Remarks.*—The preliminary discussion in the preceding chapter of the straining actions to which loaded beams and framework girders are subject will have given some idea of the importance of the effect of shearing and bending on structures, and we shall now go on to consider the question somewhat more generally.

Let us suppose any body or structure possessing, as it usually will, a longitudinal vertical plane of symmetry, to be acted on by a set of parallel forces in equilibrium symmetrically disposed with respect to this plane, as, for example, gravity combined with suitable vertical supporting forces. Then these forces will be equivalent to a set of parallel forces in the plane of symmetry in question. Let the structure now be divided into two parts, A and B, by an ideal plane section, parallel to the forces and perpendicular to their plane. Then the forces acting on A may be reduced to a single force F lying very near the section considered and a couple M, while the forces acting on B may be reduced to an equal and opposite force F lying very near the section and an equal and opposite couple M. The pair of forces are the elements of the shearing action on the section, and the pair of couples are the elements of the bending action on the section. As the nature of the structure is immaterial, we may consider these straining actions for a given vertical section quite independently of any particular structure, and describe them as the Shearing Force and Bending Moment *due* to the given Vertical Load. We shall first consider the connection which exists between the two kinds of straining action and the method of determining them for any possible load.

CONNECTION BETWEEN SHEARING AND BENDING.

29. *Relation between the Shearing Force and the Bending Moment.*—Figure 34 shows the lines of action of weights W_1, W_2, etc., placed at the successive intervals a_1, a_2, etc.

Fig. 34.



In the first division the shearing force is

$$F_1 = W_1;$$

in the second $\quad F_2 = W_1 + W_2 = F_1 + W_2,$

$$\therefore \quad F_2 - F_1 = W_2;$$

in the third $\quad F_3 = W_1 + W_2 + W_3 = F_2 + W_3,$

$$\therefore \quad F_3 - F_2 = W_3;$$

and so on for all the divisions, so that in the n^{th} division

$$F_n - F_{n-1} = W_n.$$

We express this in words by saying that *the difference between the shearing forces on two consecutive intervals is equal to the load applied at the point between the two intervals*; or it may be written

$$\Delta F = W.$$

By setting down ordinates to a horizontal base line we obtain the stepped figure as the graphical representation of the shearing force at any point of the beam. It is drawn by first setting downwards at 1 an ordinate for the shearing force on the 1st interval, and then passing along the beam to the other end, on meeting the lines of action of the successive weights the length of the ordinates is increased by the amount of the weights. In so doing we make use of the proposition which has just been proved.

This is called the *Polygon of Shearing Force*, or more generally, when the loads are continuous, the *Curve of Shearing Force*.

Next as to the bending moment. At the first point where W_1 is applied $\quad M_1 = 0,$

at the second point $\quad M_2 = W_1 a_1 = F_1 a_1;$

at the third point $M_3 = W_1(a_1 + a_2) + W_2a_2 = W_1a_1 + (W_1 + W_2)a_2$
$= M_2 + F_2a_2,$

$\therefore \quad M_3 - M_2 = F_2a_2;$

„ fourth point $M_4 = W_1(a_1 + a_2 + a_3) + W_2(a_2 + a_3) + W_3a_3,$
$= W_1(a_1 + a_2) + W_2a_2 + (W_1 + W_2 + W_3)a_3,$
$= M_3 + F_3a_3,$

$M_4 - M_3 = F_3a_3;$

and generally, $M_n - M_{n-1} = F_{n-1}a_{n-1}.$

We may express this in words by saying that the *difference between the bending moments at the two ends of an interval is equal to the shearing force, multiplied by the length of the interval.* Or the result may be written

$$\Delta M = Fa.$$

We will now take a nnmerical example and see how we may make use of this property to determine a series of bending moments.

Let AB be a beam fixed at one end, and loaded with weights of 2, 3, 5, 11, 13, 7 tons, placed at intervals of 3, 2, 3, 5, 4, 6 feet, commencing from the free end. We adopt a tabular method of carrying out the work of calculation.

W.	*F.*	*a.*	*Fa.*	*M.*
2				0
	2	3	6	
3				6
	5	2	10	
5				16
	10	3	30	
11				46
	21	5	105	
13				151
	34	4	136	
7				287
	41	6	246	
				533

First set down a column of weights applied, as shown by the figures in the column headed W. In the next column write the shearing forces. Since the shearing forces are uniform over the intervals between the weights, it will be best to write the F's opposite the spaces between the weights. Any F is found by adding to the F above it the adjacent W. In the third column we set down the lengths of the intervals. Then multiplying the F's and corresponding a's together, set the results in column 4. Lastly, we can write down the column of bending moments by the repeated addition of the Fa's—the bending moment at any point being found by adding to the bending moment at the point above the value of Fa between the points.

If instead of all the forces acting one way some of them act upwards, a minus sign should be set opposite, and all the operations performed algebraically.

The method is equally applicable however the beam is supported.

For example, let a beam 23 feet long be supported at the ends and loaded with 3, 2, 7, 8, 9 tons, placed at intervals of 2, 2, 3, 4, 5, 7 feet, reckoning from one end.

First calculate one supporting force, say at the left-hand end by taking moments about the other end. In the column of *W*'s set this for the first force, and since all the loads act in the contrary direction, put negative signs opposite them, and in writing down the next column of *F*'s add algebraically. We shall at the bottom of the column determine the supporting force at the right hand end. At the bottom of the column of *M*'s, that is at the point where the right hand supporting force acts, we ought to get a zero moment. The obtaining of this will be a test of the accuracy of the work. In this example the small difference between 89·72 and 89·81 is due to our having taken the supporting force only to two places of decimals.

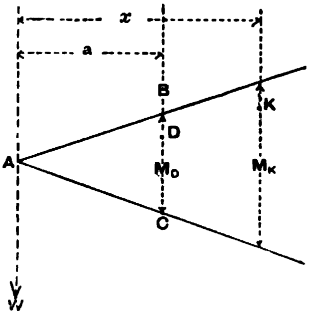
W.	*F*	*a.*	*Fa.*	*M.*
16·17				0
	16·17	2	32·34	
-3				32·34
	13·17	2	26·34	
-2				58·68
	11·17	3	33·51	
-7				92·19
	4·17	4	16·68	
-8				108·87
	-3·83	5	-19·15	
-9				89·72
	-12·83	7	-89·81	
12·83				0

Observation of the process of calculation leads us to a very important proposition, viz., *where the shearing force changes sign, the bending moment is at that point a maximum.* This will be true for all important practical cases, but exceptional cases may be imagined in which, where the shearing force changes sign, the bending moment is a minimum. Since $\Delta M = Fa$, then, so long as F is positive, M will be an increasing quantity as we pass from point to point. But where F changes to negative there M commences to diminish.

We will now explain the construction of a diagram of bending moment for a system of loads: and first let us consider how the moment of a force about any point or succession of points may be graphically expressed.

Let W be a force and D any point, and suppose the numerical magnitude of the moment of W about D known. Draw a line through D parallel to the force at a distance a (Fig. 35), and anywhere in this line take a length BC to represent on some convenient scale the moment, $M_D = Wa$, of W about D. The scale must be so many inch-tons, foot-lbs., or similar units to the inch. Then choose any point A in the line of action of the force, join AB and AC, and produce these lines indefinitely. The moment of W about any point whatever is represented by the intercept by the radiating lines AB, AC of a line drawn through the point parallel to the force. For example, the moment about $K = M_K = Wx$, where x is the perpendicular distance of K from the line of action of W.

Fig. 35.



$$\therefore \frac{M_K}{M_D} = \frac{Wx}{Wa} = \frac{x}{a}.$$

By similar triangles the intercepts are to one another in the ratio $x : a$ so that they correctly represent the moments.

We will first draw the diagram of bending moments for a beam fixed at one end and loaded at intervals along its length. Returning to Fig. 34, take a line representing the length of the beam as base line. Produce upwards the lines of action of the loads. Commence by setting up at the point where W_2 acts a line to represent the moment of W_1 about that point, that is, take 2′ 2 to represent $W_1 a_1$. If 1 2′ be joined and produced, then the intercept between this line and base line 1 5 will represent on the same scale the moment of W_1 about any point in the beam. Next at the point 3′, where 12′ cuts W_3 2′ 3′, set up 3′ 3″ to represent $W_2 a_2$ join 2″ 3″ and produce it. The intercept between 2′ 3′ and 2″ 3″ will represent the moment of W_2 about any point in the beam. Then at the point 4′, where 2″ 3″ cuts W_4, set up 4′ 4″ to represent $W_3 a_3$. Join 3″ 4″, produce it, and so on with all the weights. The polygon 1, 2″, 3″, 4″, 5″ … will be obtained, the ordinates of which measured from the base line AB will represent the bending moment at any point, due to all the weights on the beam. This is called the *Polygon of Bending Moment*. In the case of a continuous distribution of load it is called the curve of bending moment.

There is a very important relation between the polygons of shearing force and bending moment which have been drawn in all cases of loading.

The bending moment at the point 2 = $W_1 a_1$. Now referring to the shearing force diagram, we observe standing underneath the interval a_1 a rectangle whose area = $W_1 a_1$. Next, for the point 3,

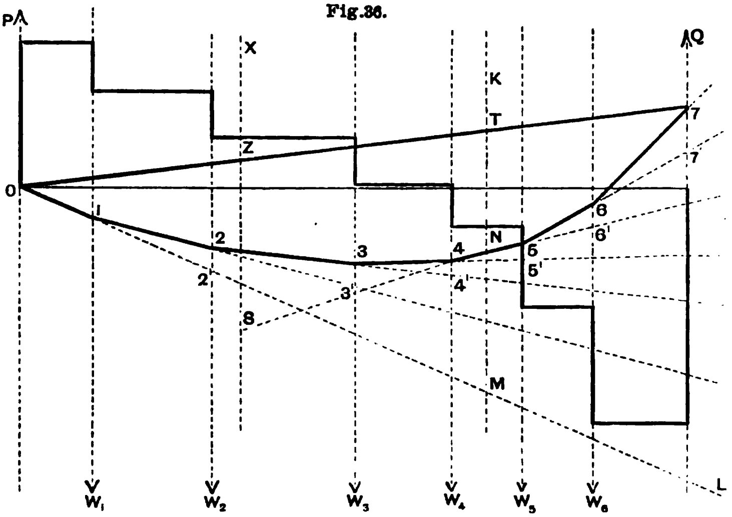
$$M_2 = W_1(a_1 + a_2) + W_2 a_2.$$

This is represented on the diagram of bending moment by the ordinate 33″. In the shearing force diagram we notice that the area under the portion of the beam from 1 to 3 consists of two rectangles, $W_1(a_1 + a_2) + W_2 a_2$. So that at this point also the bending moment is represented by the area of the polygon of shearing force, reckoned from the end up to the point 3. And so on for every point. This important deduction may be stated generally thus :—*The ordinate of the curve of bending moment at any point is proportional to the area of the curve of shearing force reckoned from one end of the beam up to that point.*

30. *Application to the case of a Loaded Beam.*—We will next take the case of a beam supported at the two ends.

First, calculate the supporting force P, set it up at the end of the base line as an ordinate, and draw the stepped polygon by continually subtracting the W's. At some point in the beam we shall cross the base line. At that point the shearing force changes sign, and there the bending moment is a maximum. The shearing force on the last interval will give the magnitude of the supporting force Q. The polygon thus drawn will be the polygon of shearing force.

Fig.36.



The polygon of bending moment may be drawn without previously determining the supporting force at either end thus :—

Commencing at *O* (Fig. 36), the point of application of *P*, draw any sloping line 0 1 2′ cutting W_1 in 1, and W_2 in 2′. Then set up

2′ 2 to represent W_1a_1, join 1 2, produce it to cut W_3 in 3′.
3′ 3 „ W_2a_2, „ 2 3, „ W_4 in 4′.
4′ 4 „ W_3a_3, „ 3 4, „ W_5 in 5′, and so on.
7′ 7 will represent W_6a_6.

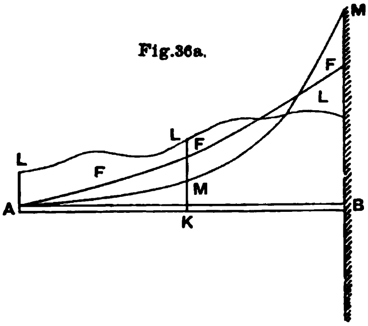
Now join 7 with the point 0, where 0 1 2′ cuts the line of action of *P*. This is called the Closing Line of the polygon of moments. Any vertical intercept of this polygon will represent the bending moment at the corresponding point of the beam. The proof of this may be stated shortly thus :—If we produce 0 1 to meet the line of action of *Q* in *L*, then *L*7 will, from what has been said before, represent the sum of the moments of all the weights *W* about the end of the beam where *Q* acts. And from the conditions of equilibrium this must equal the moment of *P* about that end. Accordingly, if we take any point *K*, the vertical intercept *MT* below it will represent the moment of *P* about *K*. This is an upward moment. The four weights which lie to the left of *K* will together have a downward moment about *K* represented by *MN*. Therefore, the difference *NT* will represent the actual bending moment at the point *K*.

It sometimes happens that we want the moment of the forces not about *K*, the section which separates the two parts of the structure, but about some other point, say *X*, in the figure. We can obtain this moment also with equal facility; for if we prolong the line 4 5 of the polygon to meet the vertical through *X* in the point *S*, we find, reasoning in the same way, that *SZ*, the intercept between the side so prolonged and the closing line, is the moment required. Polygons of moments and shearing forces may also be constructed by making use of the fundamental relations shown above to exist between them and the load, as will be seen presently, while a third purely graphical method is explained farther on, based on a most important property which they possess.

31. *Application to the case of a Vessel floating in the Water.*—We sometimes meet with cases in which the beam or structure is loaded not at intervals, but continuously, the distribution of the load not being uniform, but varied in some given way. In such a case, the diagrams of shearing force and bending moment become continuous curves. The most convenient way of expressing how the load is distributed is by

means of a curve, the ordinate of which at any point represents the intensity of the load at that point. Such a curve is called a *curve of loads*. It may be regarded as the profile of the upper surface of a mass of earth or other material resting on the beam.

Fig. 36a.



We will consider, first, the case of a beam fixed at one end and loaded continuously throughout, in a manner expressed by a curve of loads *LL*. (Fig. 36*a*.) The total area enclosed by the curve of loads will represent the total load on the beam, and between the two ordinates of any two points will be the load on the beam between the two points. Now, the area of the curve of loads, reckoned from the end *A* up to any point, *K* say, since it represents the total load to the left of *K*, will be the shearing force at *K*. If at *K* we erect an ordinate *KF*, to represent on some convenient scale the area *ALK*, and do this for many points of the beam, we shall obtain a second curve *FF*, the curve of shearing force. Having done this, we may repeat the process on the curve *FF*, and obtain the curve of bending moment. For we have previously proved that if the load on the beam is concentrated at given points, then the ordinate of the curve of bending moments is at any point proportional to the area enclosed by the curve of shearing force for the portion of the beam between the end and that point. The truth of this is not affected by supposing the points of application of the load to be indefinitely close to one another, in which case the load becomes continuous. Accordingly, if we set up at *K* an ordinate, *KM*, to represent on some convenient scale the area *AFK* of the shearing force curve, and repeat this for many points, we obtain the curve of bending moment, *MM*. Thus the three curves form a series, each being the graphical integral of the one preceding.

This process has an important application in the determination of the bending moment to which a ship is subjected on account of the unequal distribution of her weight and buoyancy along the length of the ship. On the whole, the upward pressure of the water, called the buoyancy, must be equal to the downward weight of the ship; and the lines of action of these two equal and opposite forces must be in the same vertical. But for any portion of the length, the upward pressure and the downward weight will not, in general, balance one another; so, on account of the difference, shearing and bending of the ship will be induced. In the case of a rectangular block of wood floating in water, the upward pressure of the water will, for every portion of its length, equal the downward weight, and there will be no shearing and bending action on it. But, in actual ships, the disposition of weight and buoyancy is not so simple. Taking any small portion of the length of the ship, the difference between the weight of that portion of the ship and the weight of the water displaced by that portion of the ship, will be a force which acts on the vessel sometimes upwards and sometimes downwards, according to which is the greater, just in the same way as forces act on a loaded beam producing shearing and bending. In the construction of the vessel, strength must be provided to resist these straining actions, and it is a matter of great practical importance to determine accurately the magnitude of them for all points of the length of the ship. We will select an example of very frequent occurrence, that in which at the ends of the ship the weight exceeds the buoyancy, whilst at the centre the buoyancy exceeds the weight. If the ship were very bluff ended, and carried a cargo of very heavy material in the centre hold, the distribution of weight and buoyancy would probably be the reverse of this.

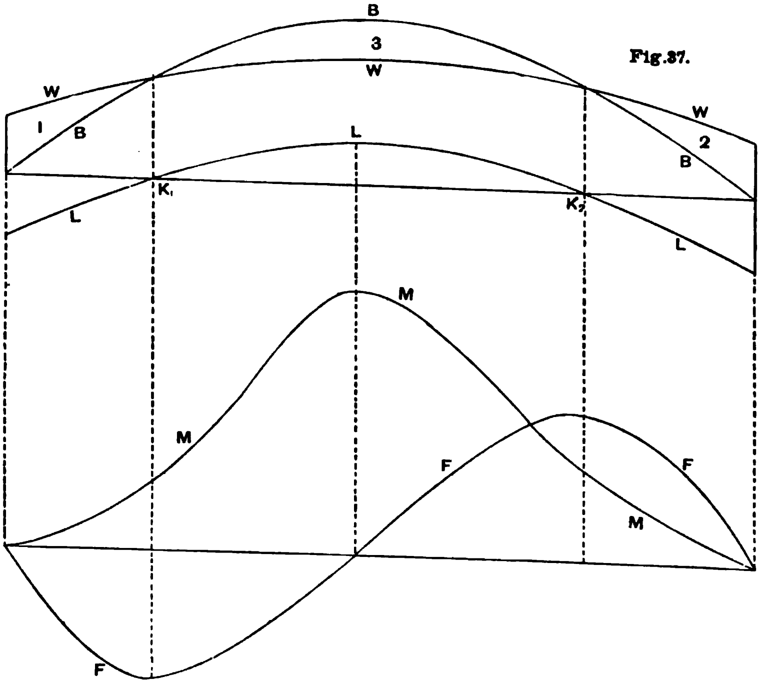


Fig. 37.

In the example the ship is supposed to be divided into any number of equal parts, and the weight of water displaced by each of those parts determined; ordinates are set up to represent those weights, and so, what is called a curve of buoyancy *BBB* (Fig. 37) is drawn. The whole area enclosed by the curve will represent the total buoyancy or displacement of the vessel, and is the same thing as the total weight of the vessel. Next we suppose that the weights of the different portions of the ship are estimated, and ordinates set up to represent these weights, then what is called a curve of weight, *WWW*, is obtained. In the figure it is set up from the same base line. The total area enclosed by this curve will also be the total weight of the ship, and must therefore equal the area enclosed by the curve of buoyancy. Thus the sum of the two areas marked 1 and 2 must equal the area marked 3. Not only must this be true, but also the centres of gravity must lie on the same ordinate. The difference at any point between the ordinates of the two curves will express by how much at the ends the weight exceeds the buoyancy, and in the middle portion by how much the buoyancy exceeds the weight, representing, in the first case, the intensity of the downward force, and, in the second, the intensity of the upward force. Where the curves cross one another and the ordinates are the same height, as at K_1 and K_2, *the sections are said to be water-borne.* If now we set off from the base line ordinates equal to the difference between the ordinates of the two curves *BBB* and *WWW*, we obtain the curve of loads *LLL*; some portions where the weight is in excess will lie below the base line, and the rest, where the buoyancy exceeds the weight, will lie above the base line. From what has been said before, the area above the base line must equal the area below. Having obtained the curve of loads, the curve of shearing force is to be obtained from it in the manner previously described, by setting up, at any point, an ordinate to represent the area of the curve *LLL* between the end of the ship and that point. In performing the operation, due regard must be paid to the fact that the loads on different parts of the ship act in different directions, and for one direction they must be treated as negative, and the corresponding area of the curve as a negative area.

Having thus determined the curve of shearing force *FFF*, the same operation must be repeated on that curve to determine the curve of bending moment. In drawing the curve of shearing force it will be found that at the further end of the ship we return again to the base line from which we started at first, for the shearing force at the end must be zero. Also the bending moment at the end must be zero. This gives us tests of the accuracy of our work.

In this example the bending is wholly in one direction, tending to make the ends of the ship droop or the ship to "hog" in the technical language of the naval architect, but in some examples the direction of bending changes once or more times. Curves of shearing force and bending moment were first explained in relation to a vessel floating in the water by the late Professor Rankine in his work on shipbuilding. It does not, however, appear that any such curves were ever constructed in any actual example until 1869, when some were drawn for vessels of war by Mr. (now Sir E.) Reed, at that time chief constructor of the Navy. The results obtained by him are described in a paper read before the Royal Society (Phil. Trans. for 1871, part 2). They now form part of the ordinary calculations of a vessel.

Since the water exerts on the vessel not only vertical but also horizontal forces, the straining actions upon her do not consist solely of shearing and bending, but include also a thrust. The horizontal pressure also produces bending in a manner which we shall hereafter explain.

32. *Maximum Straining Actions.*—The set of forces we are considering are in equilibrium, and must therefore be partly upwards and partly downwards. The downward force is the total weight W, and is generally more or less distributed, the upward force is of equal magnitude, and is usually concentrated near two or more points. In the case of the vessel, however, the upward force is distributed like the weight, though not according to the same law. In any case the greatest shearing force must be some fraction of the weight, and the greatest bending moment must be some fraction of the weight multiplied by the length l over which the weight is distributed. We may therefore express the maximum straining actions by the formulae

$$F_0 = k . W ; M_0 = m . Wl,$$

where k, m are numerical quantities depending on the distribution of the load and the mode of support. Thus for a uniformly loaded beam supported at the ends $k = \frac{1}{2}$, $m = \frac{1}{8}$. The greatest value m can have in a beam resting on supports without attachment is $\frac{1}{4}$; this occurs when the beam is supported at the ends and the load concentrated in the middle or conversely. In vessels where the supporting force is distributed m is much less; its maximum value is estimated by Mr. White at $\frac{1}{35}$ in ordinary merchant steamers.

EXAMPLES.

1. A Warren girder with 12 divisions in the lower boom is supported at the ends and loaded with 250 tons, which may be supposed to be equally distributed among all the 25 joints. Find the stress on each bar by calculating the series of shearing force and bending moments.

2. The buoyancy of a vessel is 0 at the ends and increases uniformly to the centre, while the weight is 0 at the centre and increases uniformly to the ends. Draw the curves

of shearing force and bending moment, and find the maximum values of these quantities in terms of the displacement and length of the vessel.

***Answer*—$k = \frac{1}{4}$; $m = \frac{1}{12}$.**

3. A beam, 48 feet span, is supported at the ends and loaded with weights of 6, 9, 10, 13, 5, and 7 tons, placed at intervals of 4, 5, 9, 7, 13, and 8 feet respectively, commencing at one end. Calculate the shearing force in each interval and the series of bending moments.

RESULTS FOR LEFT-HAND HALF OF GIRDER.

F	115	105	95	85	75	65	55	45	35	25	15	5	5
S	132·2	120·7	109·2	97·7	86·2	74·7	63·2	51·7	40·2	28·7	17·2	5·7	
$\Delta H = F\frac{1}{\sqrt{3}}$	66·1	60·3	54·6	48·8	43·1	37·3	31·6	25·8	20·1	14·3	8·6	2·	
H	66·1	126·4	181	229·8	272·9	310·2	341·8	367·6	387·7	402	410·6	413·4	

4. In the last question construct the polygons of shearing force and bending moment.

5. In the case of a uniformly loaded beam supported at the ends, verify the principle that the area of the curve of shearing force is proportional to the ordinate of the curve of bending moment.

6. When a beam is supported at the ends and loaded in any way, show that an ordinate at the point of maximum moment divides the area of the curve of loads into parts, which are equal to the supporting forces. Further, if a b are the distances of the centres of gravity of these parts from the ends of the beam, and l the span, show that the maximum moment is mWl where

$$\frac{1}{m} = \frac{l}{a} + \frac{l}{b}.$$

TRAVELLING LOADS.

33. We have hitherto been investigating the effect of a permanent fixed load on a structure in producing straining actions on it. We next examine the effect of a load which is not permanent, but which at different times takes up different positions on the structure, and we require to know what position of the load will produce the greatest straining action at any particular part of the structure, and also the amount of that maximum straining action.

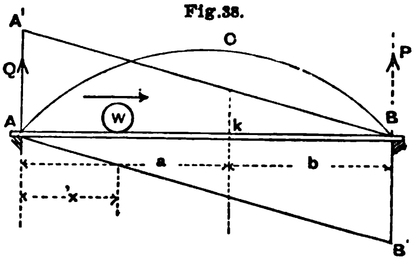


Fig. 33.

This question arises principally in the design of bridges across which a *travelling load*, such as a train, may proceed. We will take first the simple case of a beam of span l, supported at the ends and suppose a single concentrated load W to travel across it in the direction of the arrow. Let us consider any point K (Fig. 38) in the beam, distant a and b from the ends. As the load traverses the beam, each position of the load will produce a certain shearing force and bending moment at the point K. To find their greatest value let $x =$ distance of W from A, then the supporting force at $B = P = W\frac{x}{l}$. So long as the weight lies between A and K the shearing force at K will be simply P.

$$F_x = W\frac{x}{l},$$

consequently the shearing force will increase as x increases, until the oad reaches the point K. So long as the weight lies to the left of K, the tendency will be for the portion KB to slide upwards relatively to the portion AK. This we will call a positive shearing force. Therefore, putting $x = a$,

$$\text{Max. positive shearing force at } K = W\frac{a}{l}.$$

Now, supposing the weight to move onward, it will in the next instant have passed to the other side of K, and the shearing force will have undergone a sudden change. It will now be equal to the supporting force at the end B,

$$Q = W\frac{b}{l}.$$

But not only is the magnitude of the shearing force suddenly changed, but the tendency to slide is now in the other direction, and the shearing force is negative. As the weight moves farther to the right of K the shearing force diminishes, thus

$$\text{Max. negative shearing force at } K = W\frac{b}{l}.$$

Wherever we take the point K it will always be true that the maximum positive shearing force will occur when the weight lies immediately to the left of K, and the maximum negative when the weight lies immediately to the right. The maximum positive shearing force for every point in the beam may be represented by the ordinates of a sloping line AB' below the beam the length BB' being taken to represent W. And similarly the maximum negative shearing force at any point by the ordinates of the sloping line $A'B$ above AA' also being taken to represent W.

Next as to the bending moment. When the weight lies to the left of K, and is at a distance from A equal to x, the bending moment at K is given by

$$Pb = W\frac{b}{l}x.$$

This goes on increasing as x increases until the weight reaches the point K. After having passed K the bending moment at K must be differently expressed, being then

$$\frac{W(l-x)}{l}a,$$

which becomes smaller as x increases; so that the greatest bending moment at K occurs when the load is immediately over K, and then the

$$\text{Max. bending Moment at } K=\frac{Wab}{l}.$$

If the point K is taken in the centre of the beam,

$$\text{Max. Moment at centre}=\tfrac{1}{4}Wl \text{ as before.}$$

If ordinates be set up at all points to represent the maximum bending moments at these points, a parabola (ACB) will be obtained. For the expression for the maximum bending moment is just twice that previously obtained for the same weight distributed uniformly.

If there are more weights, W_1, W_2, &c., on the beam, and W_1 lie to the right of K, the shearing force at $K=P-W_1$, where P is the right-hand supporting force. Now, suppose we shift W_1 to the left of K, we shall diminish the supporting force to P' say, and this will be the new shearing force at K. The difference between P and P' will be less than W_1, and the shearing force will be increased by passing W_1 to the left of K. If we were to remove W_1 altogether the diminution of P will be less than the whole of W_1, and so the shearing force at K will be increased by so doing. We obtain the greatest positive shearing force at K when all the weights are to the left of K, but as near to K as possible. The greatest negative shearing force will occur when all the weights lie to the right of K, as near to K as possible.

The maximum bending moment at K will occur when the weights are as near K as possible, whether to the right or left. Any addition to the load, on whichever side of K it is placed, will cause an addition to the bending moment.

There is another important case, that in which we have a continuous load of uniform intensity passing over the beam, as when a long train passes on to a bridge. We observe that as the train approaches K, the supporting force at B, and therefore the shearing force at K, increases. When any portion of the weight lies to the right of K, the supporting force will be increased by a part of the weight lying to the right of K; but when we have subtracted the whole of that weight, the difference, which will be the shearing force at K, will be less than before; thus the maximum positive shearing force at K will occur when the portion AK is fully loaded, and no part of the load is on KB. To find its value we have only to determine the supporting force at *B*, by taking moments about *A*; then

$$F_K = wa\frac{\frac{1}{2}a}{l} = \frac{\frac{1}{2}wa^2}{l},$$

that is, the magnitude is proportional to the square of the distance of the point from the end *A*. It will be graphically represented by the ordinates of a parabola which has its vertex at *A*, and axis vertical, cutting the vertical through *B* in a point *B′* such that $BB' = \frac{1}{2}wl$, that is, half the weight on the beam when fully loaded. As the load travels onward the shearing force diminishes at last to zero, and then changes sign, becoming negative, the numerical magnitude increasing as the rear of the load approaches *K*. The maximum negative shearing force will occur when the portion *KB* only is loaded. The ordinates of a parabola set below the line of the beam having its vertex at *B* and axis vertical, will represent the maximum negative shearing force.

The question of maximum bending moment is more simple. It will occur at any point when the beam is fully loaded; for at any point the bending moment is the sum of the bending moments due to all the small portions into which the load may be divided, and the removal of any one of them will cause a diminution of bending action throughout the whole length of the beam. A parabola, with its highest ordinate at the centre $= \frac{1}{8}wl^2$, will represent it at any point.

34. *Counter Bracing of Girders.*—In the design of a framework girder it is very important to take account of the maximum positive and negative shearing forces due to a travelling load.

In such a structure the shearing force is resisted by the diagonal bars, and in general these bars are so placed as to be in tension, for the bar may then be made lighter than if subject to a compressive force of the same amount. Suppose the diagonal bars so arranged as to be all in tension when the girder is fully loaded, or when there is only the dead weight of the girder itself to be taken account of. There may be ample provision made for withstanding the tensile forces, and yet it will be important to examine if there may not be some disposition of the travelling load which would cause a thrust on some of the diagonals. If so, the maximum amount of this must be calculated, and the structure made capable of withstanding it. If the shearing force at any section of the girder is what we have called a positive shearing force, that in which the right-hand portion tends to slide upwards relatively to the left, then, in order that it may be withstood by the tension of a diagonal bar, the bar must slope upwards to the right. If the bar so slopes, and by the movement of the travelling load the shearing force becomes negative, then the bar will be subjected to compression. Now, it will frequently happen that in the central divisions of a girder the positive or negative shearing forces due to the dead load are less than the negative or positive shearing forces due to the travelling load, so that if those bars are arranged to be in tension under the dead load, then, on the passage of the travelling load, the stress will be changed to compression. In some cases the bars are slender and not suited to sustain compression; the shearing force is then provided for by the addition of a second diagonal, sloping in the opposite direction, which, by its tension, will perform the duty the first bar would otherwise have to perform by compression. Such a bar is called a counter-brace. We frequently see such additional bars fitted to the middle divisions of framework girders.

Again, the powers of resistance of a piece of material to a given maximum load are greater the smaller the fluctuation in the stress to which it is exposed; and therefore, in determining its dimensions, it is important to know not only the maximum but also the minimum stress to which it is exposed. This can be done on the principles which have just been explained.

EXAMPLES.

1. A single load of 50 tons traverses a bridge of 100 feet span. Draw the curves of maximum shearing force and bending moment, and give the values of these quantities for the quarter and half span.

2. A train weighing one ton per foot run, and more than 100 feet long, traverses a bridge 100 feet span. Draw the curves of maximum shearing force and bending moment, and give the values of these quantities at the quarter and half span.

3. In the last question, suppose the permanent load $\frac{3}{4}$ths ton per foot run. Find within what limits counterbracing will be required.

4. In Ex. 5, page 49, the maximum rolling load is estimated at 1 ton per foot run. Determine which of the diagonals will be in compression, and the amount of that compression, assuming a complete number of divisions to be loaded.

The two centre diagonals are the only ones which can be in compression, the maximum amount of which will be $= (3 \cdot 2 - 2)\sqrt{2} = 1 \cdot 7$. It will occur when the rolling load occupies 4 divisions only of the bridge.

5. In the last question, suppose a single load of 20 tons to traverse the bridge. Find the maximum stress, both tension and compression, on each part of the girder.

Divisions.	1	2	3	4	5
Max. tension, bottom boom, -	0	27	48	63	72
Max. compression, upper boom, -	27	48	63	72	75
Max. tension of diagonals, - -	38·1	31·1	24	17	9·8
Max. compression of diagonals, -	—	—	—	0	2·8

6. In the two preceding questions, find the fluctuation of stress on each part of the girder.

METHOD OF SECTIONS.

35. *Method of Sections applied to Incomplete Frames. Culmann's Theorem.*—The straining actions due to a vertical load may either be wholly resisted by internal forces called into play within the structure itself, or also in part by the horizontal reaction of fixed abutments: the supporting forces being in the first case vertical, and in the second having a horizontal component. The distinction is one of the greatest importance in the theory of structures, which are thus divided into two classes, Girders and Arches, including under the last head also chains. It is the first class alone which we consider in this chapter.

The general consideration of internal forces is outside the limits of this part of our work, and we shall here merely consider some cases of framework structures, commencing with that of an incomplete frame.

Incomplete frames are in general, as in Chapter I., structures of the arch and chain class, but by a slight modification we can readily convert such a frame into a girder and thus obtain very interesting results.

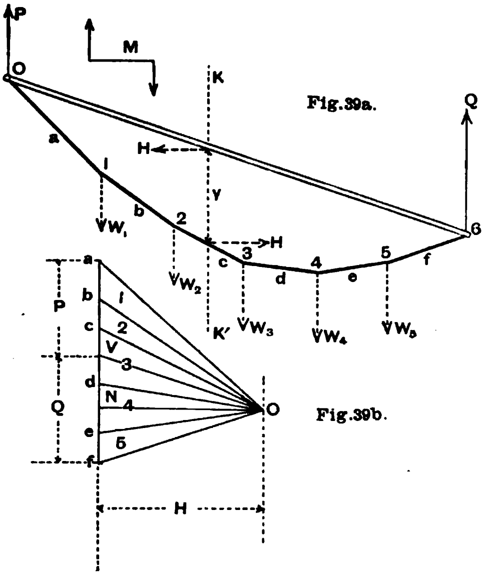


Fig. 39a.

Fig. 39b.

Fig. 39*a* shows a funicular polygon such as that in Fig. 11, page 14, except that the supports are removed and replaced by a strut *O6*. By this addition the polygon becomes a closed figure, and *O6* is therefore called its "closing line." The structure is carried by suspending rods at the joints *O6*, and loaded as shown. The construction of the diagram of forces, Fig. 39*b*, has been sufficiently explained in the article referred to, and it only remains to observe that the supporting forces *PQ* are immediately derived from the diagram by drawing *OV* parallel to the closing line, which is not necessarily horizontal. The horizontal thrust of the strut and tension of the rope is found as before by drawing *ON* horizontal.

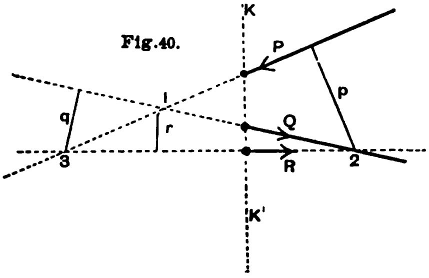
This structure may now be regarded as a girder, the load on which, together with the vertical supporting forces, produce definite straining actions *M* and *F* on any section. Let the section be *KK'* in the figure cutting one of the parts of the rope and the strut as shown in the figure: let the intercept be *y*. Consider the forces acting at the section on the left-hand half of the girder, the horizontal components of these forces are equal and opposite, acting as shown in the figure, each being *H* or *ON* in the diagram of forces. The vertical components are balanced by the shearing force, and the horizontal components by the bending moment, which last fact we express by the equation

$$Hy = M,$$

that is to say, the funicular polygon corresponding to a given load is also a polygon of bending moments, the intercept between the polygon and its closing line multiplied by the horizontal force is equal to the bending moment due to the load. Hence, by a purely graphical process, we can construct a polygon of moments, for we have only to construct a funicular polygon corresponding to the load as shown in the article already cited, and complete it by drawing its closing line. This is one of the fundamental theorems of graphical statics, a subject which of late has been extensively studied. The construction is intimately connected with the process of Art. 29 as the reader should show for himself. In its complete form it is due to Culmann and is generally known by his name, having been given in his work on graphical statics.

36. *Method of Sections in general. Ritter's Method.*— In frames which are complete the number of bars cut by the section, instead of being two only, as in the preceding case, is in general three at least.

Fig. 40.



In Fig. 40 let *KK'* be the section cutting the three bars in three points which may be considered as the points of application of three forces *PQR* due to the reaction of the bars, which balance the shearing and bending actions to which the section is subject. Resolving horizontally and vertically, and taking moments, we should—remembering that the load being wholly vertical the sum of the horizontal components must be zero—obtain three equations which would determine P, Q, R. It is, however, simpler to employ a method introduced by Ritter which enables us to obtain the value of each force at once. Let the lines of action of P, Q intersect in the point 1, Q and R in 2, P and R in 3, and let the perpendicular dropped from each intersection on to the line of action of the third force be r, p, q respectively: by measurement on the drawing of the framework structure we are considering it is always easy to determine these perpendiculars. Then taking moments about the three points we get

$$Rr = L_1; \ Pp = L_2; \ Qq = L_3,$$

where L_1, L_2, L_3, are the moments of the forces acting on the left-hand half of the structure about the points, 1, 2, 3 respectively. On page 59 it was shown how to get these moments graphically from the polygon of moments, but they also may be obtained by direct calculation.

We may write down a general formula for this method, thus—

$$Hh = L,$$

where H is the stress on any bar, h its perpendicular distance from the intersection of the two others cut by a section, and L is the moment of the forces about that intersection. The special case in which the intersection lies on the section considered so that the moment L becomes the bending moment (M) on the section, has already been considered in Chapter II. When the stress on a single bar is required as a verification of results obtained by graphical methods, or where the maximum stress due to a travelling load has to be determined, this method is often serviceable, but as a general method it is inconvenient from the amount of arithmetical labour involved.

The shearing action on the section is resisted by the components parallel to the section of the stress on the several bars. In the case of the incomplete frame of Fig. 39, p. 70, these components are given at once by the diagram of forces. In general, however, three bars, and only three, must be cut by the section if the frame be neither incomplete nor redundant; when two of these are perpendicular to the section the case is that considered in Chap. III. of a framework girder with booms parallel, in which the diagonal bars alone resist the shearing. When one bar only is perpendicular to the section, the other two collectively resist the shearing action: this case is common in bowstring and other girders of variable depth. The upper boom together with the web here resists the shearing.

When more than three bars are cut by the section, the stress in each is generally indeterminate on account of the number of bars being redundant. On this question it will be sufficient for the present to refer to Chapter II., Section II.

EXAMPLES.

1. In example 3, page 65, construct the polygon of bending moments by Culmann's method.

2. In example 6, page 32, find the stress on each part of the roof by Ritter's method.

3. In example 7, page 32, find the stress on each by Ritter's method.

4. If a parabolic bowstring girder be subject to a uniform travelling load, represented by the application of equal weights to some or all of the verticals, show that the horizontal component of the maximum stress on each diagonal is the same for all.

REFERENCES.

For further information on subjects connected with the present chapter the reader may refer to

Naval Architecture. W. H. WHITE. Murray.
Graphical Statics. LIEUT. CLARKE, R.E. Spon.
Graphical Determination of Forces in Engineering Structures. J. B. CHALMERS. Macmillan.
Graphical Statics. H. T. EDDY. Van Nostrand.

CHAPTER IV.

FRAMEWORK IN GENERAL.

37. *Straining Actions on the Bars of a Frame: General Method of Reduction.*—When the bars of a frame are not straight, or when they carry loads at some intermediate points, the straining action on them is not generally a simple thrust or pull, but includes a shearing and bending action. The present and two following articles will be devoted to some cases of this kind.

First suppose the bars straight, but let one or more be loaded in any way, and in the first instance consider any one bar *AB* (Fig. 41) apart from the rest of the frame, and suspended by strings in an inclined position. Let any weights act on it as shown in the figure, then the tensions of the vertical strings will be just the same as in a beam, *AB*, supported horizontally at the ends and loaded at the same points with the same weights. Resolve the forces into two sets, one along the bar, the other transverse to the bar. The second set produce shearing and bending just as if applied to a beam in a horizontal position, while the first set produce a longitudinal stress, which will be different in each division of the bar. Let θ be the inclination of the bar to the vertical, then the pulls on the successive divisions are

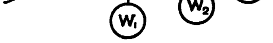


Fig.41.

$$P\,.\cos\theta : (P - W_3)\cos\theta : (P - W_3 - W_2)\cos\theta : \ldots .$$

the last being a thrust equal to $Q\,.\cos\theta$, so that the stress varies from $Q\,.\,6$ to $-P\,.\cos\theta$. Now observe that we can apply to *AB* at its ends, in the direction of its length, a thrust, H_0, of any magnitude we please without altering P and Q, but that we cannot apply a force in any other direction, whence it follows that when AB forms one of the bars of a frame, its reaction on the joint A must be a downward force, P, and a force, H_0, which must have the direction BA, while the reaction on B in like manner consists of a downward force, Q, and an equal force, H_0, in the direction AB. The downward forces P, Q, are described as the part of the load on AB carried at the joints A, B, and it is now clear that if these quantities be estimated for each bar and added to the load directly suspended there, we must be able to determine the forces H_0 by exactly the same process as that by which we find the stress on each bar of a frame loaded at the joints. The actual thrust on AB evidently varies between $H_0 - P \,.\, \cos\theta$ at the top, to $H_0 + Q \,.\, \cos\theta$ at the bottom, so that H_0 may be described as the mean thrust on the bar, while the shearing and bending depend solely on the load on the bar itself, and not on the nature of the framework structure of which it forms part, or on the load on that structure. In the particular case where the load on the bar is uniformly distributed, the forces P, Q, are each half the weight of the bar, and the thrust H_0 is the actual thrust at the *middle point* of the bar.

This question may also be treated by the graphical method of Art. 35 with great advantage. Through A and B draw a funicular polygon corresponding to the load on AB, the line OV in the diagram of forces will be parallel to AB and may be taken to represent H_0. This funicular polygon will be the curve of bending moment for the bar, and the other straining actions at every point are immediately deducible. It will be seen presently that the bar need not be straight.

For simplicity it has been supposed that the forces acting on the bar are parallel: if they be not, the reduction is not quite so simple. It will then be necessary to resolve the forces into components along the bar and transverse to the bar, the second set can be treated as above, while the total amount of the first set must be considered as part of the force applied to the joints either at A or B. Such cases, however, do not often occur, and it is therefore unnecessary to dwell on them.

The joints have been supposed simple pin joints or their equivalents, but the method used for frames loaded at the joints will apply even if the real or ideal centres of rotation of the bars are not coincident, provided only the centre lines prolonged pass through the point where the load is applied. The method of reduction just explained then requires modification. Such cases are of frequent occurrence, and the next article will be devoted to them.

jecting from the piers, technically described as "cantilevers." The joints here are at N and M. In structures of great span, in which the weight of the structure is the principal element, so that the variations in distribution are small, this type of girder is economical in weight. In a bridge over the Forth now in process of execution (1883), the central portion for each of two principal openings consists of a simple girder 350 feet span, while the cantilevers are each no less than 675 feet in length, making a total span of 1700 feet. These cantilevers are of great depth near the piers, and to provide against wind pressure, they are there likewise greatly increased in breadth, and solidly united to them. For a description of this design, which, from its gigantic dimensions and other unusual features, deserves attentive study, the reader is referred to *Engineering* for September, 1882.

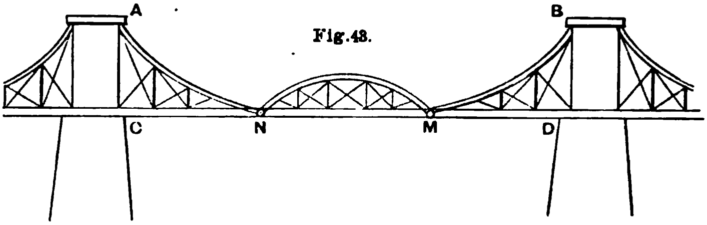


Fig.43.

39. *Hinged Arches.*—In the second section of Chapter I. certain forms of arches were considered which are simply inverted chains, and require for equilibrium a load of a certain definite intensity at each point. We shall now take the case of an arched rib capable of sustaining a load distributed in any way. We shall suppose the load vertical, and, to take the thrust of the arch, we shall imagine a tie rod introduced so as to convert it into a bowstring girder. If the straining actions at each point of the rib are to be determinate without reference to the relative flexibility of the several parts of the rib, and other circumstances, we must have, as in the case of the continuous beam, joints in some given position. The necessary joints are in this instance

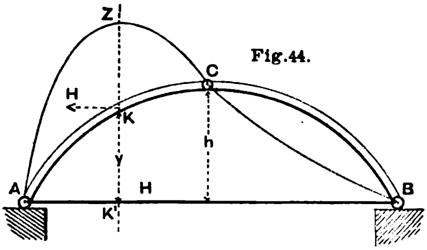


Fig.44.

three in number, and, we shall suppose, are at the crown C (Fig. 44), and one at each springing A and B.

Taking a vertical section KK' through the rib and tie, let the bending moment due to the vertical load and supporting forces be M. This bending moment is resisted, *first*, by the horizontal forces called into play; that is to say, the pull of the tie rod H at K', and the equal and opposite horizontal thrust of the rib at K; *secondly*, by the resistance to bending of the rib itself, the moment of which we will call μ. Hence if y be the ordinate of the point considered, we must have

$$M = Hy + \mu.$$

To determine H we have only to notice that at the crown where $y = h$ there is a joint, that is, $\mu = 0$,

$$\therefore \ M_0 = Hh,$$

where M_0 is the bending moment due to the load for the central section. Thus, to determine μ we have the equation

$$\mu = M_0 - M \cdot \frac{y}{h}.$$

The graphic representation of μ is very simple. Let us imagine the curve of moments drawn for the given vertical load, and let it be so drawn as to pass through A, B, and C, which is evidently always possible. Then, if Y be the ordinate of the curve,

$$M = H \cdot Y.$$

Therefore, by substitution,

$$\mu = H(Y - y).$$

So that the bending moment at each point of the rib is represented graphically by the vertical intercept between the rib and the curve of moments. In the figure, the curve $AZCB$ is the curve of moments, and KZ is the intercept in question.

Arched ribs in practice are rarely, if ever, hinged, and the straining actions on them occasioned by a distribution of the load not corresponding to their form depend, therefore, upon the relative flexibility of the several parts of the rib, and other complicated circumstances. If the position of the virtual joints be known, or the bending moments at any three points, the graphical construction just given can be applied.

Instead of a rigid arch, from which a flexible platform is suspended, we may have a stiff platform suspended from a chain. This is the case where a suspension bridge is adapted to a variable load by means of a stiffening girder. For this case it will be sufficient to refer to Ex. 3, page 86.

40. *Structures of Uniform Strength.*—In any framework structure without redundant bars, the stress on each bar may be determined as in Chapter I., by drawing a diagram of forces for any given load, W, and expressed by the formula

$$H = kW,$$

where k is a co-efficient depending on the distribution of the load. If A be the sectional area of the bar we find by division the stress per sq. inch, which must not exceed a certain limit, depending on the nature of the material as explained in Part IV. of this work. When the structure is completely adapted to the load which it has to carry, the stress per sq. inch is the same for all the bars, and it is then said to be of Uniform Strength. Uniformity of strength cannot be reached exactly in practice, but it is a theoretical condition which is carried out as far as possible in the design of the structure. Other things being equal, the weight of a structure of uniform strength is less than that of any other. Such a structure is therefore less costly, for weight is to a great extent a measure of cost.

Whenever the load is known, the weight of a structure of given type and of uniform strength can be calculated thus. Suppose A the sectional area of a piece, H, the stress on it, f, a co-efficient of strength, then

$$H = fA.$$

Next let w_0 be the weight of a unit of volume, usually a cubic inch, and assume

$$\lambda = \frac{f}{w_0},$$

then λ is a certain length, being in fact the length of a bar of the material which will just carry its own weight. Its value for various materials is given in Chapter XVIII. Then, assuming the piece prismatic and of length s, its weight is

$$w_0 As = \frac{Hs}{\lambda},$$

and therefore the weight of the whole structure must be for the same value of λ,

$$W_0 = \frac{\Sigma Hs}{\lambda},$$

the summation extending to all the pieces in the structure, and being performed by integration in a continuous arch or chain. It will be observed that s is the length of any line in the frame-diagram, and H that of the corresponding line in the diagram of forces; we have only then to take the sum of the products of these lines and divide by λ, the result will be the weight of the structure. It is, however, generally necessary to find the weights, W_1, W_2, of the parts in compression and in tension separately, because the value of λ is generally different in the two cases.

A remarkable connection was shown by the late Prof. Clerk Maxwell to exist between W_1 and W_2. Let us take a structure of the girder class and suppose the total load upon it G, and the height of the centre of gravity of that load above the points of support h. Imagine this structure to become gradually smaller without altering either its proportions or the magnitude and distribution of the load G, then G descends and does work during the descent in overcoming the resistance (T) of the bars in compression to diminution of length, while at the same time the bars in tension (P) do work during contraction. The values of T and P do not alter, for the diagram of forces remains the same, and therefore if we conceive the process to continue till the structure has shrunk to a point,

$$Gh = \Sigma Ts - \Sigma Ps = \lambda_1 W_1 - \lambda_2 W_2.$$

In particular, if the centre of gravity of the load lies on the line of support, and if the co-efficients be the same, the weights of the parts in compression and tension will be equal. A correspondimg formula may be obtained for structures of the arch-class by taking into account the thrust.

The weight of an actual structure is always greater than that found by this method. First, an addition must be made to allow for joints and fastenings. Thus, for example, in ordinary pin joints the eye of the bar weighs more than the corresponding fraction of the length of the bar, and in addition there is the weight of the pin. Secondly, in all structures there is more or less redundant material necessary to provide against accidental strains not comprehended in the useful load. Thirdly, there are local straining actions in the pieces occasioned by their own weight and other causes.

41. *Stress due to the Weight of a Structure.*—The total load on any structure consists partly of external forces applied to it at various points, and partly of its own weight: the total stress on any member is therefore the sum of that due to the external load and of that due to the weight of the structure itself. As that stress cannot exceed a certain limit, depending on the strength of the material, it necessarily follows that the stress due to the weight is so much deducted from the strength. Thus the consideration of the weight of a structure is an essential part of the subject, even if we disregard the question of cost.

The weight of each member is, of course, distributed over its whole length, and so also may be a part or the whole of the external load. Applying the general method of reduction explained in Art. 37, we suppose an equivalent load applied at each joint, and drawing a diagram of forces, we determine the mean stress, H, on the member. If the unsupported length of the bars be not too great, a matter to be considered presently, this stress will be the principal part of the straining action on the bar, and the bending may be neglected as in the preceding article.

Now, consider two structures similar in form and loaded with the same total weight, distributed in the same way, so that the only difference in the structures is in size: then the stress on corresponding bars must be the same, for the structures have the same diagram of forces. That is to say, in the formula

$$H = kW,$$

the co-efficient k depends on the type of structure and the distribution of the load upon it, but not on its dimensions. Dividing by the sectional area, the intensity of the stress is

$$p = k\frac{W}{A}.$$

Next let W_0 be the weight of the structure itself, and suppose the relative sectional areas of the several pieces the same, then

$$W_0 = w_0 . cAl,$$

where c is a co-efficient depending on the type of structure, and l a length depending on the linear dimensions of the structure. For example, in roofs and bridges l may conveniently be taken as the span. Then if k_0 be the value of k, which corresponds to the distribution of the weight of the structure, which will be the same whether the structure be large or small,

$$p_0 = k_0 . \frac{W}{A} = w_0 k_0 cl,$$

will be the stress due to the weight of the structure. In other words, the stress due to the weight of similar structures varies as their linear dimensions.

Since p_0 cannot exceed f, it follows at once that there must be a limit to the size of each particular type of structure, beyond which it will not carry its own weight. If L be that limit given by

$$L = \frac{\lambda}{k_0 c},$$

the stress due to the weight of any similar structure of smaller dimensions will be simply

$$p_0 = f . \frac{l}{L},$$

and

$$f' = f - p_0 = f . \frac{L - l}{L}$$

is the strength which may be allowed in calculations made irrespectively of weight. If the structure be of uniform strength throughout under its own weight, the value of p_0 will be the same for each member, but this is not necessarily the case, and there may be a different value of f' for each member. The actual limiting dimensions of the structure will, of course, be the least of the various values corresponding to the various members.

The conclusion here arrived at is obviously of the greatest importance for it immediately follows that in designing a roof, bridge, or other structure of great size, the weight of the structure is the principal thing to be considered in estimating the straining actions upon it, while a certain limiting span can never be exceeded. On the other hand, in small structures the straining actions due to the weight are unimportant; it is the magnitude and variations of the external load which have the greatest influence. This remark also applies to the local straining actions which produce bending in the pieces, their relative importance increases with the size of the structure, and it is necessary to provide against them by additional trussing. A large structure is therefore generally of more complex construction than a small one, as is illustrated by the various types of roof-trusses considered in Chapter I.

The difference of type of large structures and small ones, as well as the circumstances mentioned at the close of the last article, render tentative processes generally necessary in calculations respecting weight. If the type of structure and the distribution of the total load, W, be supposed known, the value of the co-efficients k and c will be known for some given member. By assuming the stress on that member equal to the co-efficient of strength f, we find

$$W_0 = W . ck . \frac{l}{\lambda}$$

a formula which gives the weight of the structure in terms of the load, but the co-efficients will generally vary according to the span. Among the circumstances on which they depend, the ratio of the vertical to the horizontal dimensions of the structure is most important. For a given span k diminishes when the depth is increased, while on the other hand c generally increases, so that for a certain ratio of depth to span, the weight of the structure is least. In ideal cases c may remain the same (Ex. 10, p. 87), but in actual structures the redundant weight of material necessary to give stiffness and lateral stability increases, so that the most economical ratio of depth to span is generally much less than would be found by neglecting such considerations. These points are illustrated by examples at the end of this Chapter and Chapter XII., where the question is again considered briefly; but for detailed applications to actual structures the reader is referred to works on bridges, in the design of which it is of the greatest importance.

42. *Straining Actions on a Loaded Structure in General.*—The results obtained in the last chapter for the case of parallel forces acting on a structure possessing a plane of symmetry in which the forces lie, may be readily extended to structures which have an axis of symmetry acted on by any forces passing through that axis and perpendicular to it. This

is the case, for example, of a beam acted on by a vertical load, and also by some horizontal forces arising say from the thrust of a roof or from wind pressure. We have then only to consider the vertical and horizontal forces separately. Each will produce shearing and bending in its own plane, which may be represented by polygons as before. The total straining action will be simply shearing and bending, and will be as before independent of the particular structure on which the forces operate. The magnitude of the straining action, whether shearing or bending, will be the square root of the sum of the squares of its components, and may therefore be readily found by construction and exhibited graphically by curves. In shafts such cases are common, and some examples will be given hereafter.

Another entirely different kind of straining action sometimes occurs in structures proper (roofs, bridges, etc.), and in machines is one of the principal things to be considered. Imagine a structure of any kind to be divided by an ideal plane section into parts A and B, and to be acted on by forces parallel to that plane. Let the forces acting on A reduce to a couple the axis of which is perpendicular to the section, the forces on B are equal and opposite, and the two equal and opposite couples tend to cause A and B to rotate relatively to each other. As already stated in Art. 16, this effect is called Twisting, and the magnitude of the twisting action is measured by the magnitude of either of the couples which form its elements.

Simple twisting sometimes occurs in practice, for example, when a capstan is rotated by equal forces applied to all the bars, but it is generally combined with shearing and bending. It is then necessary to know about what axis the twisting moment should be reckoned, which will depend on the nature of the structure. In shafts and other cases to be considered hereafter the geometrical axis is an axis of symmetry which at once determines this.

When twisting exists the shearing and bending are determined by the same method as before, for they are independent of the axis of reference. Should, however, the structure be subject to a thrust or a pull (Art 16), the axis about which the bending moment should be reckoned must be known, for it will depend on the nature of the structure.

These general observations will be illustrated hereafter, and are only introduced here to show how far straining actions can be regarded as depending solely on the external forces operating on the structure without reference to any other circumstances.

43. *Framework with Redundant Parts.*—In a complete frame, without redundant bars (pp. 11, 50), suppose a link applied to any two bars, one end attached to each. Let the link be provided with a right and left handed screw or other means of altering its length at pleasure, then by screwing up the link a pull may be produced in the link of any magnitude we please, while a corresponding stress will be produced in each bar of the frame which will bear a given ratio to the pull. Such a link may be called a straining link, and by its addition we obtain a frame with one redundant bar. The stress-ratio on the parts of a frame of this kind is completely definite, but the magnitude of the stress may be anything we please. Instead of one straining link we may have any number, and if the stress on each of these links be given, the same thing will be true. Thus it appears that a frame with redundant parts may be in a state of stress even though no external forces act upon it. This is of practical importance on account of the effect of changes of temperature. If all the bars of a frame with redundant parts are equally heated or cooled, the frame expands or contracts as a whole, but no other effect is produced; any inequality, however, causes a stress which may, under certain circumstances, be very great. This (at least theoretically) is one of the reasons why redundant parts are a source of weakness. The necessity of providing against expansion and contraction is well known in large structures resting on supports. The ground connecting the supports suffers little change of temperature, and the structure, therefore, cannot be attached to the supports, but must be enabled to move horizontally by the intervention of rollers. The magnitude of the stress produced when changes of length are forcibly prevented will be considered hereafter (Chapter XII.).

There is no essential difference between a frame the stress on the parts of which is due to the action of straining links, and a frame acted on by external forces; for every force arises from the mutual action between two bodies, and may therefore be represented by a straining link connecting the bodies. Even gravity may be regarded as a number of such links connecting each particle of the heavy body with the earth. Accordingly, if we include in the structure we are considering, the supports and solid ground on which it rests, we may regard it as a frame under no external forces, but including a number of straining links screwed up to a given stress. If the original frame be incomplete, its parts will be capable of motion, and it becomes a machine, as will be explained in Part III. of this work.

44. *Concluding Remarks.*—Various other questions relating to framework remain to be considered, especially with reference to the joints by which the parts are connected, but these, involving other than purely

statical considerations, do not come within the present division of our work, but are referred to at a later period.

EXAMPLES.

1. In Ex. 4, page 10, if the weight be supposed uniformly distributed, find the thrust, shearing force, and bending moment at each point of each rafter, and exhibit the results graphically by drawing curves.

Diagrams of shearing force will be sloping lines crossing each rafter at the centre.

Max. shearing for short rafter = 91 lbs.
,, ,, long ,, = 158·5 ,,

Diagrams of bending moment will be parabolas.

Max. moment at centre of short rafter = 117 ft.-lbs.
,, ,, long ,, = 290 ,,

2. A triangular frame *ABC*, supported at *A* and *C*, with *AC* horizontal, is constructed of uniform bars weighing 10 lbs. per foot, the length being—$AB=3$ feet, $BC=4$ feet, and $AC=5$ feet. Suppose, further, that *AB* and *BC* each carry 50 lbs. in the centre. Draw curves of thrust, shearing force, and bending moment for each bar.

3. The platform of a suspension bridge is stiffened by girders hinged at the centre and at the piers. The chains hang in a parabola, and the weight of the platform, chains and suspending rods may be regarded as uniformly distributed. Find the bending moment at any point of the stiffening girder, and exhibit it graphically by a curve when a single load *W* is placed (1) at the centre of the bridge, and (2) at quarter span.

First case. On account of *W* each half of the girder will tend to turn downwards about the ends, and will be supported by the uniform upward pull of the suspending rods. ∴ total upward pull for each $\frac{1}{2}$ girder $= W$, because the centre of action is at $\frac{1}{4}$ span. Thus each $\frac{1}{2}$ girder will be in the state of a beam loaded uniformly with *W*, and supported at the ends. Max. moment at middle of each half

$$= \tfrac{1}{8} W \times \text{half span.}$$

Second case. The upward pull of the suspending rods will still be uniform, but for each half girder will now be only $\frac{1}{2}$ *W*, found by assuming an equal action and reaction at the centre joint, and taking moments of each half about the ends. For the half girder which carries the weight the bending moment will be the difference between that due to *W* concentrated in the centre and $\frac{1}{2}$ *W* distributed uniformly.

$$\therefore \quad \text{Max.} = \tfrac{3}{16} W \times \text{half span.}$$

On the other half it will be due simply to a distributed load of $\frac{1}{2}W$. Max. $= \frac{1}{16} W \times$ half span.

4. A timber beam 24 feet span is trussed by a pair of struts 8′ apart, resting on iron tension rods forming a simple queen truss 3′ deep without a diagonal brace. The beam is loaded with 5 tons placed immediately over one of the vertical struts. Find the shearing force and bending moment at any point of the beam, supposing it jointed at the centre, and the centre only.

The thrust on each strut must be $2\frac{1}{2}$ tons; therefore, curves of shearing force and bending moment for each half of the beam are the same as those for a beam 12 feet long loaded at a point 4 feet from one end with $2\frac{1}{2}$ tons.

The problem should also be treated by the method of sections. Results should also be obtained for the case where one half the beam is uniformly loaded.

5. A beam uniformly loaded is fixed horizontally at the two ends, and jointed at two given points. Draw the diagrams of shearing force and bending moment. Show that the beam will be strongest when the distance of each point from centre is rather less than $\frac{3}{8}$ span.

6. The platform of a bowstring bridge of span $2a$ is suspended from parabolic arched

ribs hinged at crown and springing. One half the platform only is loaded uniformly with w lbs. per foot run. Show that the greatest bending moment on the ribs is $\frac{1}{16}wa^2$.

7. In the last question, if a weight of W tons travel over the bridge, how great will be the maximum bending moment produced?

8. A girder is continuous over three equal spans, and is hinged at points in the centre span midway between centre and piers. Find the virtual joints in the end spans when uniformly loaded throughout.

9. The weight of the chains, platform and suspension rods of a suspension bridge may be treated as a uniform load per foot-run, which at the centre of the bridge is double the weight of the chain. The dip of the chain is $\frac{1}{13}$th the span. The weight of iron being 480 lbs. per cubic foot, and the safe load per square inch of sectional area of chain being 5 tons, find the limiting span, and deduce the sectional area of chains for a load of $\frac{1}{2}$ ton per foot-run on a similar bridge 300 feet span.

If A = sectional area of chains at centre in sq. ins., then $\frac{20}{3} A$ = weight of bridge per foot-run in lbs.

$$\text{Horizontal tension} = \frac{65}{6} AL = 5 \times 2240 \,.\, A.$$

$$\therefore L = 1034 \text{ feet.}$$

If A' = area of one chain of the bridge 300 feet span,

$$\text{Whole load on chain} = \left(\frac{20}{3}A' + \frac{2240}{4}\right) 300,$$

$$\text{Horizontal tension} = \frac{1}{8}\left(\frac{20}{3}A' + \frac{2240}{4}\right) 300 \times 13 = 5 \times 2240A',$$

$$\therefore A' = 34{\cdot}4 \text{ sq. in each chain.}$$

Remark.—By the use of steel wire ropes and by lightening the platform and other parts of the structure as much as possible, the limiting span of suspension bridges is much increased, there being several examples of a span of 1250 feet and upwards.

10. In a girder with booms parallel and of uniform transverse section the weight of the web is equal to the weight of the booms. Assuming a co-efficient of strength of 9000 lbs. per sq. inch, and the weight of a cubic inch $\frac{5}{18}$th of a lb., show that the limiting span in feet is

$$L = 5400N,$$

where N is the ratio of depth to span.

11. The weight of a rib of parabolic form, span l, rise nl, with transverse section varying for uniform strength under a uniformly distributed load W, is

$$W_0 = \left(\frac{1}{8n} + \frac{2}{3}n\right) W \frac{l}{\lambda}.$$

This is least when $n \quad \frac{\sqrt{3}}{4} = \cdot 433$, then $W_0 = \cdot 577\, W \frac{l}{\lambda}$.

The formula fails if W_0 be nearly equal to W, for the external load would then have to be partly acting upwards to secure uniform distribution of the total load.

PART II.

KINEMATICS OF MACHINES.

PART II.—KINEMATICS OF MACHINES.

45. *Introductory Remarks.*—The object of a machine is to enable the forces of nature to do work of various kinds. In this operation some given resistance is overcome, which is accompanied by a given motion, while the driving force is accompanied by some other given motion, often at a distant place. Hence a machine may be regarded as an instrument for converting and transmitting motion. When considered under this aspect it is called a Mechanism, or sometimes a Movement, a Motion, or a Gear, the first being the scientific term, and the others occurring in practical applications.

Every mechanism consists of a set of pieces possessing one degree of freedom, that is to say, they are so connected together that when one changes its position all the rest do so too in a way precisely defined by the nature of the mechanism. Thus, for example, when the piston of a steam engine moves through any fraction of a stroke, the connecting rod, crank shaft, and the parts of any machine which it may be driving all shift their position in such a way that the connection between the various changes is completely determinate, and can be studied without reference to the work which the engine is doing, or the speed at which it is running. This branch of study is called the Kinematics of Machines.

The changes of position may be of any magnitude we please, and if they are very small are proportional to the velocities of the moving parts, hence a part of the subject, and generally an important part, is the consideration of the comparative velocities, or, as they are usually called the velocity-ratios, of the moving parts. Further, since the comparative velocities are fixed by the nature of the machine, the same must be true of the rates of change of these velocities, that is to say, the accelerations. Hence the general question is to study completely the comparative motions of the several parts of a machine, so that when the position, velocity, and acceleration of any piece are given, those quantities may be known for every other piece. It is the positions and velocities which are chiefly considered.

The converse problem is to discover the mechanisms by which any required motion may be obtained, and for this purpose the connection which exists between different mechanisms is considered. The subject therefore forms an introduction to the science of Descriptive Mechanism in which existing machines in all their vast variety are classified and studied systematically.

AUTHORITIES.

The principal treatises on the theory of mechanism are—

WILLIS. *Principles of Mechanism.* Longman.
RANKINE. *Millwork and Machinery,* Griffin.
REULEAUX. *Kinematics of Machinery.* Macmillan.

The modern form of the theory is due to Professor Reuleaux, whose nomenclature and methods are followed with some modifications, in the present work. The treatise referred to is a translation from the German by Professor A. B. Kennedy.

CHAPTER V.

LOWER PAIRING.

46. *Definition of Lower Pairs.*—Each piece of a machine is in direct connection with at least one other, and constitutes with it what is called a PAIR, of which the two pieces are said to be the Elements. The whole machine may be regarded as made up of pairs, and the nature of the mechanism depends on the nature of the pairs of which it is constructed.

In the present chapter we consider exclusively mechanism composed of pairs of rigid elements which are in contact with each other, not merely at certain points or along certain lines, but throughout the whole or part of the area of certain surfaces. Such pairs are of peculiar importance from the simplicity of the relative movement of their elements, from their resistance to wear when transmitting heavy pressures, and from their tightness under steam and water pressure. They are called Lower Pairs, and in many cases this kind of pairing is alone admissible.

In order that two rigid surfaces may be capable of moving over each other while continuing to fit, they must either be cylindrical, including under that head all surfaces generated by the motion of a straight line parallel to itself, or surfaces of revolution, or screw surfaces. In the first case the relative motion of the elements is one of translation along the line, in the second of rotation about the axis of revolution, in the third the motion of translation and rotation are combined in a fixed proportion. Hence there are three kinds of lower pairs, known as Sliding Pairs, Turning Pairs, and Screw Pairs. In each case one of the surfaces is hollow, and wholly or partly encloses the other which is solid, and the motion depends on the surfaces only, and not on the other parts of the elements which assume very various forms, according to the purpose of the mechanism. Either element may be fixed and the other move, or both elements may move in any way whatever, the relative motion is still of the same kind.

As an example of a sliding pair may be taken a piston and cylinder, in which either the cylinder may be fixed and the piston move, or the piston be fixed and the cylinder move, as in some steam hammers, or both cylinder and piston move, as in the oscillating engine. The relative motion is always a simple translation. Velocities of translation are most conveniently measured in feet per 1″ or feet per 1′, but miles per hour and knots are also used, as to which it is convenient to remember that one mile per hour is 88 feet per 1′, and one knot, that is, one nautical mile per hour, approximately 101 feet per 1′.

As examples of turning pairs may be taken a cart and its wheel, a shaft and its bearing, or a connecting rod and crank pin. The relative motion here is one of simple rotation, which may be measured by the number of revolutions (n) per unit of time, or by the speed of periphery (V) of a circle of given radius (r), or by the angle (A) turned through per unit of time. The first two modes of measurement are common in practice, the third is used for scientific purposes only. When employed the angle is always expressed in circular measure, and the three methods are therefore connected by the equations

$$V = Ar = 2\pi nr.$$

When angular velocity is used as a measure of speed of rotation, the unit of time is always 1″, but the minute and hour are common in other cases.

A screw pair consists of a screw and its nut, and the relative motion consists of a motion of translation along the axis of the screw combined with a rotation about that axis. The motion of translation is often called the "speed of the screw," and is equal to np, where p is the pitch, that is to say, the space traversed in one revolution, and n the revolutions in the unit of time. Strictly speaking, the two first lower pairs are limiting cases of the screw pair: in the turning pair the pitch is zero, and in the sliding pair infinite.

In all three cases the motion of either element relatively to the other is identically the same, and the rate of that motion may properly be called the Velocity of the Pair, whether the movement considered be translation or rotation. When the velocity of a sliding pair and a turning pair are compared, rotation may be measured by the speed of periphery of a cicle of given diameter; it is the velocity with which bearing surfaces of that diameter would rub each other. The radius of this circle may be called the "radius of reference." The velocity of a screw pair may be measured by the rate either of its translation or its rotation.

In these three simple pairs the motion of one element relatively to the other is completely defined, each point describing a definite curve. Such a pair is called a "complete" or "closed" pair, but we may have pairs in which the motion is not defined unless further constraint be applied, and the pair is then said to be "incomplete." An incomplete pair cannot be used in mechanism without employing such constraint, and this process is called "closing" the pair. A pair may be incomplete, because there is nothing to prevent the disunion of its elements, as, for example, a shaft and its bearing when the cap is removed, but it also may be incomplete in itself. Lower pairing is sometimes, though not very frequently, incomplete in this latter sense; there are three possible cases, first, when the surfaces are spherical, as in a ball and socket joint; second, where a rod fits into a hole, and is free to move endways as well as rotate; third, where a block fits in between parallel plane surfaces. The methods of producing closure will be considered hereafter.

It may be here remarked, in anticipation of what will be said hereafter, that cases of lower pairing may be imagined in which the elements are not in contact over an area but along a line. For example, a rod may fit into a square hole. It is the simplicity of the relative motion which is the essential characteristic.

The motion of the elements of a pair may be prevented by a pin key or other fastening removable at pleasure: the pair is then said to be "locked." In capstans and windlasses, provided with ratchet wheel and pawls, we have examples in which a pair is locked in one direction only.

47. *Definition of a Kinematic Chain.*—It has been already said that a machine consists of a number of parts so connected together as to be capable of moving relatively to one another in a way completely defined by the nature of the machine. Each part forms an element of two consecutive pairs, and serves to connect the pairs so that the whole mechanism may be described as a chain, of which the parts form the links. Such a series of connected pieces is called a Kinematic Chain.

The motion of any piece may be considered either relatively to one of the pieces with which it pairs, or with reference to any other piece which we may choose to regard as fixed. In the first case the rate of movement has already been defined as the Velocity of the Pair. In the second, the fixed piece is usually the frame of the machine, which unites the rest of the pieces, and is commonly attached to the earth or some structure of large size, such as a vessel. For pieces which pair with the frame the velocity of the pair is the same as the velocity of the moving element, and this element alone need be mentioned. In some common practical cases the speed of an element means the speed of one of the pairs of which it forms part. For example, the speed of piston of an oscillating engine would be understood to mean its velocity relatively to the cylinder, in other words, the speed of the "cylinder-piston pair."

In the present chapter we consider exclusively chains of closed lower pairs, so that the motion of the pairs is a simple translation, rotation, or screw motion. The motion of some of the pieces relatively to the frame may be much more complex, but this is a subject for subsequent investigation; it is the motion of the pairs alone we now consider. We shall first direct our attention to the very common and important piece of mechanism employed in direct-acting steam engines. An example is shown in Fig. 1, Plate I., p. 108, which represents a direct-acting engine of the vertical inverted cylinder type which is common in marine engines and often occurs in other cases.

Let us consider the pairs of which this mechanism is constructed. We have, *first*, a cylinder, to which are rigidly attached guides for the crosshead, and bearings for carrying the crank shaft. The cylinder-guide bars and crank-shaft bearings all form one part rigidly connected together, and must be considered as being one piece or link of the kinematic chain. It may conveniently be called the frame. *Secondly*, there is a piston, which fits and slides in the cylinder. To the piston a rod and crosshead are rigidly attached, forming practically one piece. Not only is the piston guided in the cylinder, but the crosshead also between the guide bars, and the piston rod in the stuffing box; but yet, since there are practically two pieces only which move relatively to one another, we must look on the cylinder, stuffing box, and guide bars as altogether forming the hollow element of a sliding pair, and the piston, rod, and crosshead as together forming the solid element of the pair. *Thirdly*, there is a connecting rod which is attached by a gudgeon or crosshead pin to the piston-rod head. These two parts will together compose a turning pair. At the other end the connecting rod embraces the crank pin, forming a second turning pair with it. The crank pin is one of the elements contained in the *fourth* piece of the mechanism. This piece consists of the crank pin, crank arms, and shaft with its journals. The journals turn in the bearings of the fixed frame of the machine, the first link mentioned, and so form a third turning pair. Thus the chain is complete. It consists of four links forming one sliding pair and three successive turning pairs.

The same mechanism, in a different form, is shown in Fig. 2 of the same plate which represents the air-pump of a marine engine worked, as is not unusual, by a large eccentric keyed on the crank shaft. The crank-pin is here enlarged so as to become an eccentric; and to save room, the piston rod is replaced by a trunk within which the eccentric rod vibrates. We have, however, exactly the same pairs arranged in the same way, and the difference between the mechanisms is therefore merely constructive, the motions of the parts being identical.

48. *Mechanism of Direct-Acting Engine—Position of Piston.*—This is such an important piece of mechanism that we will examine its motion somewhat fully.

First as to the relative positions of the crank in its revolution and the piston in its stroke. The position of the piston in its stroke will compare exactly with the position of the crosshead, so instead of introducing the length of the piston rod into the diagram, we may just as well determine the relative positions of the point D (Fig. 46) in its straight-line path, and P in its circular path.

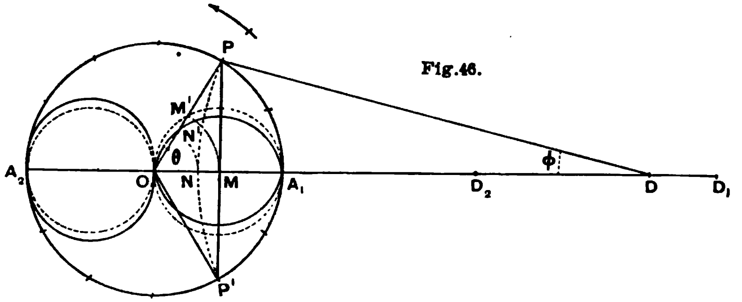


Fig.46.

Suppose the line of stroke to pass through the centre of the crank-pin circle. Let $OP =$ length of the crank arm, and PD the length of the connecting-rod. When the crank arm is in the line of stroke, away from the piston, the piston will be in one extreme position, and when the crank is in the line of stroke towards D, the piston will be in its other extreme position. The points $A_1\ A_2$ on the crank-pin circle are called the dead points. If we take distances $A_1D_1\ A_2D_2 = PD$, the length of the connecting-rod, the points $D_1\ D_2$ represent the ends of the stroke of the piston. If now we place the crank in any position OP we obtain the corresponding position of the piston by cutting the line of stroke with a circular arc of radius $= PD$ and with centre P. $DD_1\ DD_2$ will be the distances of the piston from the ends of its stroke. Since $A_1A_2 = D_1D_2$, the length of the stroke, it will be convenient to find the point A_1A_2 which corresponds to the position of the piston in its stroke. This may be readily done by striking a circular arc PN with centre D. N will be the point for $A_1D_1 = PD = ND$, therefore $A_1N = D_1D$, and the point N is the same distance from A_1 and A_2 as the piston is from the ends of its stroke.

We may just as easily solve the converse problem of finding the position of crank corresponding to any given position of the piston in its stroke. Let D be any position, cut the crank-pin circle by a circular arc of which D is the centre and DP the radius, then OP or OP' will be the corresponding position of the crank. Let the direction A_1PA_2 be the ahead direction of the crank, and let us call the motion D_1D_2 towards the crank the forward stroke, and D_2D_1 the back or return stroke of the piston, then when the piston is at D in the forward stroke the crank will be at OP, and again when the piston is at D in the return stroke the crank will be at OP'. Drop a perpendicular PM on to the line of stroke. Then the longer the connecting-rod the smaller NM will be, and the more nearly the circular arc PN will coincide with the perpendicular PM. Hence in the limiting case of an indefinitely long connecting-rod, M will be the position of the piston corresponding to the position OP of the crank. M being the position, neglecting the effect of the obliquity of the connecting-rod, and N the true position, MN is what we may call the error, or deviation due to obliquity.

In general the slide valve is worked by an eccentric, the radius of which is set at a particular angle on the shaft, so that the cut-off takes place when the crank occupies a certain angular position in its revolution, and it consequently follows that the fraction of stroke completed before cut-off takes place will be affected by the obliquity of the connecting-rod, so that in the ordinary setting of the slide valve the rates of cut-off will be different in the two strokes. This is well illustrated by Ex. 4, page 102.

We may obtain a convenient approximate expression for MN, the error due to obliquity. Referring to Fig. 46.

$$NM = DN - DM = DN(1 - \cos\phi).$$

Now the length of the connecting-rod may be conveniently expressed as a multiple of the length of the crank radius a or stroke s.

$$DN = na \text{ suppose} = \tfrac{1}{2}ns.$$

$$\therefore \quad NM = n\frac{s}{2}(1 - \cos\phi) = ns\sin^2\frac{\phi}{2}.$$

In the triangle POD, the sides being proportional to the sides of the opposite angles,

$$\text{Sin}\,\phi = \frac{OP}{DP}\sin\theta = \frac{1}{n}\sin\theta.$$

Now, the angle ϕ is in all practical cases a small angle, so we may write approximately

$$2\sin\frac{\phi}{2} = \sin\phi.$$

$$\therefore \quad NM = n\,.\,s\,.\,\frac{\sin^2\theta}{4n^2} = \frac{s}{4n}\sin^2\theta.$$

This is greatest when $\theta = 90$. $NM_{\text{max.}} = \frac{s}{4n}$.

If the connecting-rod is four times the crank, the greatest error due to obliquity $= \frac{1}{16}$ stroke.

We see that, in the forward stroke, the effect of the obliquity of the connecting-rod is to put the piston in advance of the position due to an indefinitely long connecting-rod, and, in the return stroke when the piston moves from the crank, the piston will be behind that position.

The relative positions of piston and crank may be very conveniently represented by a curve in this way. Divide the crank-pin circle (see Fig. 46) into a number of equal parts, and supposing the crank-pin at the points of division P, find the corresponding positions of the piston N. If then we take along the crank arm a distance ON' equal to ON, the distance of the piston from the centre of its stroke, and do this for a number of positions, we shall find the points N' will lie on a double-looped closed curve, shown in full lines in the figure. This may be called a curve of position of the piston. If we had supposed the connecting-rod to be indefinitely long, and had taken a distance OM' along $OP = OM$, the curve of position in such a case would have been a pair of circles, dotted in the figure, on OA_2 and OA_1 as diameters. The true curves of position will deviate from these circles more the shorter the connecting-rod. For the half stroke nearer the crank the curve will lie outside the dotted circle, and for the further half stroke inside. In Zeuner's valve diagram the obliquity of the eccentric rod is neglected, and the circles employed to show the position of the slide valve.

49. *Velocity of Piston.*—We will now pass on to the question of the relative velocity of the piston and crank-pin.

We will suppose the crank to turn uniformly at so many revolutions in the unit of time. If $n =$ number of revolutions and $a =$ length of crank-arm, $s =$ stroke.

$$\text{Velocity of crank-pin } V_0 = 2\pi a n = n\pi s.$$

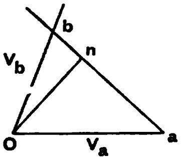
Now, as the crank-pin moves with uniform velocity, the piston undergoes continual changes of velocity, from being zero at the ends to a maximum at about the centre of the stroke. What is commonly spoken of as the speed of piston is the mean speed. If in the unit of time a complete number of revolutions are performed at a uniform rate, the mean speed will be the actual distance traversed by the piston in the unit of time. In each revolution the piston will complete a double stroke, so that speed of piston $= \overline{V} = 2ns$. This may be compared with the speed of crank-pin V_0,

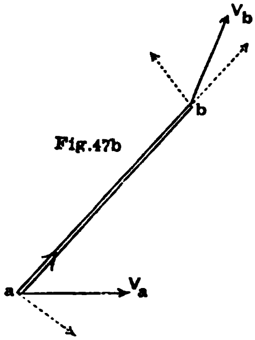
$$\frac{V_0}{\overline{V}} = \frac{\pi n s}{2ns} = \frac{\pi}{2}.$$

Next, as to the actual velocity of the piston at any point of its stroke. The piston and crank-pin are joined together by a connecting-rod of invariable length; one end of this rod has the velocity of the piston and the other that of the crank-pin. In Fig. 47*b* let *ab* be a rod, the ends of which move with velocities V_a, V_b in given directions. If one of these velocities be given, the other can be determined. For in Fig.

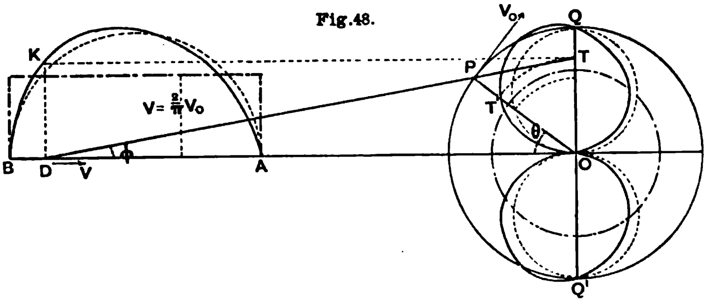
47*a* draw *Oa* parallel and equal to V_a and *Ob* parallel to V_b to meet a line *ab* which is perpendicular to the line *ab* of the first figure; then, if

Fig.47a.





we drop a perpendicular *On* on *ab*, this will be parallel to *ab* of the first figure, and must represent the resolved part of the velocity V_a along the rod. But the velocities of *a* and *b* resolved along the rod must be equal, because the length *ab* of the rod is invariable; hence *On* also represents the resolved part of V_b along the rod, and consequently *Ob* must represent that velocity in magnitude as well as in direction. The figure *Oab* is called the Diagram of Velocities of the rod, and from it we can find the velocity of any point we please either in, or rigidly connected with, the rod. We shall return to the properties of this diagram frequently hereafter: it will be sufficient now to remark that the triangle *Oab* determines the velocity-ratio of the two ends. In drawing the triangle it is generally convenient to turn it through 90°, so that the lines *ab* in the two figures become parallel, while the sides *Oa*, *Ob* become perpendicular to the velocities they represent.



In Fig. 48 *OP* is the crank arm, *PD* the connecting-rod; through *O* draw *OT* at right angles to the line of stroke to meet the connecting-rod produced in *T*, then *P* moves perpendicular to *OP*, and *D* to *OT*, therefore OPT is a triangle of velocities, so that if V be the velocity of the piston, V_0 that of the crank pin,

$$\frac{V}{V_0}=\frac{OT}{OP}.$$

This simple construction enables us very conveniently to draw a curve of piston velocity. In the first place, set off along OP a length $OT'=OT$, and do this for a number of positions of the crank. The points T' will be found to lie on a pair of closed curves, shown in full lines in the figure, passing through O and also through Q, Q', the upper and lower ends of the vertical diameter of the crank circle. Had the connecting-rod been indefinitely long, the points T' would have been found to lie on a pair of circles, of which the diameters are OQ and OQ', shown in dotted lines. On account of the obliquity of the connecting-rod, the curve of actual velocity lies outside the circle on the cylinder side of the crank, and inside the circle when the crank lies away from the cylinder.

When the crank is at right angles to the line of dead centres, the velocity of the piston is the same as that of the crank-pin, and neglecting the obliquity of the connecting-rod this will be the maximum velocity of the piston. If the obliquity is taken into account, the greatest velocity of piston occurs when the crank is inclined a little towards the cylinder, it is very approximately when the crank is at right angles to the connecting-rod, and the maximum velocity will a little exceed the velocity of the crank-pin.

The curve just described is a polar curve, the magnitude of the velocity being represented by the length of the *radii vectores* of the curve. But we may draw a curve of velocity in a different way, thus—from the end of the connecting-rod which represents the position of the piston when the crank is at OP, set up an ordinate $DK=OT$, and do the same thing for a number of positions of the piston, the curve of velocity AKB will be obtained; the ordinate of which will represent the velocity of the piston when at any point of stroke AB. The longer the connecting-rod the more nearly does the curve approximate to the dotted semicircle of which AB is the diameter. The effect of the obliquity is to make the true curve of velocity lie outside the semicircle in the first half of the stroke of the piston towards the crank, and inside for the second half of the stroke.

The mean velocity of the piston may be conveniently represented by an addition to the diagram, thus:—On the same scale that OP, the length of the arm, represents the velocity V_0 of the crank-pin, take a length to represent

$$\overline{V}=\frac{2}{\pi}V_0.$$

In the polar diagram draw a circle with O as centre and radius of this length. Where this circle cuts the polar curve of velocity the positions of the crank are given at which the actual speed of the piston is equal to its mean speed. In the second diagram of velocity, set up an ordinate to represent $\overline{V}$, and draw a line parallel to the line of stroke. It will cut the curve of piston velocity in two points.

An approximate expression for the velocity of the piston may be determined thus:

$$V = V_0 \frac{\sin OPT}{\sin OTP} = V_0 \frac{\sin(\theta + \phi)}{\cos \phi};$$

or expanding the numerator,

$$V = V_0 \left\{ \sin \theta + \cos \theta \tan \phi \right\}.$$

Since ϕ is in all practical cases a small angle, $\tan \phi$ may be written $= \sin \phi$ without sensible error.

$$\therefore V = V_0 \left\{ \sin \theta + \cos \theta \sin \phi \right\}.$$

$$\text{Now} \quad \frac{\sin \phi}{\sin \theta} = \frac{OP}{PD} = \frac{1}{n}.$$

$$\therefore V = V_0 \left\{ \sin \theta + \frac{1}{n} \sin \theta \cos \theta \right\}$$

$$= V_0 \left\{ \sin \theta + \frac{1}{2n} \sin 2\theta \right\}.$$

By differentiation with respect to the time t we obtain the acceleration of the piston. Let a be the length of the crank, then

$$\frac{dV}{dt} = \frac{dV_0}{dt} \left\{ \sin \theta + \frac{1}{2n} \sin 2\theta \right\} + \frac{V_0^2}{a} \left\{ \cos \theta + \frac{1}{n} \cos 2\theta \right\}.$$

If the length of the connecting-rod be infinite, and the crank turn uniformly, we obtain a simple harmonic motion, the deviation from which is therefore, approximately, assuming n large and dV_0/dt small,

$$\textit{Deviation} = \frac{V_0^2}{na} \cos 2\theta + \frac{dV_0}{dt} \sin \theta.$$

The graphical construction for the acceleration when the crank turns uniformly will be found in Ch. IX.

EXAMPLES.

1. The driving wheels of a locomotive are 6 feet in diameter, find the number of revolutions per minute and the angular velocity, when running at 50 miles per hour. If the stroke is 2 feet, find also the speed of piston.

Revolutions per minute, = 233½.
Angular velocity, = $24\frac{4}{9}$ per second.
Speed of piston, = 933·6 feet per minute.

2. The pitch of a screw is 24 feet, and revolutions 70 per minute. Find the speed in knots per hour. If the stroke is 4 feet, find also the speed of piston in feet per minute.

Speed of screw = 16·58 knots.
,, piston = 560 feet per minute.

3. The stroke of a piston is 4 feet, and the connecting-rod is 9 feet long. Find the position of the crank, when the piston has completed the first quarter of the forward and backward strokes respectively. Also find the position of the piston when the crank is upright.

Ans. The crank will make, with the line of dead centres, the angles 55° and 66°.

When the crank is upright the piston will be 2¾ inches from the middle of its stroke.

4. The valve gear is so arranged in the last question as to cut off the steam when the

crank is 45° from the dead points both in the forward and backward strokes. Find the point at which steam will be cut off in the two strokes. Also when the obliquity of the connecting-rod is neglected.

Ans. **Fraction of stroke at which steam is cut off is—**

·175 in forward stroke,
·118 in backward ,,
·146 neglecting obliquity.

5. Obtain the results of the two last questions for the case of an oscillating engine, 6 feet stroke, the distance from the centre of the trunnions to the centre of the shaft being 9 feet.

Ans. **Angles 68° and 51°: Cut off ·2 and ·115.**

6. In Ex. 3 construct both curves of piston velocity. If the revolutions be 70 per minute, find the absolute velocity of the piston in the positions given. Find also the maximum velocity of the piston.

Ans. **¼ stroke forward, velocity - 810 feet per 1′.**
¼ ,, back, ,, - 730 ,,
Maximum, ,, - 900 ,,

Find also the points in the stroke at which the actual speed of piston is equal to the mean speed.

Ans. **4¾ in. from commencement of forward stroke.**
6¾ in. ,, end ,, ,,

7. The travel of a slide valve is 6 in., outside lap 1 in. Find, in feet per second, the velocity with which the port commences to open when the revolutions are 70 per minute.

Ans. **Port commences to open when the valve is 1 in. from the centre of its stroke. Neglecting the obliquity of the eccentric rod, velocity of valve is then 1·72 feet per second.**

8. Show that the maximum velocity of the piston occurs when the crank is nearly at right angles to the connecting-rod, the difference being a small angle, the sine of which is $\frac{1}{n(n^2+2)}$ nearly, where n is the ratio of connecting-rod to crank.

50. *Mechanisms Derived from the Slider-Crank Chain.*—In the investigation just given it has been supposed, for simplicity, that the crank turns uniformly, but if this be not the case the curve constructed will show the ratio of the velocities of the piston and crank pin. In all cases it is the velocity-ratio of two parts, not the velocities themselves, which are determined by the nature of the mechanism. The velocities are of course reckoned relatively to the frame, but as both piston and crank pair with the frame, they are also the velocities of the piston-frame pair and the crank-frame pair (see p. 95), the crank being the radius of reference. The velocities of the other pairs will be determined presently, but in this mechanism are of less importance. We will now direct our

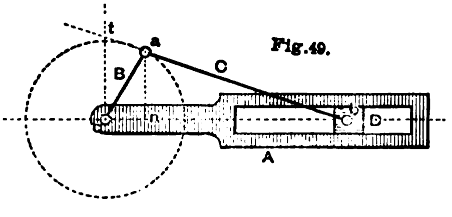


Fig. 49.

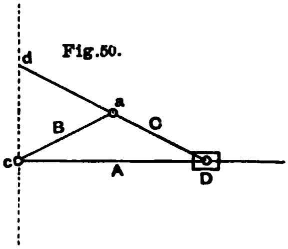
attention to other examples of the simple chain of lower pairs, of which the direct-acting engine is only a particular case. In Fig. 49, *D* is a block capable of sliding in the slot of the piece *A*. By means of a pin this block is connected with one end of the link *C*. *B* is a crank capable of rotating about a pin attached to the piece *A*, and united to *C* by another pin. Each of the four pieces of which this mechanism is composed, together with either of the adjacent pieces, constitutes a "pair," of which there are *four*, viz., three turning pairs, *AB*, *BC*, *CD*, and a sliding pair *DA*. This simple combination of pairs is known, in the modern theory of machines, as a Slider-Crank Chain.

Since the relative motions of the parts depend solely on the form of the bearing surfaces of the pairs and the position of their centres, not on the size and shape of the pieces in other respects, we may vary these at pleasure, and thus adapt the same chain to a variety of purposes. Especially we may interchange the hollow and solid elements of the pairs, a process which occurs constantly in kinematic analysis, and is called "inversion of the pair."

Again, any one of the four pieces may be fixed and the others move, so that we can obtain four distinct mechanisms from the same chain, simply by altering the link which we regard as fixed, a process called "inversion of the chain."

(1.) Let *A* be fixed, then we obtain the mechanism of the direct-acting engine already fully considered. In this, however, the connecting link *C* is much longer than the crank *B*; by supposing them equal we obtain a mechanism well known in various forms. In Fig. 50 *C* is prolonged beyond the crank pin *a* to a point *d*, such that $ad = ac$, a circle struck with centre *a* then passes through *c*, *d*, and the centre of the block, thus *cd* is at right angles to the line of stroke, so that *d*, when the crank turns, describes a straight line. This property renders the mechanism applicable as a parallel motion. It has also been used in air-compressing machinery. (See page 118.)

Fig. 50.



The various forms of the well known toggle joint, some of which will be given hereafter, are examples of the same mechanism with different proportions of *C* to *B*.

(2.) Instead of *A*, let us suppose *C* to be the fixed link, so that *A* and the other pieces have to take a corresponding motion. With this, by a change in the shape of the pieces, we are able to derive a mechanism well known in two forms. *C* being fixed, and *B* caused to rotate, *A* will have given to it an oscillating motion about the block *D*, and, at the same time, will slide to and fro on the block, the block itself having a vibrating motion about the other end of the piece *C*. Now, the relative movement of the parts of this mechanism is identical with that of the oscillating steam engine, and by a suitable alteration in the shape of the pieces, that mechanism may be derived. Thus, suppose, in the first place, the hollow element of *A* to become the solid one, in the shape of a piston-rod and piston, whilst the block *D* is enlarged into a cylinder to surround the piston, and so becomes the hollow element of the pair. The cylinder *D* will oscillate on trunnions, in bearings in the fixed piece *C*, which must be so constructed as to be a suitable frame for carrying the engine, and have bearings in which the crank shaft and crank *B* can turn.

The oscillating cylinder is in general mounted on bearings, the centre line of which coincides with the centre of the stroke of the piston, so that the distance apart of the shaft and trunnion bearings is equal to the length of the piston-rod. An example is shown in Fig. 4, Plate I.

Next let us consider the relative motions of the parts. Returning to Fig. 49 above, suppose *a*, *b*, *c*, to be the centres of the turning pairs, and draw *ct*, *an* perpendicular to the line of centres *bc*, to meet *C* and *A* in *t* and *n*, then it was shown above (page 100) that the velocity-ratio of the pairs *DA*, *BA* in the direct-acting mechanism was *ct*/*ac*, and as fixing a link makes no difference in the relative motions, this must also be the ratio of the speed of the piston of the oscillator in its cylinder, to the speed of the turning movement of the crank *relatively* to the piston-rod. Again when *C* is fixed, as in the oscillator, the link *A* (Fig. 49) slides on the block *D* with a velocity the direction of which is perpendicular to *an*, while the point *c* in it moves perpendicular to *ac*. Hence it follows that the triangle of velocities is *acn*, and therefore the velocity-ratio of piston and crank-pin is *an*/*ac*. The curve of piston velocity can be drawn as before; it differs little in form from that of the direct actor, but the maximum velocity of the piston is equal to that of the crank-pin, instead of being somewhat greater. Once more, remembering that fixing a link does not alter the relative motions, it appears that, in all cases, the velocity-ratio of the pairs *DA*, *BC* must be *an*/*ac*, so that we have determined the ratio of the speed of piston in the direct actor to the speed of the turning movement of the crank *relatively* to the connecting-rod.

Comparing our results, we see that the velocity-ratio of the turning pairs *BC*, *BA* must be *ct* : *an*, or what is the same thing *bt* : *ab*. Since the three angles of the triangle *abc* are always together equal to 180°, it is clear that the sum of the speeds of the three turning pairs must be zero, due regard being taken of the direction of rotation, and it follows, therefore, that in any slider-crank chain the speeds of the three turning pairs are as $at : ab : bt$. By the introduction of a suitable radius of reference, we may compare these velocities with that of the sliding pair. The most convenient radius to take is that of the crank, then assuming, as before, $ab = n . ac$, the velocities of the pairs are shown by the annexed table :—

Velocity Ratios in a Slider-Crank Chain.

Pair,	DA	BA	BC	DC
Velocity,	ct	ac	$\frac{bt}{n}$	$\frac{at}{n}$

In the oscillator the angular velocity-ratio of the cylinder and crank is the velocity-ratio of the pairs CD, CB and is therefore $at : bt$ or $cn : bc$. This can readily be constructed by drawing nz parallel to ac, then, since ab is constant, the angular velocity of the cylinder is proportional to az, as may be verified by an independent investigation. The result may be exhibited by a polar curve similar to the curve of piston velocity already drawn. In Fig. 51, c is the crank pin, $TST'S'$ the crank circle, az' is set off along the crank radius equal to az, then a curve with two unequal loops is obtained, which shows the law of vibration of the cylinder. The motion of the cylinder is such that, in the swing to the left, whilst the crank-pin moves along the arc $T'ST$, the angular velocity is much greater than in the return swing to the right, whilst the crank-pin moves along the arc $TS'T'$. Supposing the crank to revolve uniformly, the times occupied by the forward and return swings are as the arcs $T'ST$ and $TS'T'$, which are proportional to the angles subtended by them. By measuring or otherwise estimating these angles, the mean angular velocities in the forward and backward oscillation may be determined.

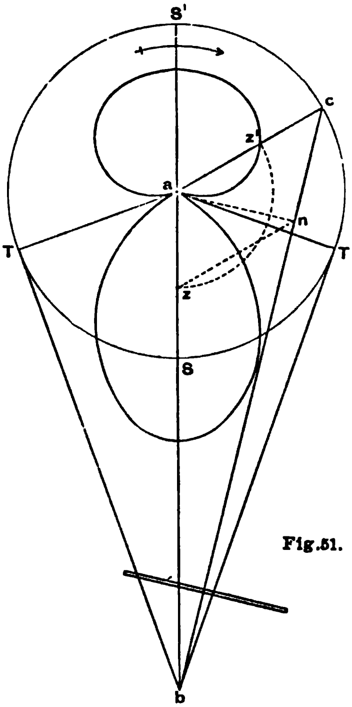
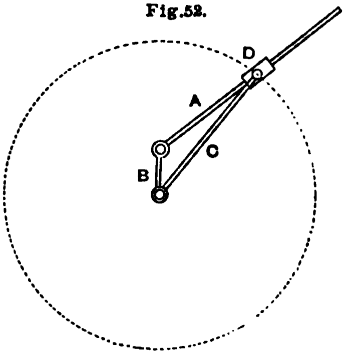


Fig. 51.

This peculiar vibration, rapid one way and comparatively slow the other, has been made use of to obtain a quick return motion of a cutting tool in a shaping machine. The velocity with which a tool will make a smooth cut in metal is limited, and since in general the tool is made to cut in one direction only, time is saved by causing the return stroke to be made more quickly. One construction of such a quick return motion may be thus described. A slotted lever *D* vibrates on a fixed centre in the frame piece *C*, its motion being derived from the revolution of a crank *B* on another fixed centre in the same frame piece *C*. The crank pin of *B* turns in the block *A*, which slides in the slotted lever *D*. There is in addition a connecting-rod, by means of which a to-and-fro motion of a headstock carrying the cutting tool is communicated from the oscillating lever, the headstock sliding in a guide.

Omitting the connecting-rod, we have the same kinematic chain, with the same fixed link *C*, as in the oscillating engine. There has been a change made only in the form of some of the pieces. What was the oscillating cylinder is now the slotted lever, and instead of a piston and rod, we have here the simple block *A* sliding in the slot. The crank *B* and frame-link *C* remain practically unaltered. The slotted lever will vibrate according to the same law which we have investigated for the oscillating cylinder, and thus with a uniform rotation of the crank, a quick return motion of the tool will be obtained. This mechanism is shown in Fig. 5, Plate I., in a form employed for given motion to the table of small planing machines.

Fig.52.



(3.) Let us next take an example in which *B* is the fixed link, and becomes the frame, its form being of course modified to suit the new conditions.

A crank arm *C* (Fig. 52) turns on a fixed centre in the frame piece *B*; so also does another arm *A* on a second fixed centre, *D* slides on *A*, being connected by a pin to the second end of *C*. Both *A* and *C* may make complete revolutions. If we suppose *C* to turn with uniform angular velocity, *A* will rotate with a very varying angular velocity, the movement of *A* in the upper part of its revolution being much more rapid than in the lower. This device has been employed by Whitworth to get a quick return motion of a cutting tool in a shaping machine. When separated from the rest of the machine, the construction may be thus described:—A spur wheel *C*, which derives its motion through a smaller wheel from the engine shafting, revolves on a fixed journal *B*, of large dimension. Standing from the face of the journal is a fixed pin placed out of the centre of the journal. On this fixed pin a slotted lever *A* rotates, in which a block *D* slides, a hole in the block receiving a pin which stands out from the face of the spur wheel. A second slot in *A*, on the other side of the pin, contains another block, which, by a screw, can be adjusted and secured at any required distance from the centre of rotation, so as to give any stroke at pleasure. This mechanism, omitting the adjustment by which the stroke is varied, is shown in Fig. 6, Plate I. The same mechanism in a somewhat different form is often employed in sewing machines to give a varying motion to the rotating hook.

(4.) The fourth possible mechanism which can be derived from the slider-crank chain is obtained by fixing the block *D*. This case is not so common as the three preceding, but in Stannah's pendulum pump, shown in Fig. 3, Plate I., we find an example. In a simple oscillating engine driving a crank shaft and fly-wheel, suppose the cylinder *D* fixed instead of the piece *C* which carries the cylinder and crank shaft. The crank and fly-wheel *B* has become the bob, and the link *C* the arm of the pendulum, from which the mechanism derives its name; *D* is a fixed cylinder, and *A* is a piston and rod. As the crank rotates the crank-pin moves up and down, while its centre vibrates in the arc of a circle.

(5.) The four mechanisms here described are all which can be obtained from the simple slider-crank chain, but an additional set may be derived by supposing that the line of stroke of the slider does not pass through the centre of the crank. A common example is found in the chain communicating motion from the piston to the beam in a beam engine. (See page 111.)

Although the mechanisms derived by inversion from a given kinematic chain may be described as distinct, it must be carefully observed that there is in reality no kinematic difference between them, the distinction consisting merely in a different link being chosen to reckon velocities from. If we consider the velocities of the pairs which constitute the chain, those velocities are always related to each other in the same way, and the same machine may be regarded sometimes as one mechanism and sometimes another. For example, suppose a direct-acting engine working on board ship; the ship may be imagined to roll so that the connecting-rod of the engine is at rest relatively to the earth, and the engine becomes an oscillator to an observer outside the ship. Dynamically and constructively, however, there is a great difference, for the fixed link is the frame, and is attached to the earth or other large body, the predominating mass of which controls the move-

Plate. I.

Fig. 1.

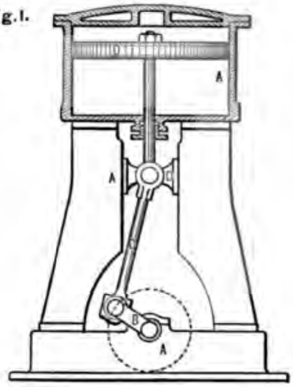


Fig. 2.

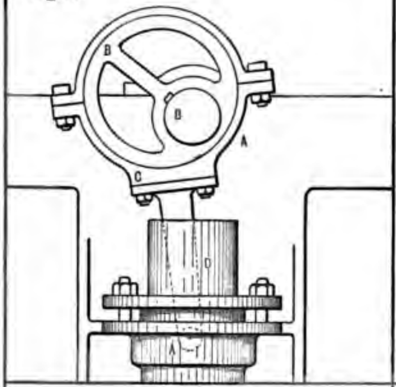


Fig. 3.

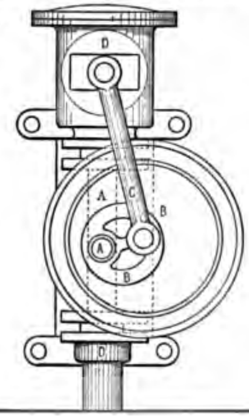


Fig. 4.

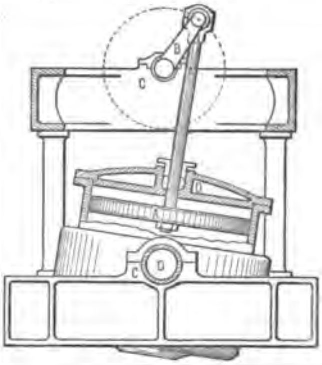


Fig. 5.

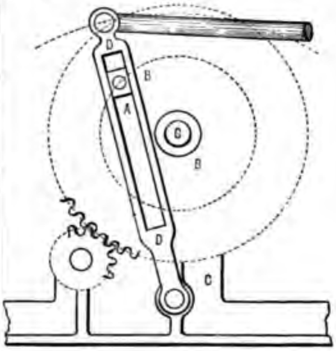
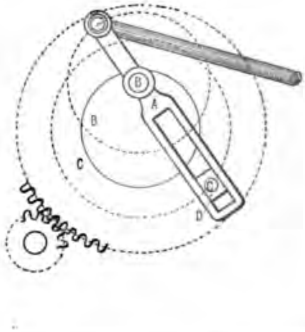


Fig. 6.



To face page 109.

ments of all bodies connected with it. To illustrate and explain the inversion of a slider-crank chain, Plate I. has been drawn. The six examples which have just been described are here placed side by side with the same letters *ABCD* attached to corresponding links, so that they may readily be recognized. It will be seen that each link assumes very various forms; thus, for example, the link *A* is the frame and cylinder in Figs. 1 and 2, a piston and rod in Figs. 3 and 4, a block in Fig. 5, and a rotating arm in Fig. 6. The relative motions of corresponding parts are, however, always the same.

51. *Double Slider-Crank Chains.*—We now pass on to the consideration of a kinematic chain consisting of two turning pairs and two sliding pairs. We will commence by showing how this chain may be derived from that previously described. Suppose the piece *D*, instead of being simply a block, is a sector shaped as shown in Fig. 1, Plate II., having a slot curved to the arc of a circle of centre *O*, while the piece *C*, which was before the connecting rod, is compressed into a block sliding in the curved slot. The law of relative motion of the parts of this mechanism will be precisely the same as in the direct-acting engine, for the block *C* will move just as if it were attached by a link, shown by the dotted line, to a point *O*, a fixed point in the piece *D*. The piece *D* will slide in *A*, just as if there were a connecting link from *C* to *O* and no sector—that is, it will slide just as the piston does in the cylinder of a direct-acting engine. Moreover, there are, in reality, exactly the same pairs in this as in the mechanism of the direct-acting engine, for *C* and *D* together make a turning pair, although only portions of the surfaces of the cylindric elements are employed.

This being so, let us now imagine the radius of the circular slot in the piece *D* to be indefinitely increased, so that the slot becomes straight, and is at right angles to the line of motion of *D*. In such a case the pair *CD* would be transformed into a sliding pair, and the mechanism would consist of two turning pairs, and two sliding pairs, and is known as a *double slider-crank chain.*

The most important example of this kinematic chain is that found in some small steam pumping engines. (Fig. 4, Plate II.) The pressure of the steam on the piston is transmitted directly to the pump plunger. The crank *B* and sliding block *C* serve only to define the stroke of the piston and plunger, and, by means of a fly-wheel, the shaft of which carries an eccentric for working the slide valve, to maintain a continual motion. The law of motion of piston and crank-pin may be readily seen to be the same as that in a direct-acting engine, in which the connecting-rod is indefinitely long. *P* being the position of the crank-pin, *M* will represent the position of the piston and reciprocating piece, and *PM* will represent the velocity of the piston at the instant, *OP* being taken to represent the uniform velocity of crank-pin. (See Fig. 48, p. 100.) In this case the polar curve of velocity would consist of a pair of circles. This motion, shown in dotted lines in Fig. 48, is called a simple *Harmonic motion*, because the law is the same as that of the vibration of a musical string.

By a change of the link which is fixed, we may now derive other well known mechanisms from this kinematic chain.

Instead of *A*, which forms part of a sliding and part of a turning pair, being fixed, let *B* be the fixed frame link. *B* contains the elements of two turning pairs, so that the frame must contain two bearings or journals. An example of such a mechanism is that known as Oldham's coupling, Fig. 5, Plate II., used for connecting parallel shafts, which are nearly but not quite in the same straight line, and which are required to turn with uniform angular velocity-ratio. Each shaft terminates in a disc, in the face of which a straight groove is cut. The two discs, *A* and *C* in the figure, with the grooves face each other, and are placed a little distance apart, with the grooves at right angles to each other. Filling up the space between them is placed a disc *D*, on the two faces of which are straight projections at right angles to one another, which fit into the grooves in the shaft discs. In the revolution of the shafts each of these projections slides in the groove in which it lies, and rotates with it. The two grooves are, therefore, maintained always at right angles to one another, and the two shafts rotate one exactly with the other.

Next, let the fixed link of the chain contain the elements of two sliding pairs, which would be obtained if we made *D* the frame piece. An interesting example of this is the instrument sometimes employed in drawing ellipses. (Fig. 2, Plate II.) Two blocks slide in a pair of right-angled grooves. By means of clamp screws a rod unites them at a constant distance from one another. Pins fitting in holes in the blocks allow the rod to rotate relatively to the blocks. Any point in the rod will describe an ellipse, as indicated in the figure.

If the link *C* be fixed, the resulting mechanism dees not differ from that derived by fixing *A*, and the three mechanisms just described are therefore all which can be obtained by inversion of a double-slider chain. In Figs. 2, 4, 5 of the plate referred to, they are shown side by side with the same letters attached to corresponding links, as in Plate I.

The directions of motion of the two sliding pairs have been supposed at right angles, but any other angle may be assumed, and mechanisms obtained which we need not stop to examine. A more important change

Plate.II.

FIG.1.

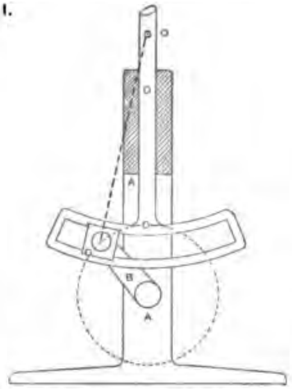


FIG.2.

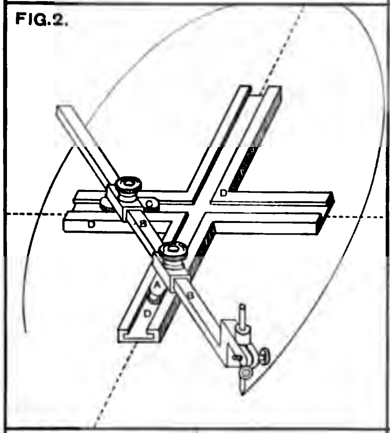


FIG.3.

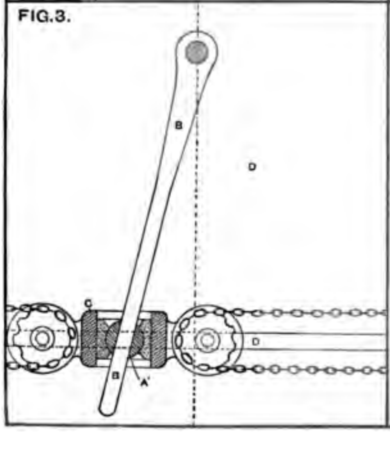


FIG.4.

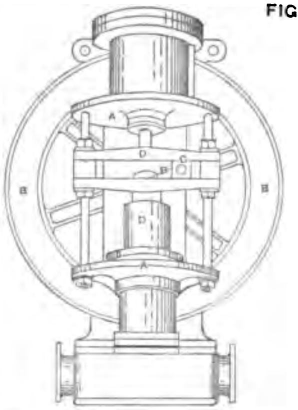


FIG.5.

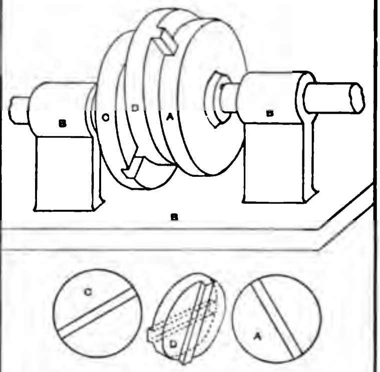
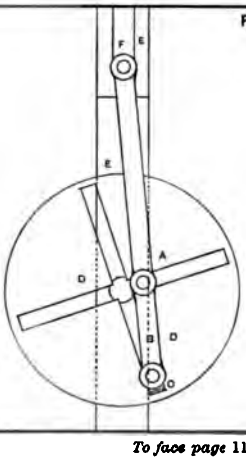


FIG.6.



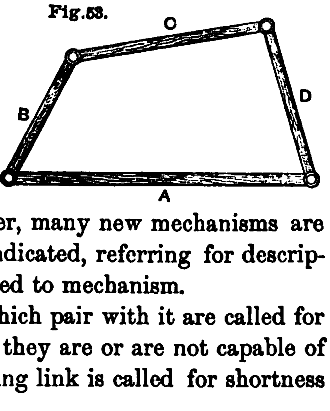
To face page 111.

is to suppose that the sliding pairs and turning pairs alternate, so that each link forms an element of one sliding and one turning pair. A mechanism known as "Rapson's Slide," employed as a steering gear in large ships, will furnish an example. Fig. 3, Plate II., shows one way in which it is applied. *A'* is an enlarged pin made in two pieces between which the tiller *B* slides while turning about an axis fixed in the ship *D*. *A'* is carried by the piece *C*, which slides in a groove fixed transversely to the ship being drawn to port or starboard by the tiller chains passing round pulleys mounted on *C*, as shown in the figure. The further the tiller is put over the slower it moves (Ex. 8, p. 120), and therefore the greater the turning moment (Ch. VIII.), a property of considerable practical value. In this kinematic chain the same mechanism is obtained whichever link is fixed.

The mechanism shown in Fig. 6 of this Plate is a compound chain, to be referred to hereafter.

52. *Crank Chains in General.*—Instead of having a chain of turning pairs connected by one or two sliding pairs, we may have turning pairs alone. The number will be four, and their axes must meet in a point or be parallel. Taking the second case, the chain in its most elementary form consists of 4 bars united by pin joints at their extremities, as in Fig. 53. It is called a crank or four-bar chain, and from it may be derived the slider crank chain already considered, in the same way as from that chain we derived the double slider chain. All the mechanisms hitherto considered may therefore be regarded as particular cases of it. In its present form, however, many new mechanisms are included, some of which will be briefly indicated, referring for descriptions and figures to works specially devoted to mechanism.

Fig. 53.



Assuming *A* the fixed link, *B* and *D* which pair with it are called for distinction cranks or levers, according as they are or are not capable of continuous rotation, while *C* the connecting link is called for shortness the *coupler*.

(1.) Let *B* be a crank and *D* a lever, then the mechanism is a "lever-crank," an example of which occurs in the common beam engine, *D* being the beam, *B* the crank, *C* the connecting-rod, and *A* the entablature, foundation, and all other parts connected therewith.

(2.) The links *B* and *D* may be equal, and *C* may be equal to *A*. This may be called "parallel cranks" when *B* and *D* are set parallel, as in the coupled outside cranks found in locomotives, or "anti-parallel cranks" when they are set crosswise, a case to be hereafter referred to. (Page 159.)

(3.) The links D and B may still both be cranks if C be greater than A, provided that the difference between B and D be not too great. The mechanism is called "double cranks," and occurs in the common drag-link coupling, and also in the mechanism of feathering paddles.

(4.) If the coupling link be too short, neither B nor D will be capable of a complete rotation. The mechanism is then a "double lever," and an example occurs in the common parallel motion to be considered hereafter.

(5.) A number of additional mechanisms may be derived by supposing the axes of the four turning pairs to meet in a point, instead of being parallel; we thus obtain a "conic crank chain." Hooke's joint is a particular case of this, but in general these mechanisms are of less importance.

53. *Diagram of Velocities in Linkwork.*—A simple construction has already been given, by means of which the velocity-ratios of the parts of a slider-crank chain are determined, and we will now consider this question for any case of linkwork in which the axes of the pairs are parallel.

Figure 54*a* represents a chain of links *zOabcd*... united by pins so as to form a succession of turning pairs. The first link *Oz* is fixed, so that the second turns about a fixed O point as centre, and therefore a moves perpendicularly to Oa, with a velocity V_a, which we may suppose known. The other points b, c, d... move in directions which we suppose given, and with these data it is required to find the magnitudes of the velocities. In figure 54*b*, from a pole O draw radiating lines perpendic-

Fig.54a.

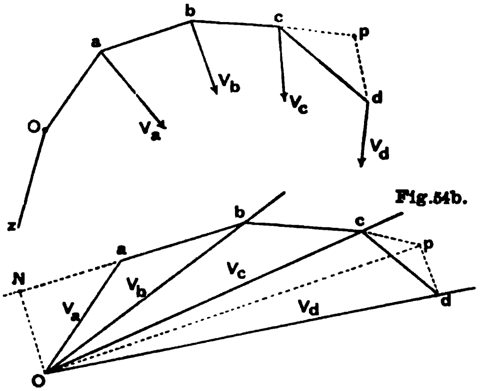


Fig.54b.

ular to the given directions, and set off on the first Oa to represent V_a, then draw ab, bc, cd... parallel to the links of the chain to meet the corresponding rays, then the lengths of those rays represent the velocities.

For drop a perpendicular ON from O on to ab, or ab produced, then ON represents the component of V_a in the direction of the second link, but this must also be the component of V_b in that direction, since ab is of invariable length; that is, Ob must represent V_b. Similarly all the other rays must represent the velocities of the corresponding points.

The figure thus drawn may be called the Diagram of Velocities of the chain. It may be constructed equally well, if the magnitudes of the velocities be given, instead of their directions, also any of the turning pairs may be changed into sliding pairs. If both ends of the chain be attached to fixed points, the diagram will evidently be a closed polygon. Its sides, when divided by the lengths of the corresponding links of the chain, represent their angular velocities, for each side is the algebraical difference of the velocities of the ends of the link perpendicular to the link.

In the four-link chain (Fig. 55*a*), consisting of two links turning about fixed centres a, d, coupled by a link bc, the diagram of velocities is a simple triangle Obc (Fig. 55*b*), the sides of which when divided by the lengths of the links to which they are parallel, represent the angular velocities of the links. Through a draw aZ parallel to cd, and prolong bc to meet it in Z, and the line of centres in T, then, since the triangle Zab is similar to the triangle of velocities, the angular velocities of the levers cd, ab will be proportional to Za/cd and ab/ab. The last fraction is unity, and therefore we have

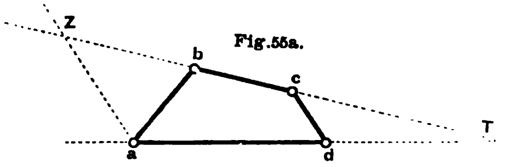


Fig.55a.

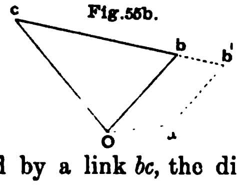


Fig.55b.

$$\text{Angular Velocity-Ratio} = \frac{Za}{cd} = \frac{aT}{dT},$$

showing that the ratio in question is the inverse of the ratio of the distance of T from the centres.

If, instead of the link ad being fixed, the chain of four bars be imagined to turn about one joint such as d, the diagram of velocities would be a quadrilateral $Oab'c$, with sides parallel to $abcd$.

Returning to the general case, let p be any point rigidly connected with one of the links of the chain, say cd, in the figure; then if we lay down on the diagram of velocities a point p, similarly situated with respect to the corresponding line cd of that diagram, it follows at once, by the same reasoning, that the ray Op, drawn from the pole O, must represent the velocity of p in the same way that the other rays represent the velocities of the points a, b... Thus it appears that for any linkwork mechanism, consisting of pieces of any size and shape connected by pin joints, the axes of which are parallel, a diagram may be constructed which will show the velocities of all points of the mechanism. By constructing the mechanism and its diagram of velocities for a number of different positions, curves of position and velocity may be drawn, such as those described in preceding articles for special cases.

54. *Screw Chains.*—We have hitherto considered only chains of turning pairs and sliding pairs, but screw pairs also occur in a great variety of mechanisms which we can only briefly indicate.

(1.) In the Differential Screw, there are two screw pairs with the same axes but of different pitch, combined with a sliding pair, forming a three-link chain. The connection between the common velocity of rotation of the screws and the velocity of translation of the sliding pair is the same as that between the rotation and translation of a screw, the pitch of which is the difference between the pitches of the actual screws. The arrangement has often been proposed for screw presses, a mechanical advantage being obtained, at least theoretically, with screws of coarse pitch, which would otherwise require a thread so fine as to be of insufficient strength. The right and left-handed screw is an example in common use.

(2.) In the Slide Rests of lathes and other machine tools, the traversing motion of planing machines, and many other cases we find a three-link chain, consisting of a screw pair, a turning pair, and a sliding pair. This may be regarded as a particular case of the preceding, the pitch of one of the screws being zero.

(3.) In presses, stearing gear, and many other kinds of machinery we find a simple screw chain employed to work a slider-crank chain. Some examples will be given hereafter.

55. *Parallel Motions Derived from Crank Chains.*—In beam engines the connecting-rod by which the reciprocating motion of the piston is communicated to the vibrating beam is necessarily short, in order to diminish the height of the machine, and therefore, if guides are employed to retain the end of the piston-rod in a straight line, there will be considerable lateral pressure which is difficult to provide against, and which involves a large amount of friction. The guides may then be replaced with advantage by some linkwork or other mechanism. Such a mechanism is called a Parallel Motion, and in the early days of engineering was employed more extensively than at the present time. In its most simple form it consists of two levers capable of turning about the fixed centres a and b. (Fig. 56.) The ends of the levers are connected by a coupling link pq, then, so long as the angular movement of the levers is not too great, there is a point in the link pq which will describe very approximately a straight line. In the first instance let us suppose the links so set that when ap_0 and bq_0 are parallel, p_0q_0 is at right angles to them. Let $apqb$ be the extreme downward movement of the levers, then p lying to the left and q to the right, there will be some point P in pq which in this extreme position lies in the straight line p_0q_0. In the upward extreme position the same point of pq will, approximately, also lie in this line. If, then, p_0q_0 be the line of stroke, and the point P be selected for the point of attachment of the piston-rod head, then this point will be exactly in the line at the middle and bottom of the stroke, and at other points will deviate but little from it.

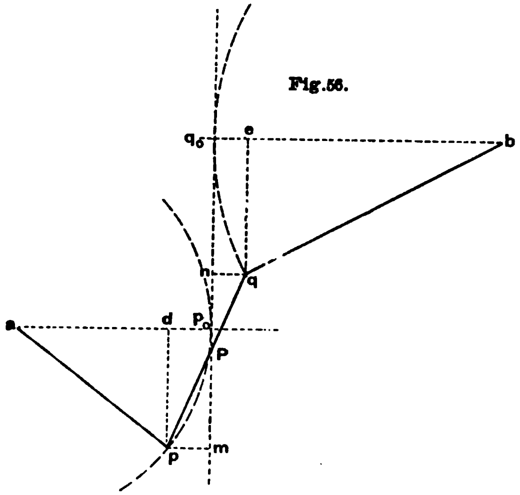


Fig. 56.

To find the point where pq intersects p_0q_0, we must first obtain expressions for the amount that the point p deviates to the left of p_0 and q to the right of q_0; these amounts being the versines of the arcs in which the points move, and shown by dp_0 and eq_0, where pd and qe are drawn perpendicular to ap_0 and bq_0. By supposing the circle of which a is the centre to be completed, it is easy to see that

$$(ad + ap_0)dp_0 = pd^2,$$

$$\therefore \quad dp_0 = \frac{pd^2}{ad + ap_0}.$$

If the angle p_0ap is not greater than 20°, we may write

$$dp_0 = \frac{pd^2}{2.ap_0},$$

the error not being greater than 1 per cent. Now, neglecting the small effect due to the obliquity of the connecting link when in the extreme positions, $pd = \frac{1}{2}$ stroke; therefore, supposing $ap = r_a$ and $bq = r_b$,

$$pm = dp_0 = \frac{(\text{stroke})^2}{8\,r_a},$$

$$qn = eq_0 = \frac{(\text{stroke})^2}{8\,r_b}.$$

Now P being the point where pq intersects p_0q_0, we have similar triangles in which

$$\frac{pP}{qP} = \frac{pm}{qn} \text{ and } \therefore \; = \frac{r_b}{r_a}.$$

Thus the point P, which has most correctly the straight-line motion, is such that it divides the coupling link into segments which are inversely proportional to the lengths of the levers. If the levers be placed into all possible positions, then in the motion the connecting link will be inverted and the point P will trace a closed curve resembling a figure of 8. There are two limited portions of this curve which deviate very little from a straight line.

We may approximate still more nearly to a straight line by a little alteration in the settling of the levers. Suppose the centres of vibration a, b, are brought a little nearer together so that the line of stroke bisects the two versines dp_0 and eq_0. Then when the levers are parallel, the link slopes to the left upwards, whereas at the ends of the stroke the link will slope to the right upwards. At two intermediate positions about quarter stroke from the ends, the link will be vertical. If we choose the point P as previously described, the maximum deviation will be only about one fifth of its former amount. In practice, the final adjustment of the centres of motion is performed by trial.

The use of parallel motions is almost exclusively confined to beam engines. In that case bq will be the half length of the beam of the engine, and in order that the angle through which the beam vibrates should not exceed 20° above and below the horizontal, the length of the beam should not be less than three times the stroke. The radius rod may be somewhat shorter than the half beam, but should not be less than the stroke, or the error in the motion of P will be too great. This mechanism will, therefore, occupy a considerable space. To economise space, and also to provide a second straight-line path to guide the air-pump rod, a modification of the mechanism is made use of.

In Fig. 57, be being the half length of beam, a point q is chosen so that

$$\frac{bq}{be}=\frac{\text{stroke of air pump}}{\text{stroke of piston}},$$

and a parallelogram of bars $qeQp$ provided, united by pins. The point p is jointed to the end of the radius rod ap vibrating on the fixed centre a. Consequently there will be some point P in the back link qp which will describe very nearly a straight line. This point is such that

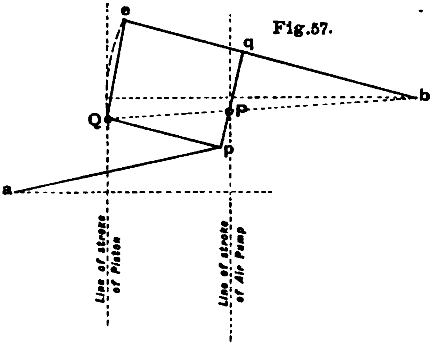


Fig.57.

$$\frac{Pp}{Pq}=\frac{bq}{ap}.$$

Now, if the proportions of the links are such that bPQ is a straight line, bQ/bP will be constant, and therefore the path described by Q will be an enlarged copy of the path described by P. That is to say if P moves approximately in a straight line, then Q will do so also. If then the radius rod is of suitable length we provide a point Q for the attachment of the piston-rod, and also a point P for the attachment of the air-pump rod. To find this length we have

$$\frac{bq}{pQ}=\frac{qP}{pP},$$

whence multiplying by the preceding equation

$$bq^2 = pQ \times ap,$$

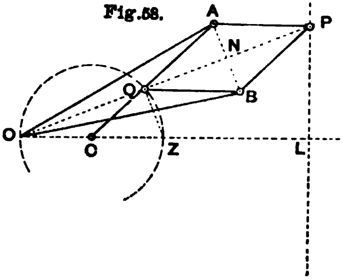
$$\text{or Length of radius rod } = \frac{(bq)^2}{eq}.$$

The parallel motion just described which was introduced by Watt is the only one much used in practice, but there is another form which possesses great theoretical interest because it is exact and yet involves only turning pairs. Scott Russell's parallel motion (Fig. 50, page 104) is exact, but as it involves a sliding pair its accuracy depends on the exactness with which the guides of the slides are constructed. Now, a straight edge or a plane surface can only be constructed by a process of copying from some given plane surface or by trial and error, whereas a circle can be described by a pair of compasses independently of the existence of any other circle. Hence an exact parallel motion, with turning pairs only, enables us theoretically to trace a straight line in the same way that a circle is traced with compasses. It has long been known that this could be done by a circle rolling within another twice its diameter, but this method does not satisfy the necessary conditions, and it was not till 1872 that Col. Peaucellier invented a linkwork mechanism for the purpose. This mechanism consists of two equal bars OA, OB, jointed to each other at O, and at A, B to a parallelogram of eqnal bars $AOBQ$, so that OQP are in a straight line (Fig. 58). This being so then, however the bars are placed there will always be some fixed relation existing between OQ and OP. Thus drop a perpendicular AN on OP, then $OQ = ON - QN$ and $OP = ON + NP$. Also since $AQ = AP$, $QN = NP$,

$$\therefore\ OQ \,.\, OP = ON^2 - QN^2.$$

But $ON^2 = OA^2 - AN^2$ and $QN^2 = QA^2 - AN^2$, therefore $OQ \,.\, OP = OA^2 - QA^2$, and is a constant quantity for all positions; that is to say, if we cause Q to move over any curve, then P will describe its reciprocal.

Fig. 58.



We can now show how this mechanism may be employed to draw a straight line. Let O be a fixed centre and PL be the straight line which it is required to describe. Draw the perpendicular OL on PL. Then the mechanism being placed in any position with P at any point on the line to be drawn, draw QZ at right angles to OQ. Bisect OZ in C and attach Q to C by means of a jointed rod which can turn on the fixed centre C. The circle which Q describes during the motion of the bars will have OZ as a diameter, for OQZ is a right angle, and therefore the angle in a semicircle. We observe now that we have similar triangles OQZ and OLP.

$$\therefore\ OL = \frac{OP.OQ}{OZ};$$

but $OZ = 2.OC$ is a constant quantity and so is the product OP, OQ.

$$\therefore OL \text{ is constant.}$$

That is to say, wherever P is, the length of the projection of OP on the perpendicular OL is a constant quantity. This can be true only so long as P lies in the perpendicular line PL. Thus, by the constrained motion of Q in a circle passing through O, P is caused to move perfectly in a straight line.

This mechanism has been applied to a small engine used for ventilating the House of Commons.

56. *Closure of Kinematic Chains. Dead Points in Linkwork.*—A kinematic chain, like a pair (p. 95), may be "incomplete," that is, the relative movements of its links may not be completely defined. It then cannot be used as a mechanism without employing some additional constraint, a process called "closing" the chain. In order that a chain may be closed it must be endless, and the number of links must not be too great; for example, in a simple chain of turning pairs with parallel axes we cannot have more than 4 links. If there be 5 the motion of any one link relatively to the rest will not be definite, but may be varied at pleasure.

So also a chain may be "locked" either by locking one of the pairs of which it is constructed; or by rigidly connecting two links not forming a pair; it then becomes a frame, such as was considered in a previous part of this book.

A chain is often incomplete or locked for special positions of its links, though closed and free to move in all other positions; this, for example, is the case at the dead points which occur in most linkwork mechanisms. A well known instance is that of the mechanism of the steam engine, in which the chain is locked and the direction of motion of the crank indeterminate when the connecting rod and crank are in the same straight line. This instance further shows that it is necessary to distinguish between the two directions in which motion may be transmitted through the mechanism, for the dead points in question would not occur if the crank moved the piston instead of the piston the crank. A piece, then, which transmits motion is called a "driver," in relation to the piece to which motion is transmitted, which is called a "follower," terms which will be frequently used hereafter in cases where both pair with the fixed link. (Compare Arts. 47, 86.) The dead points in a mechanism may be passed either by the union of two similar mechanisms, with dead points in different positions, as in a steam engine with a pair of cranks at right angles, or by aid of the inertia of the moving parts. This last method involves what is called "force-closure," a term which will be explained presently.

EXAMPLES.

1. The stroke of an oscillating engine is 6 feet, and the distance between the centre line of the trunnions and the centre of shaft is 9 feet. Find the maximum and mean angular velocity of the cylinder in each of its two oscillations as compared with that of the crank. Find also the velocity of the piston at half stroke as compared with the speed of piston.

Ans. Maximum angular velocity-ratios of cylinder and crank, $\frac{1}{2}$ and $\frac{1}{4}$.
Mean „ „ ·277 and ·178.

$$\frac{\text{Velocity of piston at } \frac{1}{2} \text{ stroke}}{\text{Mean speed of piston}} = \frac{1·54}{1}.$$

2. The travel of the tool of a shaping machine is to be 9 inches, and the maximum return, three times the maximum cutting, velocity. Neglecting the obliquity of the connecting link, find the proper proportions of the quick return motion (Fig. 5, Plate I.). Find also the revolutions per minute for a maximum cutting velocity of 6 inches per second, and compare the times of cutting and return.

Ans. The length of slotted lever = 9 inches.
Distance apart of centres = twice length of crank.
Revs. per minute of crank = 19·1.

$$\frac{\text{Time of cutting}}{\text{Time of return}} = \frac{2}{1}.$$

Remark.—The lever vibrates through 60°, an inconveniently large angle. Whitworth's mechanism would be preferable, as it readily gives an even greater inequality.

3. In Whitworth's quick return motion find the proportions that the maximum return may be three times the maximum cutting velocity, and compare the times of cutting and return.

4. In example 1, draw curves showing the angular velocity and position of the piston for any position of the crank.

5. In example 2, draw a diagram of velocities for any position of the mechanism, and hence trace curves showing the velocity of the tool at any point of the cutting and return stroke.

6. In question 1, p. 102, supposing two pairs of driving wheels coupled, the lengths of cranks 1 foot, find the velocity of the coupling-rod in any position. First, relatively to the locomotive; second, relatively to the earth.

7. In Ex. 6, p. 103, find in feet per second the maximum and minimum velocity of rubbing of the crank pin, assuming its diameter 12 in., and the revolutions 30 per 1′. Draw a curve showing this velocity in any position of the crank.

8. In Rapson's Slide (p. 111), if the tiller be put over through an angle θ, show that the velocity-ratio of tiller and slide varies as $\cos^2\theta$, and draw a curve of velocity.

9. In a drag-link-coupling the shafts are 6 in. apart, the drag-link 1 foot long, and the cranks each 3 feet long. By construction, determine the four positions of the following crank when the leading crank is on the line of centres, and at right angles to the line of centres.

10. The length of the beam of an engine is three times the stroke. Supposing the end of the beam when horizontal is vertically over the centre of the crank shaft at a height equal twice the stroke, and the crank also is then horizontal, find the length of connecting-rod and the extreme angles through which the beam will sway. Adjust the crank centre so that the beam may sway through 20° above and below the horizontal.

Length of rod = 2·06 stroke. The beam sways $22\frac{1}{2}$° above the horizontal, and 17° below.

11. The depth of the floats of a feathering paddle wheel is $\frac{1}{8}$th the diameter of the wheel, and the immersion of the upper edge in the lowest position $\frac{1}{4}$th the depth of the float. Assuming the stem levers $\frac{3}{5}$ths the depth of the floats, find the position of the centre of the collar to which the guide rods are attached. Determine the length of the rods, and draw the float in its highest position.

If O be centre of wheel, K centre of collar, $OK =$ ·054 of diameter of wheel, and is horizontal (very approximately).

Length of guide rods = 1·01 radius of wheel.

12. In Ex. 9, find the angular velocity-ratio of the shafts when the cranks are in the positions mentioned. Find also the maximum and minimum angular velocity-ratio.

13. In Oldham's coupling, show that the centre of the coupling disk revolves twice as fast as the shafts, and hence show how to give two strokes of a sliding piece for one revolution of a shaft.

14. In a simple parallel motion the lengths of the levers are 3 feet and 4 feet respectively, and the length of the connecting link is $2\frac{1}{2}$ feet. Find the point in the link which most nearly moves in a straight line, and trace the complete curve described by this point as the levers move into all possible positions, the motion being set so that, when the levers are horizontal, the link is vertical.

Ans. The required point in link is $17\frac{1}{7}$ in. from the 3-feet lever.
and $12\frac{6}{7}$ in. ,, 4-feet ,,

15. In a beam engine the stroke of piston is 8 feet, of air pump $4\frac{1}{2}$ feet, length of beam 24 feet, the front and back links of the parallel motion being 4 feet. Find the proper length of radius rod, and the point in the back link where the air-pump rod should be attached.

Ans. Length of radius rod = 8 feet $8\frac{1}{8}$ inches.
Point of attachment of air-pump rod = 2 ,, 3 ,, below beam.

16. Suppose in last question the parallel motion set for least deviation from a straight line, find the correct positions of the centre lines of air pump and piston, and the position of the centre of motion of the radius rod.

Ans. Horizontal distances from centre of beam—
Line of stroke of piston, - 11 feet 8 inches.
,, air pump, 6 ,, $6\frac{3}{4}$,,
Centre of motion of radius rods, 15 ,, $1\frac{1}{8}$,,

REFERENCE.

A good collection of linkwork and other mechanisms, some of which do not occur in the larger works cited on page 92, will be found in the 4th edition (1880) of Professor Goodeve's *Elements of Mechanism*. Much valuable information on the details of machine design is contained in a treatise on Machine Design by Professor W. C. Unwin, M.I.C.E. (Longman.)

CHAPTER VI.

CONNECTION OF TWO LOWER PAIRS BY HIGHER PAIRING.

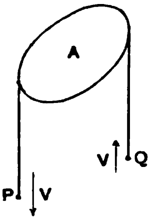
SECTION I.—TENSION AND PRESSURE ELEMENTS.

57. *Preliminary Remarks—Tension Elements.*—If one of the elements of a pair be not rigid, or if the contact be not of the simple kind considered in the preceding chapter, the pairing is said to be "higher," because the relative motion of the elements is more complex. Higher pairing is seldom employed alone; it is generally found in combination with lower pairs, the elements of which it serves to connect. The most important case is that where a chain of two lower pairs is completed by contact between their elements or by means of a link which is flexible or fluid. Motions may thus be produced in a simple way which are impossible or difficult to obtain by the use of lower pairing alone. The present chapter will be devoted to mechanisms derived from chains of this kind, the fixed link being generally a frame common to the two lower pairs. The velocities of each of the pairs are thus the same as those of their moving elements. We commence with the case of non-rigid elements.

A body which was incapable of resistance to any kind of change of form and size would of course be incapable of being used as part of a machine, for it could not furnish any constraining force whereby the motion of other pieces could be affected, but if it resists any particular kind of change it will supply a corresponding partial constraint which may be supplemented by other means. The first case we take is that of a flexible inextensible body, such as is furnished approximately by a rope, belt, or chain. This is called a Tension Element, being capable of resisting tension only, and it is plain that when any two points are connected by it, their distance apart, measured along the element itself, must be invariable so long as the rope remains tight. If the rope be straight, it may be replaced by a link, and we obtain the mechanisms already considered, but we now suppose it to pass over a surface of any form.

In Fig. 59*a*, let A be a fixed body of any shape, round which an inextensible rope PQ passes, the ends hanging down. If P moves

Fig. 59a.



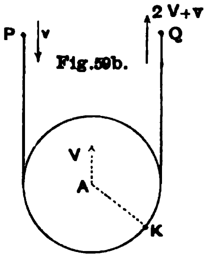
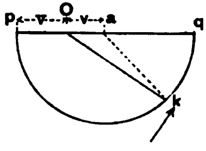


Fig. 59b.

Fig. 59c.

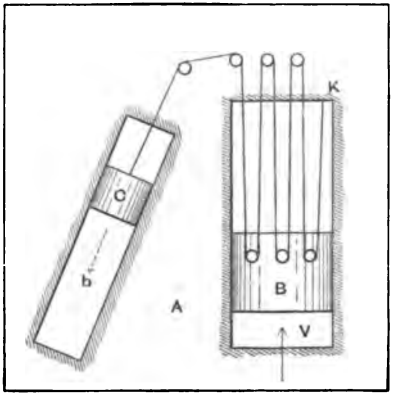


downwards with velocity V, Q moves upwards with the same velocity, the rope slipping over A at all points with velocity V. In practice A is generally circular, and is mounted on an axis, upon which it revolves. We have then a "pulley block," of which A is the "pulley" or "sheave," and the rope causes it to rotate instead of slipping over it, but this makes no difference in the motion, and the only object of the arrangement is to diminish friction and wear.

Next suppose the pulley moveable (Fig. 59*b*), and imagine P attached to a fixed point, while Q moves upwards with the same velocity V relatively to A as before. Then A must move upwards with velocity V, because its motion relatively to the fixed point P is unaltered, and hence Q moves with velocity $2V$. More generally, if P, instead of being fixed, moves downwards with velocity v, Q must move upwards with a velocity $2V+v$, or to express the same thing otherwise—*the difference of velocities of the two sides of the rope is twice the velocity of lifting*—a principle applicable to all questions relating to pulleys. The velocity of rotation of the pulley is $V+v$, its radius being the "radius of reference" (Art. 46). The motion of rope and pulley may be represented by a diagram of velocity. Thus, in Fig. 59*c*, describe a semi-circle with radius equal to $V+v$, then the radius of that circle represents the velocity of rotation or the velocity of any point in the rope relatively to the centre of the pulley. The actual velocity of any point K in the rope is found by compounding this with V, the velocity of the centre of the pulley. The pole of the diagram is therefore a point O, distant V from the centre of the circle, so that if k be the point in the diagram corresponding to the point K of the rope, Ok represents the velocity of K.

58. *Simple Pulley Chain—Blocks and Tackle.*—We have now a simple means of solving one of the most important problems in mechanism—namely, to connect two sliding pieces with a constant velocity-ratio.

Fig. 60a.



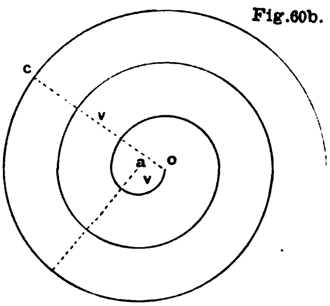
In Fig. 60*a*, *B*, *C* are pieces sliding in guides attached to a frame-piece *A*, thus forming two sliding pairs with one link common. In *B* a number of pins are fixed, and in *A* an equal number placed as in the figure, so that a rope passing round them as shown may form a number of plies parallel to *B*'s motion.*

The rope is attached at one end to *C*, and led to the nearest fixed pin, over a guide pin placed so that this part of the rope may be parallel to *C*'s motion, while the other end is attached to a fixed point *K*. The effect of this arrangement is that when *C* moves in the direction of the arrow, *B* also must move with a velocity which is readily found by the principle just explained, for the difference of velocities of the two parts of each ply must be the same, being twice the velocity of *B*. Thus reckoning from the fixed end, if *B*'s velocity be *V*, the velocities of the several parts of the rope must be

$$0,\ 2V,\ 2V,\ 4V,\ 4V,\ 6V,\ 6V\ldots\ldots,$$

so that if there are n pins in *B*, the velocity of the other end of the rope must be $2nV$, and the velocity-ratio $2n : 1$. The diagram of velocities consists of a number of semi-circles (Fig. 60*b*), the lower set struck with centre *a* and the upper with centre *O*, where *O* is the pole and *Oa* the velocity of lifting.

Fig. 60b.



The simple kinematic chain here described may be inverted, by fixing *B* or *C* instead of *A*. In the blocks and tackle so common in practice, the pins are replaced by moveable sheaves, usually, but not always, of equal diameters, and placed side by side so as to rotate on the same axis. Some of the various forms they assume will be illustrated hereafter. The diagram of velocities shows that, if the diameters of the sheaves are proportional to the diameters of the circles shown in the diagram, they will have the same angular velocity, and may therefore be united into one, an idea carried out in White's Pulleys.

---

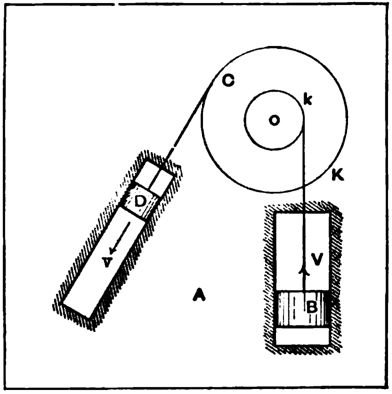
* This figure is taken, with some modifications, from the second edition (1870) of Willis's Mechanism.

In all cases the mechanism which we have been considering (Fig. 60*a*) is a closed kinematic chain only so long as the rope remains tight. One method of securing this would be to supply a second rope passing under another set of pins below *B* (not shown in the figure) and led to the other side of *C* by a suitably placed guiding pulley; we should then, by tightening up the ropes, have a self-closed chain similar to those considered in the preceding chapter. In practice, however, forces are applied to *B* and *C* which produce tension in the rope; thus, for example, when employed for hoisting purposes the weight which is being lifted keeps the rope tight. This is the simplest example of what is called force closure, where a kinematic chain, which is not in all respects closed, is made so by external forces applied during the action of the mechanism. In practical applications the principle of force-closure is carried still further, for the guides which compel the pieces *B* and *C* to move in straight lines are usually omitted. In the case of *B* the weight and inertia of the load which is being raised or lowered supply sufficiently the necessary closure, while in the case of *C* the end of the rope may be guided by the hand.

59. *Wheel and Axle.*—When mechanical power is employed for hoisting purposes, the end of a rope is frequently wound round an axle, the rotation of which raises or lowers the weight, and this leads us at once to a different and equally important method of employing tension elements—namely, by attaching one end to a fixed point in the cylindrical surface of an element of a turning pair. The rope in this case passes over the surface and is guided by it, but does not slip over it as it does over the pins of the previous arrangement. The most useful case is that where the transverse section of the surface is a circle, and the direction of the rope always at right angles to the axis of rotation; then it is clear that the motion of the surface is the same as the motion of the rope.

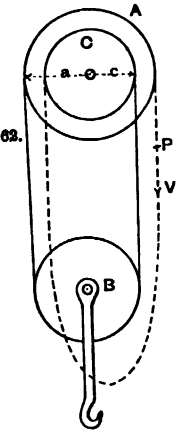
The well known Wheel and Axle is a combination of two chains of this kind. In its complete ideal form it consists of two sliding pairs *AB*, *AD*, with planes parallel and one link *A* (Fig. 61) common. A rope is attached to *D* and, passing partly round a wheel, is attached to it at a fixed point *K* in its circumference; a second rope is attached to *B*, and passing partly round an axle, is attached to a fixed point *k* in its circumference, the two ropes lying in parallel planes. The wheel and axle are fixed together, and form with *A* the turning pair *AC*. We have thus a second means of connecting two sliding pieces so that their velocity ratio may be uniform, for the velocities of *B* and *D* must be inversely as the radii of the wheel and the axle. As before, the ropes must be kept tight, also the guides of the pieces *B* and *D* may be omitted and replaced by force-closure, and this will be necessary if the wheel is to make more than one revolution, for then a lateral movement is required to enable the rope to coil itself on the surfaces.

Fig. 61.



In practical applications the second rope is generally omitted and the wheel turned by other means; the lateral movement is sometimes provided for by permitting the axle to move endways in its bearings, but more often, in cases where the load is not free to move laterally, the effect of a moderate inclination of the rope to the axis is disregarded. We may, however, escape this difficulty by the use of force-closure of a different kind. Instead of attaching the rope to a fixed point in the surface, let it be stretched over it by a force at each end, there will then be friction between the rope and the surface, which will be sufficient to prevent slipping if the tendency to slip be not too great.

Fig. 62.



The Differential Pulley is a good example of the application of these principles. As is shown in Fig. 62, there are two blocks, of which the upper, which is fixed, carries a compound sheave, consisting of two pulleys *A* and *C*, of somewhat different diameters, fixed to one another. The lower block carries a single sheave *B*, the diameter of which should theoretically be a mean between those of *A* and *C*, in order that the chain may be vertical. The chain is endless, and passes round the pulleys in the manner shown, so that when the side *P* is hauled downwards with a given velocity *V*, it will raise the lower block *B* with a velocity which we will now determine.

In passing around *A* and *C* the chain is not capable of slipping.

To ensure its non-slipping the periphery may be recessed to fit the links of the chain. In passing around B the slipping is immaterial; the raising of B would take place with the same velocity, whether there were an actual slipping of the chain round the circumference, or whether B were a rotating pulley.

When the point P is hauled downwards with velocity V, it necessitates the rotation of A, and with it of C. Thus the left hand portion of the chain passing round B will be hauled upwards with the same velocity as the point P downwards, and the right hand will descend with a velocity which is less in the ratio of the radii, c, a, of the united pulleys, and thus on the whole there will be an ascending motion given to B. Now, since the upward velocity of B is ½ the difference between the velocities of the two portions of the chain,

$$v = \tfrac{1}{2}\left(V - \frac{c}{a}V\right) = \frac{V}{2}\left(1 - \frac{c}{a}\right) = \frac{a-c}{2a}V.$$

Thus, by making the difference between a and c small, the relative velocity of B to P may be made as small as we please.

This apparatus, in a somewhat modified form, is much employed. It is called Weston's Differential Pulley Block, and possesses the valuable property that the weight will not descend when the hauling force is removed, for reasons which will be explained hereafter (Ch. X.).

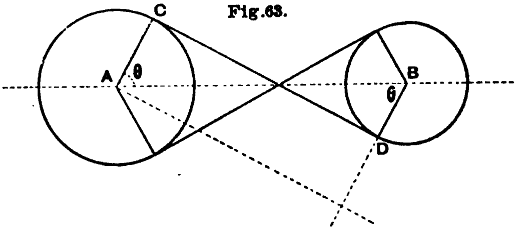
60. *Pulley Chains with Friction-Closure. Belts.*—A tension element may also be employed to connect the elements of two turning pairs. The most important case is that where two shafts are connected by an endless belt passing over a pair of pulleys and stretched so tightly that the friction between belt and pulley is sufficient to prevent slipping. If the belt were absolutely inextensible the speed of centre line of the belt would be the same at all points, and therefore the angular velocities of the pulleys would be inversely as their radii each increased by half the thickness of the belt. This mode of connection is unsuitable where an exact angular velocity-ratio is required, for even though the belt may not slip as a whole, yet it will be seen hereafter (Chap. X.) that its extensibility causes a virtual slipping to a greater or less extent. In the case of leather belts, the error in the angular velocity-ratio due to this cause is said to be about 2 per cent.

There are two ways in which the belt may be wrapped around the pulleys, being either *crossed or open.* If the belt is crossed, the pulleys will run in opposite directions of rotation. The crossed belt embraces a larger portion of the circumference of the pulleys than the open belt, and there is thus less liability to slip.

There is a proposition of some importance connected with the length of a crossed belt, which it will be useful to give here.

AC and BD (Fig. 63) being radii, each drawn at right angles to the straight portion of the belt CD, will each make the same angle θ with the line of centres. Thus the portion of the belt in contact with the pulley $A = (2\pi - 2\theta)\, r_A$ and that in contact with the pulley $B = (2\pi - 2\theta)\, r_B$.

Fig. 63.



The length not in contact $= 2 \,.\, CD = 2(r_A + r\ ) \tan\theta$.

Thus whole length of belt $= 2(\pi - \theta + \tan\theta)(r_A + r_B)$.

Now $\cos\theta = \dfrac{r_A + r_B}{AB}$.

Thus, if the distance AB between the centres is a constant quantity, and if, further, the sum of the radii $r_A + r_B$ is constant, then the angle θ will be constant. That being so, the total length of the belt will be a constant quantity.

This property is made use of when it is desired to connect two parallel shafts with an angular velocity-ratio, which may be altered at pleasure. A set of stepped pulleys, such as are shown in Fig. 1, Plate III., are keyed to each shaft, and the belt being shifted from one pair to another of the pulleys, the angular velocity-ratio is altered at will. If the belt is crossed, then the same belt will be tight on any pair of pulleys, if the sum of the radii is the same for each pair. This does not hold good for open belts. The actual length of belt required in any given example is best found by construction.

The tightness of the belt necessary to effect closure by friction of this kinematic chain may be produced simply by stretching the belt over the pulleys so as to call into play its elasticity, but the axis of rotation of one pulley is sometimes made moveable, so that the belt may be tightened by increasing the distance apart of the shafts, while in other cases an additional straining pulley is provided. The belt may then be tightened and slackened at pleasure, a method frequently used in starting and stopping machines.

In order that the belt may remain on the pulleys they must be provided with flanges, or, as is more common in practice, they must be slightly swelled in the middle, for when the shafts are properly in line, a belt always tends to shift towards the greater diameter. Great care, however, is necessary in lining the shafts that each side of the belt lies exactly in the plane of the pulley on to which it is advancing. Thus, for example, if the shafts be in the same plane, they must be exactly parallel, otherwise the belt will shift towards the point of intersection. This remark, however, does not apply to the receding side of the belt, and the shafts may make a considerable angle with each other, subject to the above restriction.

Friction-closure is always imperfect, because the magnitude of the friction is limited, but this is often a great advantage, since it permits the chain to open when the machine encounters some unusual resistance, which would otherwise produce fracture. By the use of grooved pulleys provided with clips the friction may be increased to any extent, so that great forces may be transmitted, but these devices are only suitable for low speeds, as in steam-ploughing machinery. Slipping may be avoided altogether by the employment of gearing chains, the links of which fit on to projections on the pulleys; force-closure is here replaced by chain-closure, and the action is in other respects analogous to toothed gearing. The speed is limited, as will be seen hereafter.

61. *Shifting of Belts. Fusee Chain.*—By the use of drums of considerable length as pulleys, the belt may be shifted laterally at pleasure. This principle is much employed in practice, as for example—

(1.) To stop and set in motion a machine.—The drum on one of the shafts is divided into two pulleys, one fast and the other loose on the shaft.

(2.) To reverse the direction of motion.—The drum is divided into three pulleys, the centre one fast, the two end ones loose on the shaft. Two belts, one crossed and the other open, are placed side by side. By shifting the belt either is made to work on the fast pulley at pleasure.

(3.) To produce a varying angular velocity-ratio.—The drums are made conical instead of cylindrical. The fusee employed in watches to equalize the force of the main spring is a common example.

The kinematic character of these devices will be considered in the next chapter.

62. *Simple Hydraulic Chain. Employment of Springs.*—Incompressible fluids may be employed to connect together two or more rigid pieces forming a class of elements which may be called "pressure elements," since they are capable of resisting pressure only. The pressure must be applied in all directions, and the fluid must therefore be enclosed in a chamber which pairs with the different pieces to be connected. For constructive reasons lower pairing must generally be adopted, and almost all cases are included in the following investigation.

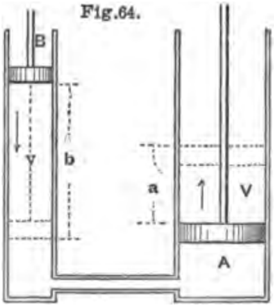


Fig.64.

Supppose two cylinders, each fitted with a piston (*A* and *B* in Fig.

64), to be connected by a pipe, the space intervening between the pistons being filled with fluid. Then when the piston B moves downwards with velocity v, the piston A will rise with velocity V, which is easily found by considering the spaces traversed by the two pistons in a given time. Let A, B be the areas of the pistons, a, b the spaces traversed, then, since the volume of the fluid remains the same, we must have $Aa = Bb$, and therefore

$$\frac{V}{v} = \frac{a}{b} = \frac{B}{A}.$$

The chain here considered, in which the elements of two sliding pairs are connected by a fluid, is kinematically identical with the arrangement of Fig. 60, p. 124, the replacement of a tension-element by a pressure-element constituting merely a constructive difference between the mechanisms. In the hydraulic press, in pumps, in water-pressure engines driven from an accumulator, and in other cases this kinematic chain is of constant occurrence, and will be frequently referred to hereafter. Combinations of an hydraulic chain with blocks and tackle are common in hydraulic machinery. Some examples will be found on page 451.

Springs, compressible fluids, and even living agents, are employed in mechanism, not only in a manner to be explained hereafter as a source of energy, by means of which the machine does work, but also in force-closure, and especially for the purpose of supplying the force necessary to shift pieces which open and close, or lock and unlock kinematic chains, and so produce changes in the laws of motion of the mechanism. The force of gravity, which, as has already been shown, frequently produces closure, should be regarded as the tension of a link of indefinite length connecting the frame-link of the mechanism with the link we are considering. The inertia of moving parts likewise gives rise to forces which are not unfrequently applied to similar purposes. Examples will be given in a later section.

EXAMPLES.

1. A shaft making 90 revolutions per minute carries a driving pulley 3 feet in diameter, communicating motion by means of a belt to a parallel shaft, 6 feet off, carrying a pulley 13 inches diameter. Find the speed of belt and its length—1st, when crossed, and 2nd, when open. Find also the revolutions of the driven shaft, allowing a slip of two per cent.

Speed of belt = 847·8 feet per minute.
Length when crossed = 19 feet 2 inches.
" open = 18 " 8 "
Revolutions of the follower = 244¼

2. Construct a pair of speed pulleys to give two extreme velocity-ratios of 7 to 1 and 3

to 1, and two intermediate values. The belt is to be crossed and the least admissible diameter is 5 inches.

Velocity-ratios	$\frac{21}{3}$	$\frac{17}{3}$	$\frac{13}{3}$	$\frac{9}{3}$,
Diameter of pulleys	5 35	6 34	7½ 32½	10 30.

3. The diameters of the compound sheave of a differential pulley block are 8 inches and 7 inches respectively; compare the velocities of hauling and lifting.

Velocity-ratio = 16 to 1.

4. In a pair of ordinary three-sheaved blocks compare the velocity of each part of the rope with the velocity of lifting.

5. In a hydraulic press the diameter of the pump plunger is 2 inches and that of the ram 12 inches, determine the velocity-ratio.

SECTION II.—WHEELS IN GENERAL.

63. *Higher Pairing of Rigid Elements.*—We next consider pairs of rigid elements in which the relative motion is not consistent with continuous contact over an area. The elements then touch each other at a point or along a line which is not fixed in either surface, but continually shifts its position. The form of the surfaces is not then limited as in lower pairing, but may be infinitely varied, with a corresponding variety in the motion produced.

This kind of pairing occurs when a chain of two lower pairs is completed by simple contact between their elements. In the double slider-crank chain shown in Fig. 4, Plate II., of the last chapter, let us omit the block C and enlarge the crank-pin so as just to fill the slot. By so doing the relative motions of the remaining parts will be unaltered, but we shall have three pairs instead of four, the turning pair BC and sliding pair CD being replaced by a single higher pair BD. This process is called *Reduction* of the chain, and when higher pairing is admissible the reduced chain serves the same purpose as the original, but with fewer pieces. The crank-pin and slot are in contact along a line only which during the motion continually shifts its position. In practice, the elements not being perfectly rigid, the contact extends over an area, but this area is of very small breadth, and consequently if heavy pressures are to be transmitted at high velocities the wear is excessive. If we trace the development of pieces of mechanism we observe that in the earlier stages higher pairing is much employed for the sake of simplicity of construction, but is gradually replaced by lower pairing. Nevertheless, where the object of the machine is mainly to transmit and convert motion rather than to do work, or where the velocity of rubbing is low, higher pairing may be employed. In many cases it is necessary, because the required motion cannot be produced by any simple combination of lower pairs.

Higher pairing of rigid elements may be divided into two classes according as the surfaces in contact do or do not slip over one another, just as in the case of tension elements considered in the last section. In the first case the contact is spoken of as Sliding Contact and in the second as Rolling Contact. In rolling contact the difficulty of wear does not occur, and hence it is always used when possible. The relative motion of the two elements is determined by considering that as the surfaces do not slip, the space moved through by the line of contact along each surface must be the same. Or as we may otherwise express it, if *A*, *B* be a pair of elements in rolling contact, the velocity of the surface of *A* must be the same as that of the surface of *B*.

64. *Rolling Contact.*—Rolling contact may be employed for the communication of motion between two shafts, the centre lines of which are either parallel or intersect, by means of surfaces rigidly attached to the shafts. In the first case the surfaces are cylindrical and in the second conical, the apex of the cone being the intersection of the shafts. By far the most important case, and the only one we shall here consider, is that in which the transverse sections of the surfaces are circular. Portions of the surfaces are used, as in Figs. 64*a*, 64*b*, and are pressed together by external forces, so that sufficient friction is produced to prevent the slipping of the surfaces. In other words, force-closure is necessary, as in the case of connection by a belt. This being supposed, it will immediately follow that the velocity of the two surfaces at the points of contact is the same, and hence, as before, the

Fig. 64a.

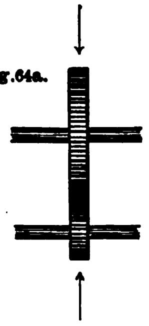
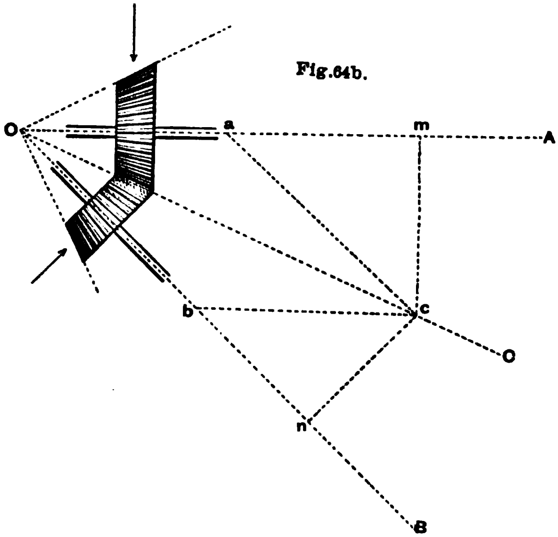


Fig. 64b.



angular velocity-ratio of the shafts is inversely proportional to the radii of the wheels. In the case of intersecting shafts, the surfaces are frustra of cones called "bevel," or, if the semi-angle of the cone be 45°, "mitre wheels," and their radii may be reckoned as the mean of that at the inner and outer periphery. The shafts revolve in opposite directions, unless one of the surfaces be hollow so that the other may be inside it, in which case the corresponding wheel is said to be "annular." When it is inconvenient to use an annular wheel, the same result may be obtained by transmitting the motion through an intermediate or "idle" wheel. If the radius of a wheel be infinite, it becomes a "rack," and the surface a plane.

In the case of bevel wheels the corresponding cones may be found, when the centre lines of the shafts and the angular velocity-ratio are given, by a simple construction. In Fig. 64*b*, let OA, OB be the centre lines of the shafts, and let distances Oa, Ob be marked off upon them in the ratio of the required angular velocities. Complete the parallelogram $Oacb$, then OC must be the line of contact of the required cones. For drop perpendiculars cm, cn, on OA, OB, then

$$\frac{cm}{cn} = \frac{\sin aOc}{\sin bOc} = \frac{Ob}{Oa},$$

so that the radii of any frustra of the cones employed for wheels will be inversely as the angular velocities of the shafts.

The particular case may be mentioned in which one of the cones becomes a plane; the corresponding wheel is then a "crown" or "face" wheel. The shaft of a wheel which is to work correctly with a crown wheel must be inclined to the plane of that wheel at an angle depending on the angular velocity-ratio required, a restriction not generally attended to, especially in the earlier stages of machinery in which face wheels were of common occurrence.

If, as generally happens, it is required to transmit a working force of a considerable amount, then the friction between the two circumferences will be found not to be sufficient to prevent slipping taking place, unless a considerable pressure to force the shafts together is employed, which involves an excessive friction on the bearings. In what is known as "frictional gearing," this is partially avoided by the use of wheels with triangular grooves fitting each other as the thread of a screw fits into its nut; but, in general, to prevent slipping, teeth are cut on the two peripheries, and the motion is transmitted by the gearing together of the teeth. Since this is a substitution for the rolling contact of two surfaces, it is required to so design the number and form of the teeth that the wheels on which they are cut shall turn one another with the same constant angular velocity-ratio as that due to the two original surfaces. If recesses are cut in each wheel, and projections be added between the recesses so as to fit into the corresponding recesses of the other wheel, then the two wheels may be placed to gear together at such a distance that the two original surfaces would have been in contact and would have rolled together. In the case of a pair of toothed wheels, such a pair of imaginary surfaces which will roll together with the same angular velocity-ratio as that obtained from the toothed wheels, are called *pitch surfaces.* Considering first the case of parallel shafts, the transverse sections of these surfaces are called *pitch circles,* and their point of contact is called the *pitch point.* The radii of these pitch circles must be to one another in the inverse of the velocity-ratio. The circumference of each circle is to be divided into a number of equal parts, which will include a tooth and a recess. The length of each part measured along the pitch circle is called the *pitch.* Let p = *pitch*, and n = number of teeth, d = diameter, then

$$p = \frac{\pi d}{n}.$$

The thickness of each tooth is made a little less than $\frac{1}{2}p$ to allow the clearance necessary for easy working. The magnitude of the pitch which governs the thickness of the teeth must be determined from considerations as to their strength. If n' = number of teeth in the second wheel, and d' = its diameter, then the pitch being the same for each wheel

$$p = \frac{\pi d}{n} = \frac{\pi d'}{n'}.$$

The distance apart of the shafts is generally adjusted to allow the pitch to be some exact number of inches, half, or quarter inches. The pitch is to be measured along the pitch circle, and is not the cord of the arc, as is sometimes stated.

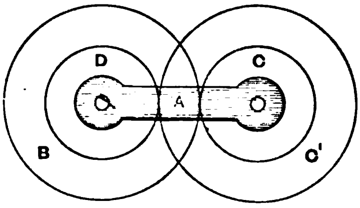
In some small wheels used for spinning machinery, another kind of *pitch* is referred to. The diameter of the pitch circle is divided by the number of teeth, and the result is called the *diametral pitch.* In the smallest class of wheelwork used in clocks, the dimensions of the teeth are stated as so many to the inch. The proper form of teeth will be considered farther on.

65. *Augmentation of a Kinematic Chain. Trains of Wheels.*—Another important application of rolling contact is to diminish friction by the intervention of rollers, hence called Friction Rollers. Thus the friction between the elements of a sliding pair, subject to heavy pressure, will be so great as to require a great force to overcome it, but if rollers be placed between the elements the friction is greatly reduced, as will be seen hereafter. In this case sliding friction is wholly replaced by rolling friction; in carriage wheels the sliding velocity which, without the wheel, would be the actual velocity of the carriage, is reduced to that at the periphery of the axle, that is to say, in the ratio of the diameters of the axle and the wheel. The sheaves of an ordinary pulley block are examples of the same principle. In all these cases where additional pieces are added to a kinematic chain, in order to reduce friction or to serve some other non-kinematical purpose, the chain is said to be "augmented."

Chains are frequently augmented for purely constructive reasons; thus, if the velocity-ratio of a pair of shafts is great, the diameters of a single pair of wheels necessary in order to obtain it will be inconveniently large or small. A train of wheels is then resorted to. This is also the case where the shafts to be connected are too near or too far apart; in the latter case bevel wheels and an intermediate transverse shaft may be employed.

When, however, the shafts to be connected are in the same straight line, a train of wheels is kinematically necessary, and forms virtually a new mechanism. This is a common case in practice when a pulley or wheel is loose on a shaft, and it is required to connect the wheel and the shaft so as to revolve with different velocities. Such a train is shown in Fig. 65 in a simple ideal form. *B* and *D* are two wheels turning on the same centre but disconnected. *C*, *C'* are two wheels gearing with *B* and *D* and turning about another centre but united. The two centres are connected by the frame-link *A*. When *B* revolves it drives *C*, and *C'* drives *D*. If the numbers of teeth in these wheels be denoted by the letters which distinguish them, and the velocity of *B* be unity, the velocity of *C* or *C'* will be *B*/*C*, and that of *D* will be *BC'*/*DC*. Let it now be observed that the wheels *B* and *D* form a pair, the velocity of which will be the difference between the velocities of these wheels. We have then altogether four turning pairs in this train of wheels, the relative velocities of which are—

Fig.65.



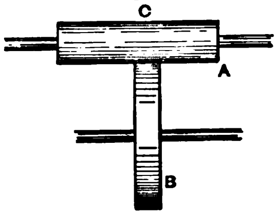
Pair, -	*BA*	*CA*	*DA*	*DB*
Velocity,	Unity	$-\frac{B}{C}$	$\frac{BC'}{CD}$	$-1+\frac{BC'}{CD}$

One of the wheels in this train may be annular, and all may be bevel; in either case the wheels *C*, *C′* may be equal, and the train reduced to three wheels, though the number of simple pairs remains as before four. Examples are given in the figures of Plate III.

Either this or any other train of wheels may be inverted by fixing one of the wheels instead of the frame-link, the resulting mechanism is then called an Epicyclic Train; the velocity-ratios of the various pairs are unaltered, and are therefore shown by a table similar to that given above. Should the angular velocity of any wheel be required relatively to the fixed wheel, we have only to add to the velocity of the corresponding pair the velocity of the frame-link. Some examples of epicyclic trains are shown in the figures, but for detailed descriptions we must refer to a work on mechanism. Their use in compound chains will be further referred to in the next chapter.

66. *Wheel Chains involving Screw Pairs.*—In a simple wheel chain (Fig. 66) consisting of a wheel *B*, a pinion *C*, and a frame-link *A*, not shown on the figure, suppose *C* to be of considerable length, then there will be nothing to prevent the endways movement of *B* in its bearings if they be supposed cylindrical. This circumstance is often taken advantage of in machinery in shifting wheels in and out of gear, but the case to be examined here is that in which the endways movement is given by independent means during the action of the mechanism. The simplest example is a three-link chain derived from the train of wheels just considered by changing the turning pair *BA* into a screw pair; *B* then travels endways through the pitch of the screw in each revolution. The pinion *C* sometimes slides on the shaft which carries it, but quite as often it is made long enough to permit the necessary traverse of *B*. A well known example of this mechanism is that of the feed motion common in drilling and boring machines, in which the train of wheels of the last article is used with *B* and *D* nearly equal, so that the velocity of the pair *BD* is very small. *B* is attached to the nut and *D* to the screw, so that *BD* is a screw pair. *D* then traverses through *B* by a space each revolution which may be made very small.

Fig. 66.



To illustrate and explain preceding articles Plate III. has been drawn, giving examples of trains of wheels, especially of the differential trains of Fig. 65.

Fig. 1 shows the slow motion of a lathe. *D* is a wheel keyed on the mandrel and connected with *B*, the driving pulley, when the motion is not in use. *B* rides loose on the mandrel, and by means of a pinion gears with *C*, a wheel on the same shaft with *C′*, which gears with *D*. *CD* being large compared with *BC′*, the speed of the mandrel is

Plate.III.

FIG.1.

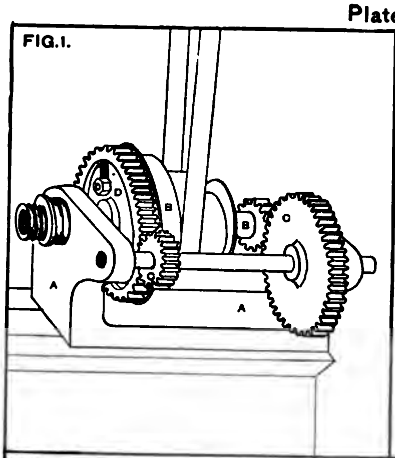


FIG.2.

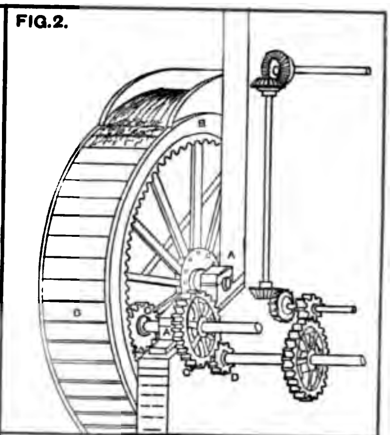


FIG.3

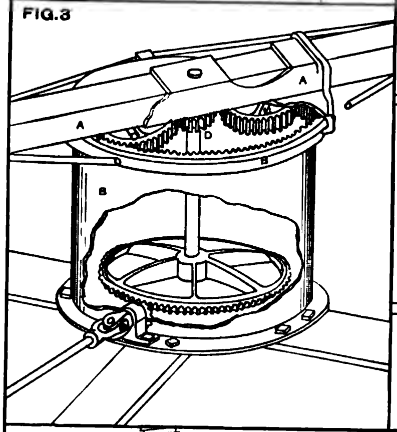


FIG.4.

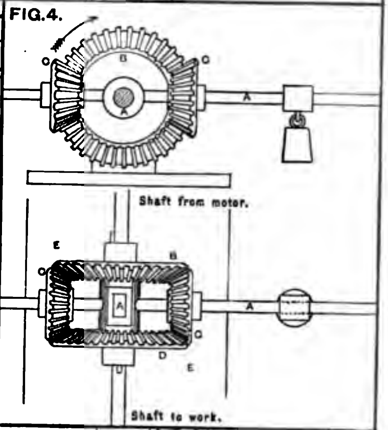


FIG.5.

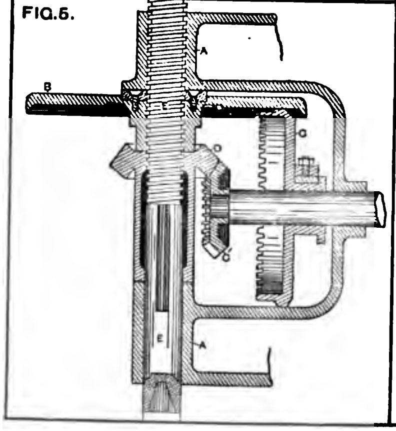
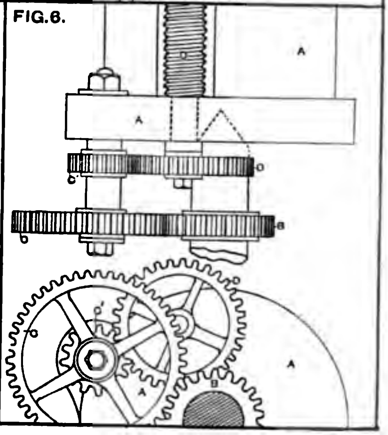


FIG.6.



To face page 137.

much less than that of the pulley. For lighter work *CC′* are thrown out of gear by an endways movement of the shaft.

Fig. 2 represents the train of wheels by which the slow movement of a water-wheel is multiplied and transmitted to all parts of a factory. *B* is now an annular wheel attached to the water-wheel gearing with *C*, *C′* with *D*, and so on. A vertical shaft *F* with bevel wheels transmits the motion to the upper floors. The bearings of the secondary shafting are omitted for clearness, but they all form part of a frame-link *A*, which is fixed.

In Fig. 3 the kinematic chain is inverted. *B* is a fixed annular wheel, *CC′* are of equal diameter and reduce to one wheel, which, however, is in duplicate, in order to balance the driving forces. This epicyclic train is applied to many purposes. In the example shown the frame-link is a long arm, at the end of which a horse is attached, and a rapid motion thus given to the central pinion *D*. The motion is further multiplied by the bevel gear shown below, and applied to drive a thrashing machine or some similar purpose. The same mechanism is employed as a purchase in capstans and tricycles.

In Fig. 4 the train consists of three bevel wheels, *BCD*, *C* and *C′* reducing to one, as in the preceding case. The simple chain consists of these wheels and the train arm *A*. When *A* is fixed the wheels *B* and *D* turn in opposite directions with equal velocities; when *B* is fixed *A* revolves with half the velocity of *D*. The mechanism is much employed, but usually as a compound chain, and as such will be considered in the next chapter. The example shown is a dynamometer.

Fig. 5 represents the feed motion of a drilling machine. *A* is the frame of the machine in which rotates the vertical drill spindle *E* driven by a pair of mitre wheels *D* and *C′* from a horizontal shaft. A screw thread is cut on the spindle, of which *B* forms the nut. If *B* and *D* rotate at the same speed the drill moves neither up nor down, but any difference will result in a motion of the screw pair *BE*, and will thus give the necessary feed or raise the drill out of the hole. In the example chosen *B* is driven by a flat disc gearing by friction with a wheel *C′* turning with *D* (Naish's patent). This wheel, by means of a lever, can be moved along the shaft so as to gear with *B* at any radius at pleasure, and can therefore be set so as to raise or lower the drill at any required speed. The contact between *C*, *D* here is not pure rolling (p. 133); but as *C* is of small breadth the error is not of practical importance.

In Fig. 6 the same kinematic chain is employed as an epicyclic train to give motion to the cutters of large boring machines. The cylinder to be bored is fixed, and the boring bar rotates on the lathe centres. The wheel *B* is fixed; *D* is attached to the end of a long screw, which on turning causes a nut *E* (not shown in the figure) to traverse slowly, carrying with it the cutting tools. The train arm *A* rotates and carries on it the wheels *C*, *C′*, and *D*.

EXAMPLES.

1. The diameter of pitch circle of a wheel is 4 feet, and the number of teeth 120. Find the pitch.

Pitch = 1¼ inches.

2. Two shafts about 4 feet apart are to be connected by spur wheels, the velocity-ratio being 4 to 1. Find the diameters of the wheels and also the number of teeth, assuming the pitch to be 2 inches.

Ans. The numbers of teeth in wheels are 30 and 120, and the exact distance apart of the shafts = 47¾ inches.

3. The diameter of the pitch circle of the annular wheel by means of which a water wheel communicates motion to a mill, is to be as nearly as possible 24 feet. The pitch is to be 4 inches. Find the diameter and the number of teeth in the wheel. The velocity of the periphery is to be 5¼ feet per second and the first motion shaft is to make 30 revolutions per minute. Find the necessary diameter of pinion and the number of teeth in it.

Ans. The number of teeth in the annular wheel = 226, and its exact diameter is ¼ inch less than 24 feet. The number of teeth in the pinion is 32, making the revolutions per minute somewhat less than 30. The diameter of pinion = 40¾ inches.

4. A pair of shafts, the centre lines of which intersect at an angle of 60°, are to be connected by bevel wheels so as to revolve, the one at 250 and the other at 90 revolutions per minute. Find the pitch surfaces.

Angles of cones 90° and 30°.

5. Two shafts intersecting at an angle of 75° are connected by a crown wheel gearing with a pinion. What must be the velocity-ratio?

6. The weight of a revolving turret rests on a ring of friction rollers, the axes of rotation of which radiate horizontally from the axis of the turret: find the angle at which the rolling surfaces must be bevelled. Compare the rates of rotation of the ring and the turret.

7. The feed motion of a boring machine consists of a nut working on a screw cut on the spindle of the drill or borer which is raised or lowered whilst the nut turns on it. The nut carries a wheel of 96 teeth which gears with one of 35. When the drill is at work the wheel of 35 teeth is secured to one of 36 on the same axis, and this latter gears with one of 95 teeth secured to the spindle of the drill. The screw has four threads to the inch. Determine the depth of hole bored per revolution.

$$\text{Depth of hole bored per revolution} = \tfrac{1}{4}\ \text{inch}\left(1 - \frac{35 \times 95}{36 \times 96}\right) = {\cdot}0095\ \text{inch.}$$

8. The train of wheels in the preceding question is used as an epicyclic train by fixing the wheel of 96 teeth. Find the direction and number of revolutions of the train arm for each revolution of the spindle.

Ans. For each revolution of 95 wheel forwards, the arm turns backwards through

$$\frac{95 \times 35}{96 \times 36 - 95 \times 35} = 25{\cdot}4 \text{ revolutions.}$$

SECTION III.—TEETH OF WHEELS.

67. *Preliminary Explanations.*—Even though the number of teeth in a pair of wheels be such as to give the correct mean angular velocity-ratio due to the rolling together of the pitch circles, yet if they be of improper form they will jam or work roughly.

Theoretically the form of the teeth of one of a pair of wheels may be chosen at pleasure if a proper corresponding form be given to the teeth of the other; the problem of rightly determining the form is therefore one which admits of many solutions. We commence with some general explanations applicable to all forms of teeth.

The diagram (Fig. 67) shows a section of a pair of spur wheels in gear, with three teeth in action, the lower wheel being the driver. *BTB*, *ATA* are the pitch circles in contact at the pitch point *T*. $ST = TL$ is the pitch, being the distance of a point in one tooth from the corresponding point in the next consecutive measured along the pitch circle. The teeth as shown in the figure partly project beyond the pitch circle and fit into corresponding recesses in the other wheel, so that each tooth is divided into two parts, a part within and a part without the pitch circle. The corresponding acting surfaces are called the Flank and the Face of the tooth respectively. In annular wheels the flank is outside and the face inside the pitch circle. The teeth commence action before reaching the line of centres by the flank of a tooth of the driver *A* coming into contact with the face of a tooth of the follower *B*, as shown at *C* in the diagram, and gradually approach that line till after the wheels have turned through a certain arc, which measured on the pitch circle is called the Arc of Approach; they are then in contact at *T* the pitch point. After passing the line of centres they remain in contact till the wheels have turned through a second arc called the Arc of Recess and then cease contact as shown at *D*, the face of a tooth of the driver being always in contact with the flank of a tooth of the follower. The sum of these arcs is called the Arc of Action, and must be great enough to permit at least two teeth to be in contact at once. Their magnitudes depend on the projection of the teeth beyond the pitch circle, a quantity which is called the Addendum of the corresponding wheel, the arc of approach depending on the addendum of the follower, and the arc of recess on the addendum of the driver.

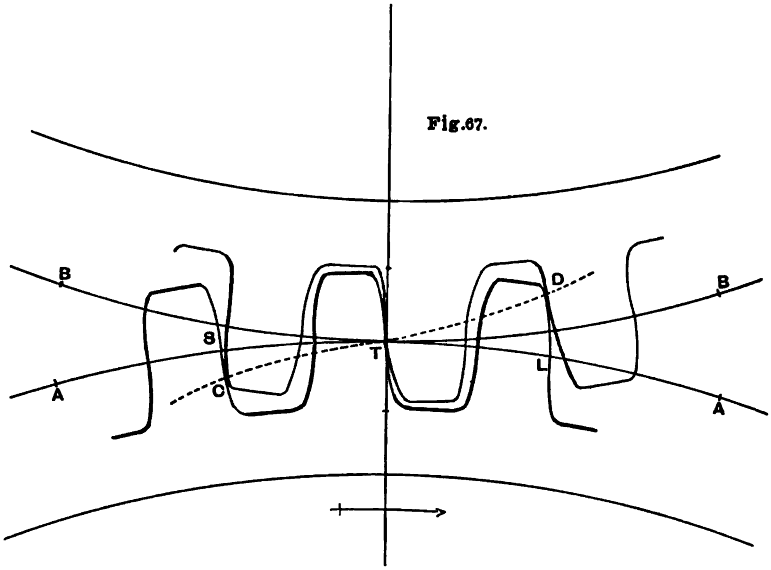


Fig. 67.

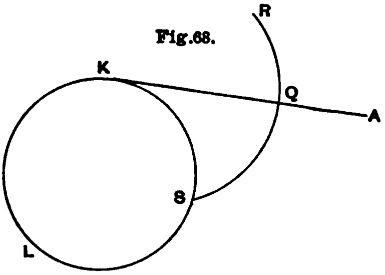


Fig. 68.

68. *Involute Teeth.*—The question of the form of the teeth requires much explanation to render it completely intelligible; we shall only

give a brief sketch, referring for full details to the works cited on page 92. Some points will be further considered at a later period. We commence with what are known as Involute Teeth.

Imagine a string AKL wound on a cylinder (Fig. 68). If the string be gradually unwound, the string being kept tight all the time, a point Q of the string will trace out a curve SQR called the *Involute of the Circle.* Instead of causing the string to be unwound around the fixed circle we may if we please move A in a fixed straight line and cause the unwinding to take place by the revolution of the circle. If now a piece of paper be fixed to and revolve with the circle, the same involute curve will be traced on it as before.

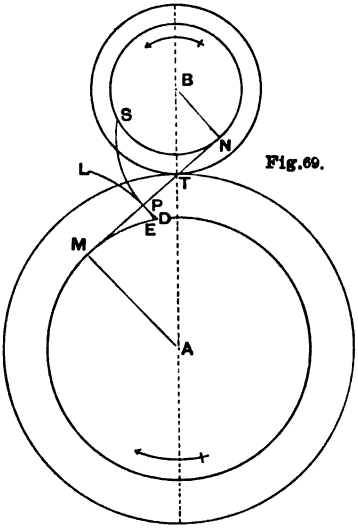


Fig. 69.

Now let A and B (Fig. 69) be two circles not in contact which are each capable of revolution about its centre. If we connect them by a crossed belt, of which one half is shown in the diagram by the line MTN, each will be capable of driving the other with a constant angular velocity ratio, namely, the inverse ratio of the radii. If, therefore, T be the point where the belt crosses the line of centres,

$$\frac{A_A}{A_B} = \frac{r_B}{r_A} = \frac{BT}{AT}.$$

Now, with centres A and B and radii AT and BT, describe circles which touch one another. These two circles would turn one another by rolling contact with the same angular velocity-ratio as that due to the belt. If we were to form teeth on the two wheels and cause them to turn one another by the gearing of the teeth, then the two circles passing through T may be regarded as the pitch circles of the two wheels.

Now to trace the form of the teeth. Attach a pencil (P) to any point of the belt and fix a piece of paper to the wheel A so that it may turn with it, then the pencil will trace on the paper the curve EPL, being an involute of the circle A. Similarly, if we imagine a piece of paper attached to B, an involute DPS of the circle B will be traced on that. These two curves will be in contact at the tracing point P, and will always remain in contact as the circles turn. If, therefore, we construct teeth of this form with any given pitch, and then remove the belt, the two toothed wheels will drive one another with the constant angular velocity required. In this form of tooth the face and flank are one continuous curve, which is a property practically confined to involute teeth. From this fact a practical advantage follows. By the continual action of the teeth together they wear and cause a looseness of fit, which may be remedied by bringing the centres of the wheels more nearly together, and this without altering the smooth action of the teeth or the exact uniformity of the angular velocity-ratio. In no other form of tooth occurring in practice is this possible.

The line of action of the mutual pressure between the teeth is always along the tangent line to the two base circles, from which the teeth are generated, thus tending always to force the axles apart. If the angle between this line and the common tangent to the two pitch circles, or as it is called, the "obliquity," be large, much friction in the bearings would result. On this account the obliquity is made as small as possible, not being allowed to exceed 14½° or 15°. With this a limit is introduced to the smallness of the number of teeth which may be used. The action of the teeth must always be along the line *MTN*, and hence cannot extend beyond the point *N*. If it is essential that when two teeth are in contact at the pitch point another pair of teeth should just be coming into action whilst a third pair are just ceasing action, then the length of the arc of the pitch circle which corresponds to an arc on the base circle equal to *TN* will be the greatest length that can be given to the pitch of the teeth, and when the obliquity is 14½° there will be about twenty-five such pitches on the pitch circle, and hence the number of teeth cannot be less than twenty-five.

Having given the pitch circles we first lay off, through the pitch point, the line of oblique action which is to be allowed, and then draw the base circles touching this line. The involutes of the base circles will give us the form of the teeth. The thickness of the tooth is to be taken a little less than half the pitch, and the addenda of the teeth such as to give a sufficient number of teeth in contact at the same time. (Art. 71.)

All involute teeth of the same pitch and obliquity will work together; they have never been much used in practice, although there appears to be no reason why they should not be in cases where it is not necessary to have less than twenty-five teeth. Their wear is said to be greater than that of teeth of other kinds.

69. *Path of Contact the Pitch Circle.*—In involute teeth the tracing point is attached to a belt stretched over pulleys, and therefore describes a straight line on paper, which is fixed to the line of centres so as not to revolve with either wheel. Now, the tracing point is also the point of contact of the two teeth, and therefore the path of this point, or, as it is conveniently called, the "path of contact," is a straight line. Teeth of any shape may be traced by this method if, instead of simply stretching the belt over the pulleys, we pass it over a fixed curve between the pulleys, so that the tracing point describes the curve in question instead of a straight line, provided the fixed curve be such that the curves traced on the rotating circles touch one another. In other words, we may assume various "paths of contact" at pleasure and obtain teeth which will work together correctly. We shall next suppose the tracing point attached to the circumference of a rotating wheel, in which case the path of contact is a circle.

In the use of toothed wheels the earliest idea was, for simplicity of construction, to form the smallest wheel of a number of cylindrical pins projecting from a disc. Supposing one of a pair of wheels to be so constructed, it is required to determine the proper form of the teeth for the other wheel.

On the wheel *B* (Fig. 70) let pins be placed at equal distances, with their centres on the pitch circle, and in the first place suppose the pins indefinitely small, being mere points. Now, if at one of the points. *P* a pencil be attached, then if *B* be caused to roll without slipping over the surface of *A* kept fixed, the pencil *P* will trace a curve on a piece of paper attached to the wheel *A*. The same curve will be drawn if we cause one wheel to drive the other without slipping, the centres *A* and *B* being fixed, while the paper is attached to *A* and turns with it. If the tracing point started from the pitch point *T*, then the curve *KP* will have been drawn on the paper, which, by the further rotation of the circles, will be produced to *Z*. This curve is called an Epicycloid, and will be the proper form of teeth for the wheel *A* to drive the pinion *B*. For the pin *P* will be always in contact with the tooth *KZ* as the wheels revolve with uniform angular velocity-ratio. We complete the form of the teeth by drawing a similar curve *ZS* for the other face, *SK*

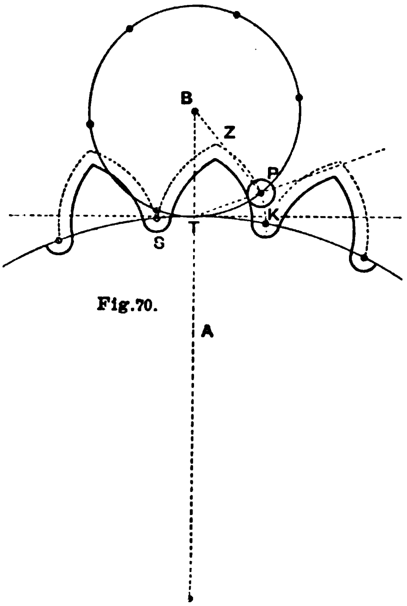


Fig. 70.

being the pitch, in order to enable the wheels to be turned in the opposite direction if necessary. Placing a number of such teeth on the pitch circle *A*, we see they all touch one another at the roots on the pitch circle. The reason is because we have imagined the pins of *B* to have no definite dimensions, but to be mere mathematical points. In practice some definite dimensions must be given to the pins of *B*. In such a case the proper form for the teeth of *A* is derived from the previous construction by drawing a curve which at all points shall be at a distance from the epicycloid, when measured along the normal, equal to the radius of the pin. Below the pitch circle *A* a semi-circular recess must be formed, as shown by the full curve in figure.

These teeth possess the peculiar property of having faces but no flanks. The consequence is that, the toothed wheel *A* being the driver, the action of the teeth is wholly after the line of centres; there is no arc of approach, but only an arc of recess. On this account the pin-wheel must always be the follower, for if it be the driver the action of the teeth would be wholly before the line of centres, in consequence of which the friction is said to be more injurious.

The angle which *PT* makes with the common tangent is, as in the case of involute teeth, called the "obliquity"; it is now no longer constant, but varies from zero, when *P* passes the line of centres at *T*, to a maximum value when *P* escapes. It is easily seen that this angle is always one-half the angle *PBT*, which *PT* subtends at the centre of the pin-wheel, and hence the obliquity increases uniformly as the wheels turn; its mean value may be taken at half the maximum, and is limited in the same way as in involute teeth to about 15°, so that the greatest value of the angle *PBT* may be taken as 60°.

If the two sides of the teeth are alike, as in the figure, the pin then comes to the point of the tooth at *Z*. This circumstance determines the smallest number of pins which can be used, for one pin must not escape before the next comes to the line of centres; that is to say, *PT* cannot be greater than the pitch, the pitch then must not be greater than one-sixth the circumference of the pin-wheel, whence it appears that the least number of pins is six.

Pins are now rarely employed unless in clock and watch work; they have the great practical disadvantage that the toothed wheel to work with them must be specially designed, as it will work with only one diameter of pinion.

If we imagine a pin-wheel to work with an annular wheel, the teeth may be traced in the same manner as shown in Fig. 71 (p. 144), to which the same letters are attached. The point *P* now traces out a curve called a Hypocycloid, the general character of which may be seen by joining P to F, the other extremity of the diameter TF of the circle B; for since the angle FPT must be a right angle, the angle APT will be greater than a right angle if, as in the figure, F lies between A and T, and less than a right angle if F lies beyond A. Thus the hypocycloid must reduce to the radius AK if F coincides with A, that is, if the diameter of the pin-wheel be half the diameter of the annular wheel; while, for smaller diameters, it forms a curve always concave towards T. Hence it appears that to work with a pin wheel of half its diameter the teeth of the annular wheel should be constructed simply by drawing radii of the pitch circle. With a larger diameter of pin-wheel the teeth would be undercut, and therefore weak; the annular wheel must be the driver as before.

In all epicycloids and hypocycloids the normal to the curve at the tracing point P passes through the point of contact T of the circles considered—an important geometrical property, which we shall presently make use of, and hereafter prove.

70. *Path of Contact any Circle.*—Teeth traced in the way just described are wholly within the pitch circle, and this circumstance suggests that

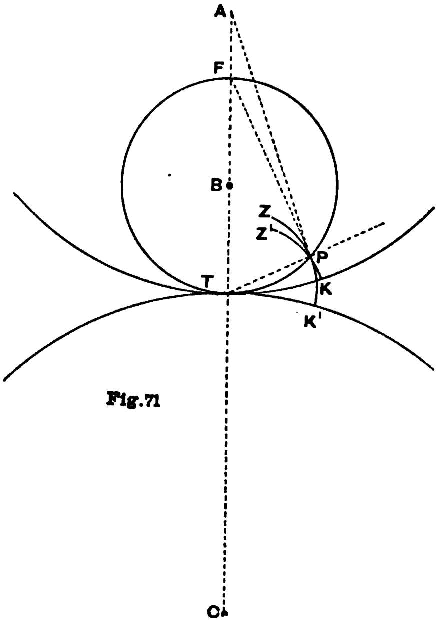


Fig. 71

by a combination with the preceding case, where they were wholly without, a form may be found which may be more suitable for practical use.

In Fig. 71 a third circle *C* is shown, touching the two others at the same pitch point *T*. The three circles *ABC* turn each about its own centre without slipping. Imagine paper attached to *A* and *C* and rotating with them, while a pencil *P* is attached to *B* as before; then *P* will trace out two curves as in the case of involute teeth, one outside the circle *C*, the other inside the circle *A*. *A*'s curve will be an hypocycloid *KPZ*, starting from *K* in the circle *A*, while *C*'s curve is an epicycloid *K'PZ'* starting from *K'* in the circle *C'*. Now these curves will, as in involute teeth, touch one another, having a common normal *PT*, and hence it follows that, while the circles turn with uniform angular velocity-ratio, the curves will always be in contact, and may be taken as face and flank of a pair of teeth. Thus it appears that we can obtain the faces of the teeth of *C*, and the flanks of the teeth of *A*, by causing a third circle *B* of any diameter to rotate within the circle *A*. If the diameter of *B* be half the diameter of *A*, the flanks for *A* will be simply radial lines, but if it be less they will be concave towards *T*, the effect of which is that the teeth will spread out at the root, which is desirable on the score of strength. We can now imagine the faces of the teeth of *A* and the flanks of the teeth of *C* to be traced by another circle *B'* rotating within *C* instead of within *A*. The diameter of this circle need not be the same as that of *B*; it may, for example, be half the diameter of *A*, while *C*'s diameter is half that of *B*; if so, the flanks of the teeth of both wheels will be radial. Teeth with radial flanks have the disadvantage of weakness, especially when the number of teeth is small, because the thickness at the root is less than that at the pitch circle, and they are, besides, only capable of working correctly with wheels specially designed for them. In order that a set of wheels of this kind may be interchangeable, it is necessary that the circles *B'B* be of equal diameter and the same for all the set. This diameter should not be larger than half that of the smallest wheel of the set, for if it is, the flanks of the teeth of the small wheels will be undercut and consequently weak, while, on the other hand, it should be as large as possible, for otherwise the teeth of the large wheels will be too thick at the roots and too thin at the points, a form which is found to be unfavourable to good wearing. Hence the diameter chosen for *B* is half that of the smallest wheel of the set, the flanks of which will be radial. As *B* is a pin-wheel, its smallest circumference is six times the pitch (Art. 69), and the smallest wheel of the set has consequently 12 teeth; but if no wheel is required with so small a number of teeth as this, it will be better, for the reason stated above, to take a larger describing circle.

71. *Addendum and Clearance of Teeth.*—In any form of teeth it is clear from what has been said that the point of contact travels along the path of contact DT (Fig. 67, page 139) from the pitch point T to the end of the tooth at D, where the contact ceases. The length of the path of contact thus traversed is equal to the arc of recess in all kinds of cycloidal teeth, and less than that arc in a given ratio in involute teeth. By stepping off a suitable length on the path of contact then, we can find the end of the tooth for any given arc of recess, and the distance of this point from the pitch circle A of the driver is what we have already defined as the "addendum" of that wheel. The position of this point on the flank of the tooth of the follower B gives the working length of flank necessary. Similarly the length of face in the follower and flank in the driver depend on the arc of approach. The depth of the recesses between the teeth, however, must be made greater than is necessary for working length of flank, in order to allow the ends of the teeth to clear; the amount usual in practice appears to be about one-fifteenth the pitch.

The allowance necessary in practice for clearance in the thickness of the teeth depends on the degree of accuracy attainable in construction. The value formerly employed for teeth shaped by hand was one-eleventh the pitch, but the best modern teeth are machine cut, and a much smaller amount is sufficient. Less clearance is required for involute teeth than in teeth of other kinds. The setting out of bevel teeth is not theoretically more difficult than in the case of spur gear, but their accurate execution by a machine is far from easy. If the machine operate by straight cuts like an ordinary shaping machine, the tool must be mounted so that the line of cut always passes through the apex of the pitch cone. Gear cutting machines generally employ revolving cutters formed to fit the space between two teeth. Much ingenuity has been expended on giving the cutter a lateral movement to suit the bevel, but an exact bevel tooth cannot be formed in this way.

72. *Endless Screw and Worm Wheel.*—When two shafts are to be connected which are not parallel, and the centre lines of which do not intersect, it is necessary to resort to skew bevel, or screw, teeth. Only one case of this kind need be mentioned here as being of common occurrence, namely, the endless screw and worm wheel employed when the shafts are at right angles, and a slow motion of one of them is desired. In a common screw let the thread be so formed that the longitudinal section of the screw thread shows a range of teeth like those of a rack which would gear with a given spur wheel. Let the teeth of the wheel be set obliquely at an angle equal to the pitch angle of the screw; strictly speaking they also are screw threads, the pitch angle of which is the complement of the pitch angle of the screw. Then

the screw and wheel will gear together, and the wheel moves through one tooth for each revolution of the screw. Like screws in general, this combination is non-reversible unless the pitch of the screw be coarse (Ch. X.), and for this reason, and on account of its simplicity, is much employed in practice. The method of constructing the teeth of a worm wheel is explained in a work by Prof. Unwin, cited on page 121.

1. A pair of wheels have 25 and 120 involute teeth respectively, and the addendum of each is $\frac{3}{10}$ths the pitch. Find the arcs of approach and recess in terms of the pitch, assuming the obliquity $14\frac{1}{2}°$, the large wheel being the driver. (See Art. 71.)

***Ans.*—Arc of approach = ·89 × pitch.**
Arc of recess = 1·12 × pitch.

2. If the arcs of approach and recess in involute teeth are each to be equal to the pitch, show that the addenda of the wheels should be calculated by the approximate formula

$$\text{Addendum} = \left(\frac{1}{4} + \frac{3}{n}\right) \times \text{pitch},$$

where n is the number of teeth.

3. A pair of wheels have 25 and 120 teeth respectively, the flanks being in each case radial. Find the addendum of each wheel that the arcs of approach and recess may each be equal to the pitch.

***Ans.*—Addendum of driver = ·283 × pitch.**
Addendum of follower = ·178 × pitch.

4. Connect two shafts which are not parallel and which do not intersect by bevel wheels, an intermediate idle wheel being admissible.

SECTION IV.—CAMS AND RATCHETS.

73. *Reduction of a Crank Chain by Omission of the Coupling Link.*—A pair of spur wheels in gear form a particular case of a three-link kinematic chain consisting of two lower pairs with parallel axes, two elements of which are united and generally form the frame-link, while the other two pair by contact.

Such a chain may be derived from the four-link crank chain of Art. 52, page 111, by omission of the coupling link, a process of reduction which has already been employed on page 131.

In Fig. 72*a*, *ab*, *dc* are levers turning about fixed centres and connected by a coupling link *bc*, all three links being in one plane as in the article referred to. Imagine now the crank-pins at *b* and *c* enlarged until they touch one another as shown by the dotted circles and then remove the coupling link. Suitable forces being applied to close the chain by keeping the surfaces in contact, the link *bc* may be removed without in any way altering the motion, and therefore the angular velocity-ratio will still be as before $aT : dT$, where T is now the intersection of the common normal at the point of contact with the line of centres. Now the instantaneous motion of the levers cannot be affected by the shape of the pins except at the point of contact, and it therefore follows that if we replace the pins by any surfaces such as those indicated by the full lines in the figure, which have the same common normal at the point of contact, the result will be the same.

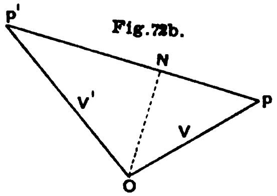


Fig. 72b.

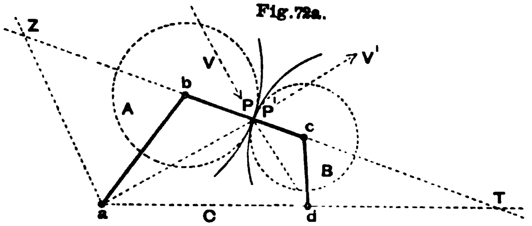


Fig. 72a.

We may reach this conclusion directly by constructing a diagram of velocities for the two pieces in question. For let *P*, *P′* be points in the profiles which at the instant considered coincide by becoming the point of contact. Then *P*'s velocity in the direction of the normal must be the same as that of *P′*, for otherwise the surfaces would interpenetrate or move out of contact. If then from a given point *O* (Fig. 72*b*) we draw *Op*, *Op′* parallel to the lines *aP*, *dP′*, to meet a parallel to the normal in *pp′*, it follows by the same reasoning as in the case of link-work that *Opp′* is a triangle of velocities of which the sides *Op*, *Op′* represent the velocities of *P*, *P′*. Hence drawing *aZ* parallel to *dP′* it appears as before that the angular velocity-ratio of the lines *aP*, *dP′* is *dT* / *aT*, and these lines are fixed in the rotating pieces so as to have the same velocity-ratio.

The third side *pp′* of the triangle of velocities represents in this case the velocity with which the surfaces rub against one another, for dropping the perpendicular *ON* the segments *Np*, *Np′* represent the resolved part of the velocities along the common tangent. Suppose *A*, *A′* to be the angular velocities of the pieces, *V*, *V′* the actual velocities of *P*, *P′*, then by similar triangles

$$\frac{pp'}{PZ}=\frac{Op}{aP},$$

that is, if *v* be the velocity of rubbing,

$$\frac{v}{PZ}=\frac{V}{aP}=A,$$

from which we obtain

$$v = A \,.\, PZ = A(TZ - PT).$$

But it was shown above that

$$A \,.\, aT = A' \,.\, dT\,;$$
$$\therefore\ A \,.\, TZ = A' \,.\, PT\,;$$

hence

$$v = (A' - A)PT.$$

This formula suppose the pieces to turn in the same direction, as in the figure. If they turn in opposite directions, as in a pair of toothed wheels,

$$v = (A + A\ )PT,$$

a simple and important result which we shall hereafter verify.

It follows at once that for rolling contact the point of contact must lie on the line of centres, and that for a constant angular velocity-ratio T must be a fixed point. Thus in all forms of teeth for wheels the common normal at the points of contact of the teeth must always pass through a fixed point on the line of centres, as is easily seen to be the case in the examples already considered. The velocity with which the teeth slide over one another is given by the above formula.

The diagram of velocities may when necessary be completed by laying down on it the velocities of all points rigidly connected with either rotating piece as explained before in the case of linkwork.

74. *Cams with Continuous Action.*—In toothed wheels the revolution of one wheel is always accompanied by that of the other in the same or in opposite directions, according as the gearing is inside or outside, or, in other words, the directional relation is always the same. We now pass on to cases in which the directional relation varies, the continuous rotation of one piece being accompanied by an oscillating motion of the other. The rotating piece is then called a "Cam," or sometimes a "Wiper."

Cams are of two kinds. In the first the contact is continuous, and the oscillating motion produced is completely defined by the form of the cam; while, in the second, the contact is only during the forward vibration of the oscillating piece, while the backward vibration is produced by other causes. In both kinds force-closure is common, and sometimes indispensable.

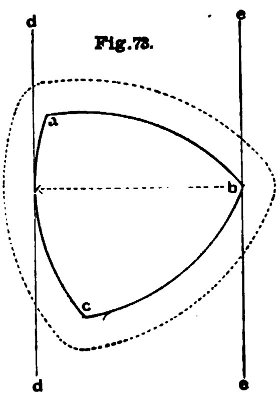
We shall now give some examples of cams of the first kind. Fig. 1, Plate IV. (p. 154), represents a sliding piece C, to which a reciprocating movement is given by a cam B, which rotates about an axis O, perpendicular to the direction of the sliding motion, the chain being completed by the frame-link A. Suppose, in the first instance, that the cam presses against a pin placed in the piece so that a line joining it to the centre of rotation gives the direction of the sliding motion.

As the cam turns in the direction of the arrow, *C* moves downwards to a certain limiting position, after which contact will cease unless some force be applied to keep it pressed against the surface. With suitable force-closure, however, supplied by the spring shown in the figure, *C* will return upwards to a second limiting position, and so on, continuously oscillating to and fro.

By properly taking the shape of the cam, any required relation may be obtained between the motions of the cam and slider; we have, in fact, only to draw a curve of position such as that constructed in Fig. 46, page 97, showing the position of the sliding piece for each position of the rotating piece. This curve will be the proper profile for the cam. In practice the chain is usually augmented by the addition of a friction roller, and the shape of the cam is modified by cutting away its surface to a depth equal to the radius of the friction roller, as was done in the case of the teeth of a wheel which drives a pin-wheel.

Force-closure, though common, is not necessary for the action of a cam chain of this kind; it may be avoided in two ways, both of which occur frequently in practice, though the mechanism would not always be described as a cam. First, the pin of the last example may be made to work in a slot cut in the face of a cam-plate, the centre line of the slot being formed to the profile of the original cam. Secondly, a slot may be cut in the sliding piece at right angles to the direction of sliding, and the cam may fit into the slot. Thus, for example, the cam may be a pin or an eccentric of any size; the chain is then merely a reduced double-slider crank motion, as explained on page 131. With other forms of cam other kinds of motion may be obtained; a common example is the Triangular Eccentric formed by three circular arcs (Fig. 73), each struck from one of the corners of an equilateral triangle *abc*. Such a curved triangle will fit between the sides *dd*, *ee* of a rectangular slot, and may therefore be used as an eccentric by fixing it to an axis passing through any point in it. In practice a figure would be used with rounded off corners, derived by striking small circular arcs with centres *a*, *b*, *c*, and uniting them by larger arcs having the same centres, thus obtaining a profile shown by dotted lines in the diagram, possessing the same essential property of uniform breadth, so that it will fit a rectangular slot of somewhat larger size. The mechanism is shown in Fig. 3, Plate IV.; it is sometimes used for a

Fig. 73.



valve motion, the opening and closing of the valve taking place more rapidly than with a common eccentric. It has also been used in the "man engine" employed in mines to enable the miners to reach the surface without the fatigue of ascending ladders.

In these, as well as all other cam motions, a triangle of velocities can be constructed by the general method explained in Art. 73, and hence curves can be drawn showing the comparative velocities of the cam, the slider, and the rubbing between the two.

75. *Mechanisms with Intermittent Action.*—In all cases of higher pairing by contact between rigid elements, the closure of the chain is imperfect in the absence of external forces, for an exact fit between the surfaces, even if it exist originally, is soon destroyed by wear during the action of the mechanism. Thus, for example, when a pin works in a groove, as in the last article, the smallest loosenesss of fit will prevent the grooved piece from exactly following the movement of the pin when the contact passes from one side to the other of the groove. The same effect is produced by the clearance necessary for the safe action of the teeth of a wheel. In cam mechanisms, where the contact is continually changing from one side to the other, the chain opens for a short interval at every change unless force-closure be employed as described above. The pair of which the oscillating piece forms an element, is locked by friction during the interval.

Suppose now that the groove is purposely made of much greater dimensions than the pin, the oscillating piece will remain at rest for a considerable interval, and will thus have an intermittent motion. The same thing occurs in wheels which work by the successive action of a number of teeth when some of the teeth in one of the wheels are removed. The pair which moves intermittently may be locked during the interval of rest either by friction or by the special means described in the next article.

Intermittent motions of both the cam and wheel class occur frequently in mechanism. Two common examples may be mentioned.

(1.) A wheel with one tooth may be employed to turn another wheel with any number of teeth through a small space at each revolution.

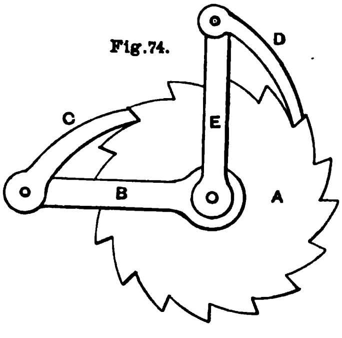
(2.) A wheel with one or more teeth may move a sliding piece alternately backwards and forwards.

In all cases, during the interval of motion, we have a chain of the kind already described which closes at the commencement of the interval. The closure is accompanied by a shock which renders such mechanisms unfit for the transmission of considerable forces, and limits the speed at which they can be run. (See Ch. XI.)

76. *Ratchets.*—The oscillating motion of the piece C may be a turning instead of a sliding motion, as is often the case in shearing machines for example, but no new principle is here involved, and we now proceed to the second class of cam motions in which the forward vibration alone is subject to the action of the cam, while the backward vibration is effected by independent causes, generally by means of springs or of gravity. In such cases the forward vibration follows the same laws as in cams of the first kind, but during the backward vibration the oscillating piece forms a distinct machine by itself, working by means of energy supplied by the cam during the forward movement. In tilt hammers and stampers the work of the machine is done in this way and we need not here further consider them; but the object may be merely to shift the position of the piece and so to lock or unlock a pair, to open or close a kinematic chain. The piece is then called in general a Ratchet, though it may receive other names according to circumstances, and a chain in which it occurs is thus known as a Ratchet Chain.

(1.) The shifting piece may lock a turning or a sliding pair in one or both directions. A common latch for example rises to permit a gate to close and then drops into its place and fastens the gate until again raised by external means.

Fig. 74.



The piece C (Fig. 74) forming a turning pair with a fixed piece B fits in the hollows of the teeth of a wheel A which also pairs with B. The teeth are formed as in the figure so as to permit A to move in one direction by raising C till it drops by the action of a spring or by gravity into the next hollow. In the other direction the pair AB is locked. C is then called a pawl, and the arrangement is the ordinary one employed in windlasses, capstans, and lifts to prevent the machine reversing when the hauling power is removed.

(2.) Two shifting pieces may be employed to lock alternately two pairs which have a common element. This is the ratchet mechanism proper from which the name of the class is derived.

Returning to Fig. 74, A, B, C are the same as in the previous case, E is an additional piece which pairs with B: in the figure the axis of the pair has been supposed concentric with A, but this is not necessary: D is the ratchet pairing with E and at the same time fitting like C into the teeth of the ratchet wheel. If now an oscillating movement be communicated to *E*, the ratchet wheel *A* will be locked alternately with *B* and *E* according to the direction of motion of *E*. Accordingly *A* has an intermittent movement moving with *E* in its forward oscillation and resting in the backward. Instead of a pawl *C*, friction may be relied on to lock *AB* in the backward movement as in the common ratchet brace, but the nature of the mechanism is the same always. It sometimes happens that the pairs *AB*, *BE* are not concentric; the chain *ABED* is then an ordinary four-link chain which opens when moved in one direction and closes when moved in the other, while the pair *CA* unlocks and locks as before, so as to permit *A* to move intermittently. In both cases the movement is single acting, but two such chains may be employed which move in opposite directions and open and close alternately; the movement may then be described as double acting. The well known "Levers of Lagourousse" (Fig. 6, Plate IV.) is a double acting ratchet mechanism in which the two chains have all the links common except the ratchets. The ratchet wheel then moves continuously in one direction, and the locking pawl *C* may be omitted. The ratchet wheel employed in the case of a turning pair may of course be replaced by a rack when a sliding pair is required, but no new principle is here involved.

(3.) The shifting piece may be connected with a pendulum or balance wheel which vibrates in equal times. Time may be thus measured by unlocking a kinematic chain at intervals. In clocks and watches a tooth of the ratchet wheel escapes from the action of the ratchet at each vibration or semi-vibration; the mechanism is therefore called an escapement.

(4.) In pumps various kinds of ratchet mechanisms are universal. The common reciprocating pump is a true ratchet mechanism, the column of water being locked and pairing with the plunger alternately; it may be single or double acting. It is needless to say that the ratchet is here called a "valve."

77. *Other Forms of Ratchet Mechanism.*—In all the examples of the preceding article the shifting piece is not subject to the action of the rest of the mechanism during its return oscillation, but it may also be worked by a cam movement of the first kind, or by linkwork mechanism: the slide valves of a steam engine are a familiar instance. Also it may be worked by external agency instead of by the machine itself, as in all kinds of starting and reversing gear. The ratchet chains form a large and interesting class of mechanical combinations, but their discussion would be out of place here.

78. *Screw Cams.*—The three-linked chain of Art. 73 may have the axes of its lower pairs inclined at an angle instead of parallel and a number of mechanisms of the cam class may thus be derived which are analogous to those already considered. Some of these may also be derived from a screw chain, and may here be briefly mentioned.

Let us take a simple screw chain consisting of a sliding pair, a turning pair, and a right-handed screw pair. Let the screw be of several threads, and let a fraction of the pitch be employed. The screw and the nut may then be alike as shown in Figure 2, Plate IV., each resembling a crown wheel with ratchet teeth. When the movement has taken place through the fraction of the pitch in question, the teeth escape and the nut may be moved back endways by force closure, or by a second screw and nut similar, but left-handed. This movement, which is the only possible cam motion with lower pairing, has been employed to work the shears in a reaping machine,* and is also well known as a clutch.

In its original form the chain consists of a sliding pair *AB*, a screw pair *BC*, and a turning pair *CA*; the piece *A* may however be omitted, and we obtain a two-link chain consisting of a screw pair *BC*, the elements of which are united to those of an incomplete lower pair, *B* and *C* both sliding and turning during the forward motion and simply sliding during the backward motion. Now imagine one of the screw surfaces replaced by a simple pin, then the other may be made of any form we please, and the elements of the incomplete pair will have a cam motion following any given law. A valve motion common in stationary engines is an example. *B* is a revolving crown wheel on which is a projection which raises the rod *C* at the proper time for opening or closing the valve. The "swash" plate usually given in treatises on mechanism is another example.

Plate IV., the figures in which are not taken from actual examples, represents some of the cam and ratchet mechanisms referred to in this section. Fig. 1 is a "heart cam," so called from its shape, in which the sliding and rotating pieces are connected with uniform velocity-ratio. Fig. 2 is the screw cam just described. Figs. 3 and 4 are two forms of the triangular eccentric motion (p. 150). Fig. 5 shows a ratchet motion (p. 152) in a form common in the feed motions of machine tools: the direction of movement of the ratchet wheel *A* is here reversible by putting over the ratchet *D* into the dotted position. Fig. 6 is referred to on page 171.

EXAMPLES.

1. A reciprocating piece moves in guides under the action of a cam attached to a shaft which rotates uniformly, and the centre of which lies in the line of motion. Trace the

---

* Journal of the Franklin Institute for March, 1880.

Plate.IV.

FIG.1.

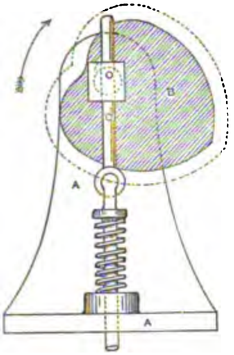


FIG.2.

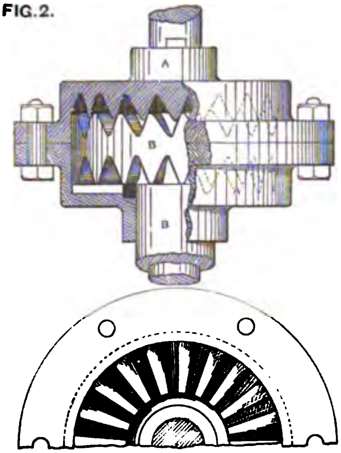


FIG.3.

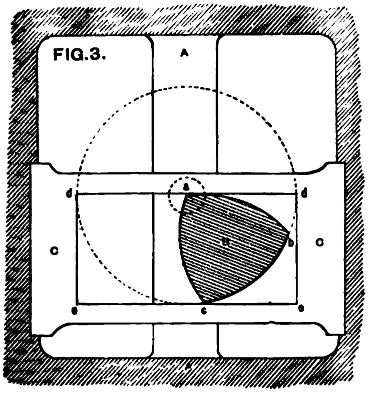


FIG.4.

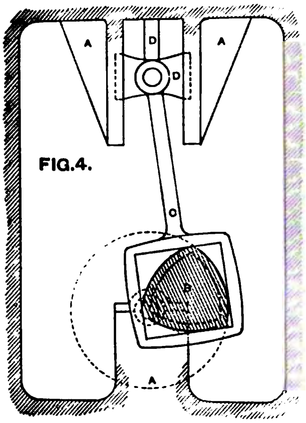


FIG.5.

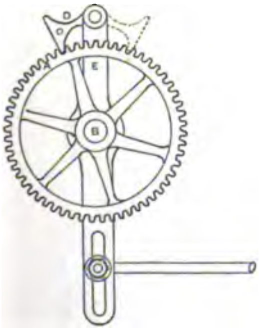
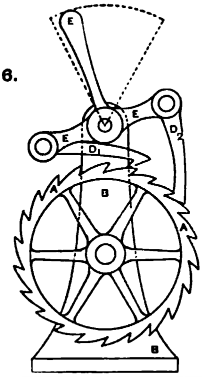


FIG.6.



To face page 154.

form of the cam that the piece may slide uniformly and make one complete movement in each revolution. Suppose a friction roller used of diameter equal to $\frac{1}{6}$ stroke, and suppose also that the least radius of the cam is $\frac{1}{4}$ the stroke.

2. A stamper is raised by a cam such that the rise takes place uniformly during a part of the revolution of a shaft which is distant from the stamper half the rise. Trace the proper form of cam, and find the fraction of the revolution in which the rise takes place.

The best solution is that in which the profile of the cam has the form of the involute of the dotted circle, whose radius is half the lift of the stamper; for then the pressure of the cam on the pin is always in the vertical direction. The rise takes place whilst the cam turns through an angle, the arc of which is equal to twice the radius, or $\frac{1}{\pi}$ of a revolution.

3. Draw a curve of velocity for a reciprocating piece moved by a uniformly rotating triangular eccentric.

CHAPTER VII.

MECHANISM IN GENERAL.

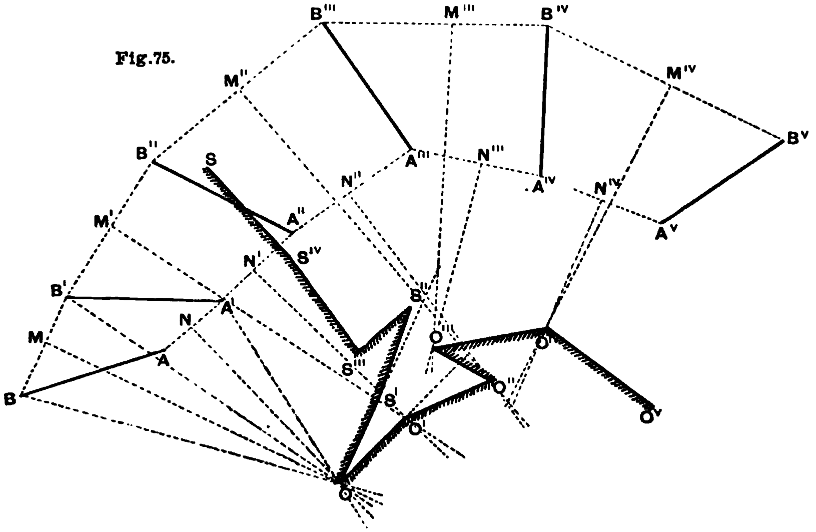
79. *Plane Motion in General. Centrodes.*—In the two preceding chapters the mechanisms considered have been composed either wholly of lower pairs or else of two lower pairs connected by higher pairing. The velocity-ratios of the various lower pairs have been considered and diagrams of velocity have been drawn for the complete mechanism, but without attempting to form any idea of the comparative motion of pieces which do not pair with each other, or which form the elements of a higher pair. It will now be necessary to consider the comparative motion of two pieces more generally.

First, suppose the two pieces to move in such a way that a plane attached to one moves parallel to a plane fixed in the other. The motion is then the same as that of a plane area which slides on a fixed plane, and may be called for brevity "plane motion." If the position of any two points in the moving area be given, all the rest can be found, and the motion is therefore completely defined by the movement of the straight line joining these points.

Let AB, $A'B'$, $A''B''$ ...(Fig. 75) be successive positions of such a line. Join AA', BB', and from the middle points of these lines draw perpendiculars NO, MO to meet in O, then $OA = OA'$ and $OB = OB'$, from which it can be proved that $AOB = A'OB'$, so that AB might be moved to $A'B'$ by attaching it to a plane area, and rotating that area about O as a centre. Obtain similar centres $O'O''O'''$ ...for the succeeding changes of position, then it is clear that the motion of AB, and therefore of the plane area to which it is attached, may be completely represented by the rotation of the area about the centres O, O', O'' ... in succession through certain angles which are given, being the inclinations to each other of the successive positions of AB.

Next, through O draw OS', making it equal to OO' and inclined to OO' at the first angle of rotation, $S'S''$ equal to $O'O''$ and inclined to it at an angle equal to the sum of the first and the second angle of rotation, and so on; we thus obtain a second polygon $OS'S''\ldots$, the sides of which are equal to those of the original polygon $OO'O''\ldots$

Fig.75.



Imagine this polygon rigidly attached to AB so as to move with it, then during the motion the polygon will rotate about O till S' reaches O', then about O' till S'' reaches O'', and so on in succession; that is to say, the changes of position of AB may be produced by the rolling of one polygon upon the other. Thus, by properly determining the polygons, any given set of changes of position of a plane area may be produced at pleasure by rolling the moveable polygon on the fixed one.

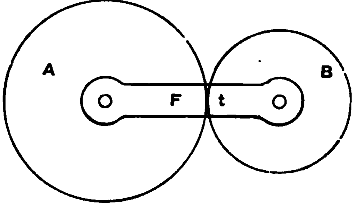
Now imagine the moving area to become fixed in its original position, and let the originally fixed area move by rolling the polygon $OO'O''\ldots$ which is attached to it upon the polygon $OS'S''$ which is now fixed. Evidently the two areas take up the same relative positions, and we obtain the very important proposition that any set of changes of relative position of two areas may be obtained by the rolling of one polygon upon another. If the positions are taken at random the polygons may have acute angles as at O'' in the diagrams, but they may also be such as would occur in a continuous motion, and the polygons will then reduce to continuous curves when the positions are taken very near together. Thus every continuous plane motion of two pieces is represented by the rolling of one curve upon another, the point of contact being a centre about which either piece is for the instant rotating relatively to the other. These curves are called Centrodes, and the point is called the Instantaneous Centre. Whenever the directions of motion of two points in a moving piece are known, the instantaneous centre is at once determined by drawing perpendiculars to intersect. During the motion it traces out two curves, one in the moving piece, the other in the fixed piece, which curves are the centrodes of the motion.

80. *Axoids. Elementary Examples of Centrodes.*—Any two bodies moving in the way described may be divided into slices by planes parallel to the plane of motion, the centrodes of which will of course be all similar and equal, so that we may regard them as the transverse sections of cylindrical surfaces in contact with each other along a generating line. The surfaces are called Axoids, and the line the Instantaneous Axis. The relative motion of the bodies is represented by the rolling of the axoids upon one another, endways motion being supposed prevented.

Any two parts of a mechanism have a relative motion which is completely defined by the nature of the mechanism, as has been sufficiently explained already; and it follows, therefore, that they must have given axoids, the nature of which completely defines the motion of the pieces. In every kinematic chain there are as many sets of axoids as there are sets of two pieces, and these surfaces are the same for all the mechanisms derived from that chain by inversion. These remarks apply even when the motion is not plane, as will be seen further on.

First. Take the case of a pair of spur wheels *AB* in gear, *F* being the frame-link (Fig. 76), forming the three-link chain considered in the last chapter. Let the pitch circles touch at the pitch point *t*, then, as before explained, those circles roll together without slipping, and therefore must themselves be the centrodes, the pitch surfaces being the axoids. Hence the point *t* is the instantaneous centre of *B*'s motion relatively to *A*, or *A*'s motion relatively to *B*. We shall return to this immediately, but for the present merely remark that if the centres of *A* and *B* move up to each other, the pitch circles reduce to points, and the axoids become coincident straight lines, the point *t* is fixed in *A* and *B*, the two pieces then become a turning pair. In lower pairing, then, the axoids are coincident straight lines, which are at infinity if the pair be sliding. The case of a screw pair in which the motion is not plane will be referred to further on.

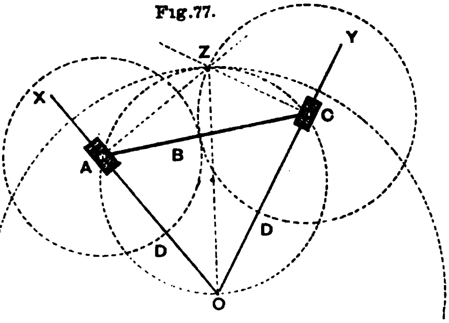
Fig. 76.



Secondly. Take the case of a double-slider chain; there are here four pieces which may be taken two and two in six ways; there are, therefore, six sets of axoids. Four of these, however, are only the four axes of the four lower pairs, and it remains to determine the other two.

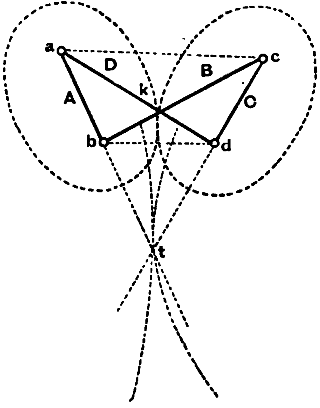
In Fig. 77 the blocks *A*, *C* are connected by a link *B* and slide on a piece *D* along lines *OX*, *OY*, forming the chain described fully in a former chapter. The blocks *A*, *C* form two turning pairs with the link *B*, and the velocities of these pairs are equal because *B* makes angles with *OX*, *OY*, the difference of which is constant. The centrodes for *A*, *C* are therefore equal circles, the centres of which are the centres of *A*, *C* (see note, p. 171). Next, to find the centrodes of *B*, *D*, through those centres draw perpendiculars to *OX*, *OY* to meet in *Z*, then *Z* is the instantaneous centre for *B* when *D* is fixed, and for *D* when *B* is fixed. First, suppose *B* fixed, then the angle at *Z* is the supplement of the angle at *O*, and is therefore constant, so that *Z* traces out an arc of a fixed circle, of which *OZ* is the diameter. Next, suppose *D* fixed, then, since *OZ* is constant, *Z* traces out a circle, the centre of which is *O*. Thus the centrodes of *B*, *D* are two circles, one half the diameter of the other; the large circle is fixed to *D*, and the small circle to *B*.

Fig. 77.



Thirdly. In the four-link chain *A*, *B*, *C*, *D*, consisting of four turning pairs with parallel axes, the sections of which are represented by the points *a*, *b*, *c*, *d* (Fig. 78); suppose opposite links equal, but set so as not to be parallel. This is the case referred to already (page 112) as "anti-parallel" cranks.

Fig. 78.



Joining *ac*, *bd* by the dotted lines in the figure, the quadrilateral *abdc* has two sides and two diagonals equal, hence the triangles *bac*, *cda* must be equal in every respect, so that *bd* is parallel to *ac*. Hence if *k* be the intersection of the diagonals, and *t* the intersection of the sides, $ak = ck : bk = dk : bt = dt$, from which it appears that

$$ak + bk = ck + kd = ad = bc$$

and

$$at - dt = ct - bt = ab = cd.$$

Suppose, now, *A* to move while *C* is fixed, then *a* moves perpendicular to *ad*, and *b* moves perpendicular to *bc*, so that *k* must be the instantaneous centre for the motion of *A* relatively to *C*, or for that of *C* relatively to *A*. Now, in the first case, it appears from what has been said that *k* traces out an ellipse, of which *c* and *d* are foci, while, in the second, it traces out an equal ellipse, of which *a* and *b* are foci. Thus the centrodes for the motion of *A* and *C* are equal ellipses, as shown in the diagram. In like manner the centrodes for the motion of *B* and *D* are the equal hyperbolæ traced out by the point *t*.

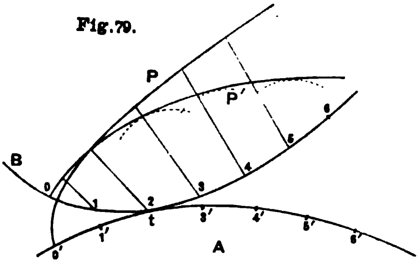
The four other pairs of centrodes are the points *a*, *b*, *c*, *d*, which are the centres of motion of the four turning pairs.

81. *Profiles for given Centrodes.*—Any given motion of one piece relatively to another may be produced in an infinite number of ways. One way of doing this is by rolling contact, for if the motion is given the centrodes will be given, and by forming the profiles so as to represent the centrodes, and applying forces to press the pieces together and cause them to roll on one another without slipping, the given motion may be produced. But if slipping be permitted, the same motion may be produced, at least theoretically, by assuming any form whatever for one profile and properly determining the other.

(1.) Let a given profile be attached to the moving piece, and as it rolls into different positions let that profile be traced on paper attached to the fixed piece. If the positions be taken near enough together, a curve may be drawn through their ultimate intersections which will envelope them all, and if a profile formed to that envelope be attached to the fixed piece, the two pieces will fit one another and yet be capable of relative motion of the prescribed kind.

(2.) To apply the foregoing process a model would be necessary, but by a simple modification, a geometrical construction may be obtained. In Fig. 79, *A* and *B* are the pieces, which move so that the centrodes are 0, 1, 2, 3 ..., 0′, 1′, 2′, 3′ ..., curves which are shown touching at the point *t*. *P* is a profile of given form attached to *B*; it is required to find a profile attached to *A*, which will always remain in contact with *P*, and so be capable of moving it in the required way by simple contact.

Fig. 79.



Divide the centrode of *B* into arcs of equal length, starting from *O*, the point where P intersects it, and let 2 be the point of contact at the instant considered. Divide the centrode of A into equal arcs, stepping from 2 in both directions, then 0′, 1′, 3′, 4′ ... are points in A's centrode, which correspond to 0, 1, 3, 4 ... in B's centroid, being during the motion points of contact in succession. From 1, 2, 3 ... drop normals on to the curve P, and with these normals as radii trace circular arcs with centres 1′, 2′, 3′ ... ; the envelope of these arcs must be the required profile P'.

(3.) Instead of assuming one profile and determining the other to suit it, it is generally more convenient to employ some method of determining pairs of profiles which satisfy the required conditions.

In Fig. 80 A, B are the centrodes as before, C is a third curve, theoretically of any form, which rolls on A and B, always touching these curves at their point of contact t. P is a tracing point which is attached to C and traces out two curves during the motion, one on A, the other on B. First, suppose A fixed, then, since t is the instantaneous centre of the motion of C, Pt must be normal to the curve NP traced out on A. Similarly supposing B fixed, Pt is normal to the curve MP traced out on B. Thus the two curves touch one another at the point P and therefore may be taken as profiles which will give the required motion. If A, B, C be circles, this construction becomes that already considered when discussing the form of teeth for a wheel. This and the preceding method show clearly that the condition which the two profiles must always satisfy is that the common normal at the point of contact must always pass through the pitch point as already proved otherwise for the special case of wheel teeth.

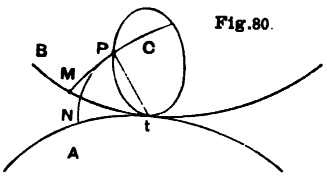


Fig.80.

Not every pair of curves which satisfy the geometrical conditions could actually be used as profiles, either for centrodes, or, in the cases just mentioned, to give a required motion, because there is nothing in the geometrical construction which excludes an interpenetration which would not be physically possible in the areas of which the profiles form the boundaries, but an infinite variety of forms can be found, for given centrodes, which might be so used.

In all cases in which the centrodes are known for the relative motion of two pieces, one of which is fixed, the velocity-ratio of any two points (a, b) in the moving piece is known for each position of the pieces. For, joining the two points to the instantaneous centre O, the ratio of the distances Oa, Ob must be the velocity-ratio in question, since the moving piece is for the moment turning about O. It is easily seen that the triangle *Oab* is similar to the triangle of velocities constructed as in Art. 49, p. 100.

82. *Centrodes for a Higher Pair Connecting Lower Pairs.*—Among the infinite variety of profiles which correspond to given centrodes it is frequently possible to find some which are closed curves, one completely surrounding the other. If these curves be used as the external and internal boundaries of two areas, the two pieces thus formed will fit one another and be capable of no motion except that of the prescribed kind without requiring any additional constraint. In Figure 4, Plate IV., a form of the triangular eccentric motion is shown, which has been occasionally used and which furnishes an example. On reference to Art. 74 it will be seen that such an eccentric will exactly fit a square within which it is enclosed, and therefore forms with it a higher pair which is "complete" in itself.

Complete higher pairs are very unusual in mechanism, higher pairing being employed almost exclusively to complete a chain of lower pairs as in the preceding chapter. It is then generally "incomplete," the necessary constraint being furnished by the rest of the kinematic chain to which it belongs, as for example in the triangular eccentric motion shown in Figure 3, Plate IV. The general problem in mechanism is not to connect two pieces in a given way, but to convert the motion of a given pair into the motion of a different pair—that is to say, to connect two pairs so as to have a prescribed relative motion. This will be further considered presently, but we must first return for a moment to a question considered in the last chapter.

In the three-link chain of Art. 73 we have two lower pairs AC, BC with axes parallel, connected by simple contact between A and B at the point P (Fig. 72, p. 148). Draw the common normal PT to meet ad in T, then when B is fixed the motion of a is perpendicular to ad, and the motion of P perpendicular to PT, therefore T must be the instantaneous centre for the motion of A relatively to B. Let v be the velocity of rubbing at P; A, A' the angular velocities of the pairs AC, BC: further let $ad=l$ and $PT=z$; then, since B is fixed and A is rotating round T,

$$\frac{v}{z}=\frac{\text{velocity of } a}{aT}=A'.\frac{l}{aT}.$$

Similarly supposing A fixed,

$$\frac{v}{z}=\frac{\text{velocity of } d}{dT}=A.\frac{l}{dT},$$

from which it appears that

$$\frac{A}{A'}=\frac{dT}{aT};\quad v=z(A'-A),$$

results which agree with those obtained in the article cited by a different method.

The centrodes in this case, as well as in that of the four-link chain from which it was derived by reduction, may be traced graphically by plotting the position of *T* for a number of positions of the pieces, but they are known curves only in exceptional cases such as those of Art. 80, and generally have infinite branches which render their use inconvenient.

When the point *P* lies on the line of centres it coincides with *T*, and the velocity of rubbing is zero; the centrodes are then no other than the profiles themselves of *A* and *B*. The curves are then said to roll together: a particular example is that of the equal ellipses of Art. 80 which are not unfrequently used to connect two revolving shafts with variable angular velocity ratio. In this case the velocity-ratio is the ratio of the focal distances of the point of contact, but by properly determining the profiles it is theoretically possible to give any velocity-ratio to the shafts at pleasure. The question, however, is not one of much practical interest.

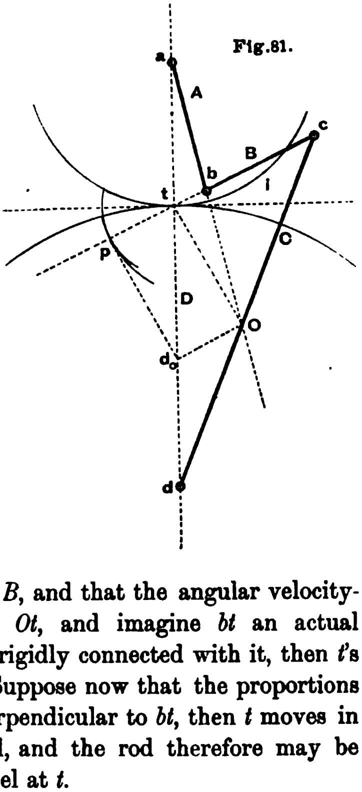


Fig. 81.

83. *Construction of Centres of Curvature of Profiles—Willis's Method.*

—In the four-link chain *ABCD* shown in Fig. 81, *D* is the fixed link and *B* the coupling link: *a*, *b*, *c*, *d* are sections of the axes of the pairs which are supposed parallel.

If the coupling link *bc* be prolonged to meet the line of centres *ad* in the point *t*, and *ab* to meet *cd* in *O*, it appears as in previous cases that *O* must be the instantaneous centre of *B*, and that the angular velocity-ratio of *A* and *C* is *dt* : *at*. Join *Ot*, and imagine *bt* an actual prolongation of the bar *bc*, so that *t* is rigidly connected with it, then *t*'s motion will be perpendicular to *Ot*. Suppose now that the proportions of the links are taken so that *Ot* is perpendicular to *bt*, then *t* moves in the direction of the length of the rod, and the rod therefore may be imagined to slide through a fixed swivel at *t*.

This reasoning shows that the levers *A* and *C*, when in this position, will move for a short interval with uniform angular velocity-ratio, and the movement of a pair of wheels in gear is thus imitated by a linkwork mechanism.

Let us now form a reduced chain by omission of the coupling-link, and we shall be able to solve the important problem of finding a pair of circular arcs which will serve for the profiles of a pair of teeth in contact. For this purpose, with centres *b* and *c*, strike arcs through any point *p* on *cbt* produced, and let these arcs be rigidly connected with *A* and *C* respectively; the coupling-link may now be removed, and *A* imagined to drive *C* by direct contact of the arcs. Evidently wherever *p* is, the pieces will move for the moment with uniform angular velocity-ratio and pitch point *t*. The uniformity, however, is only momentary, because the position of *O* changes, and to trace the profiles with accuracy it would be necessary to perform the construction for a succession of positions of *cbt*, hence the face and flank of a pair of teeth in contact cannot be exactly represented by a pair of circular arcs. When it is sufficiently approximate to do so, the arcs are found by assuming a mean position for the point *p*, and a mean value for the obliquity *i*, found by experience to give good results. The method here described was invented by the late Professor Willis, and the value of *i* recommended by him was $\sin^{-1}\cdot 25$, or about $14\frac{1}{2}^\circ$, being about the actual mean value of the obliquity in cycloidal teeth of good proportions. Also the value of *pt* was taken by him as half the pitch, *p* being then about midway between the pitch point *t* and the point of the tooth.

Having made these assumptions, it still remains to fix the position of the point *O*, which may be taken anywhere on a line through *t* inclined at $14\frac{1}{2}^\circ$ to the line of centres. This is done by observing that *O* must be the same for all wheels *D* intended to work with a given wheel *A*, and that teeth never should be undercut (Art. 70); that is, *c* and *b* must lie on the same side of *t*. Hence in the smallest wheel intended to work with *A*, *c* is at infinity, so that if d_0 is its centre, d_0O is parallel to *pt*, and therefore perpendicular to *Ot*. The flank of the tooth in this case becomes a radius d_0p. The position of *O* is thus completely determined for all the wheels of a set when the pitch is given.

Willis's method is of great theoretical interest, and has consequently been given here, but the form of teeth obtained is not always sufficiently approximate. It may, therefore, with advantage be replaced by other methods, as to which the reader is referred to a work by Professor W. C. Unwin on Machine Design.

84. *Sphere Motion.*—When a body moves about a fixed point its motion is completely represented by that of a portion of a spherical shell of any radius which fits on to a corresponding sphere, and moves on it just as in the case of plane motion. Everything which has been said respecting plane motion also applies to sphere motion, but the axoids are conical instead of cylindrical surfaces, the centrodes spherical instead of plane curves, and all straight lines are replaced by great circles of the sphere on which the motion is imagined to take place. The corresponding crank chains are called "conic" crank chains, the axes of the pairs lying on a cone instead of a cylinder.

85. *Screw Motion.*—In the plane motion of two pieces, endways motion of the cylindrical axoids is supposed to be prevented by some suitable means. Let us now remove this restriction and imagine the axoids to slide endways, while continuing to roll together, the relative movement will now not be completely defined, but additional constraint will be required. In the first place take the case of a lower pair in which the axoids are coincident straight lines; if endways sliding be permitted we obtain an incomplete pair, unless the nature of the surfaces in contact define the relation between the endways motion and the rolling motion. In the simple screw pair the two are in a fixed ratio, in the screw cams of Art. 76 they have a varying ratio. In every case of non-plane motion with cylindrical axoids, not only must the axoids be given, but also a connection between the endways sliding and the motion of rotation.

In the most general case possible the instantaneous axis changes its direction as in spherical motion, its position as in plane motion, and in addition there may be an endways sliding. This is expressed by the rolling and sliding of certain surfaces on one another, which are now neither cylindrical nor conical. These surfaces are in all cases of the kind known as "ruled" surfaces, being generated by the motion of a straight line, along which they touch each other. The surfaces are still called Axoids, and the line is the Instantaneous Axis. The hyperboloidal pitch surfaces for wheels connecting two shafts which do not intersect are examples of this kind; but for the discussion of this question, which is not of very common occurrence, the reader is referred to the works already cited.

86. *Classification of Simple Kinematic Chains.*—On observing the action of any mechanism, several of the pieces of which it is constructed may be readily distinguished as having functions different from the rest. These pieces, like the rest, occur in pairs, and may be described as such, though the pairing is not necessarily kinematic. First, one or more perform the operations which are the object of the mechanism, these may be called the Working Pairs, as, for example, the tool and the work in machine tools, the weight raised and the earth in the hoisting machines.

Second, one or more form the source from which the motion is transmitted, as, for example, the crank handle and frame of a windlass, the piston and cylinder of a steam engine. These may be called the Driving Pairs. Thirdly, various subsidiary working pairs carry out various operations incidental to the working of the machine. The object of the mechanism is always to convert the motion of the driving pairs into that of the working pairs.

The simplest case is that in which the motion has only to be transmitted without alteration; a single pair will then suffice. Thus, by means of a long rod sliding in guides or turning in bearings, a motion of translation or rotation may be transmitted to a distance only limited by non-kinematical considerations. By use of flexible elements—among which should be included the flexible shafts recently introduced—the direction may be altered at pleasure and any desired position reached.

If, however, the magnitude of the motion is to be altered, a mechanism must be employed in which at least one element of the driving and working pairs is different. The driving pairs are usually kinematic lower pairs and the working pairs are so very frequently, and this is why so many of the simplest and most important mechanisms are examples of the connection of lower pairs. The peculiar motions of lower pairs being translation and rotation, a number of mechanisms may be classed as examples of the conversion of rotation into translation or rotation and conversely, with uniform or varying directional relation or velocity-ratio. This is especially the case when, as so frequently happens, the driving and working pairs have a common link which is fixed.

It has been shown, however, that many apparently different mechanisms are in reality closely connected, being derived from the same kinematic chain. Mechanisms are therefore to be classed according to the kinematic chains to which they belong. The number of simple chains actually employed in mechanism is limited by the preceding considerations to those already described, which are ranged by Reuleaux in the following classes, the names of which are derived from the most important piece in some example of common occurrence:—

(1) Crank chains.
(2) Screw chains.
(3) Pulley chains.
(4) Wheel chains.
(5) Cam chains.
(6) Ratchet chains.

In the first two are included all combinations of sliding, turning, and screw pairs; in the third, all cases where tension or pressure elements are employed; in the fourth, all cases of connection by contact where the directional relation remains the same; in the fifth, all cases where it varies; while in the last, all combinations are included where, by action of a shifting piece, the law of motion is periodically varied.

87. *Compound Kinematic Chains.*—In a complete machine, the motions required are generally too complex to be carried out by a single kinematic chain of this simple kind; it is necessary to combine together a number of such chains, and we conclude this part of the subject with some general remarks on such combinations which may all be regarded as compound chains derived from two or more simple chains by union of their links.

(1.) From any two closed chains a third may be derived by uniting two links. The links must have the same relative motion, for otherwise the chains would lock each other, and they generally form a pair.

This is one of the commonest of all combinations. When two machines are driven from the same shaft, or when the same shaft is driven by two separate engines, we have examples in which the driving pairs or the working pairs are common, but the mechanisms are otherwise independent. Further, in every complete machine we find, in addition to the principal chain which does the work, a number of auxiliary chains which carry out various operations necessary to the working of the machine. Thus, in the steam engine, besides the slider-crank or other mechanism which turns the crank, we have the valve motion which governs the distribution of steam, the air pump motion which produces the vacuum in the condenser, and frequently others as well. Each of these auxiliary mechanisms has a pair in common with the principal chain which serves as a driving pair, but the chains are otherwise independent. Again, in trains of mechanism which, as previously remarked (page 135), are frequently simple chains augmented for non-kinematical reasons, a number of such chains are arranged so that the working pair of one chain is the driving pair of the next in succession. A train of wheels or the mechanism of a beam engine are examples already referred to, in which one link is common to all the separate chains, but cases occur in which this is not so, as, for example, the well-known Lazy Tongs.

The case here considered is that where the movements of various driving pairs have to be transmitted to various working pairs, but no new motion is required in a working pair other than could be produced by a simple chain. All such combinations may be called Trains, and may be divided into "converging," "diverging," and "transmitting" trains.

(2.) If two closed chains have only one link common they are completely independent, like two machines standing on the same floor, but disconnected. It might, therefore, be supposed that nothing was obtained that was new. In fact, however, this is a combination which is as common as the preceding, being employed to give a motion to a working pair which is too complex to be produced by simpler means, or which requires to be varied at pleasure. The working pair consists of two elements, one of which is supplied by one chain, the other by the other, and the motion of the pair is thus a combination of the motions of the two independent chains. Completely new motions are obtained in this way, and they may be varied at pleasure by alteration of either or both of the primary motions.

Take, for example, the common planing machine. The working pair consists of the table, upon which the work is mounted, and the tool. To the first a reciprocating movement is communicated by means of a suitable kinematic chain connecting it with the driving shaft. The other is mounted on a slide rest, forming an element of a screw chain, which gives it a horizontal movement. This chain has one link in common with the principal chain, but is otherwise independent. In the ordinary working of the machine this chain is locked by friction, except at the end of each reciprocating movement of the table when it moves to take the next cut. The tool thus traces out a complete plane surface.

In this example the common link is fixed, but this need not be the case, and in fact in the planing machine a third independent chain is added to adjust the tool vertically, the tool being mounted on a vertical slide having an independent movement. Also, one element of the working pair may be fixed, and both movements given to the other, which is common to both chains. Double and treble chains of this kind occur whenever it is necessary to move the elements of the working pair into all possible positions. In cranes of all kinds we find a treble movement, one to raise and lower the jib, a second to swing the jib round, and a third to raise or lower the load. In traversing cranes the three movements are rectangular, as in the planing machine. In either case we find the methods employed by mathematicians to define the position of a point in space by rectangular or polar co-ordinates exactly imitated by the mechanism.

The elements of the working pair need not be wholly disconnected as we have hitherto supposed, they may form an incomplete kinematic pair. Thus if the axoids be cylindrical, endways motion may still be possible and may be given by an independent chain. A common example is a drilling machine, the working pair in which consists of a table on which the work is mounted, and a spindle carrying the drill which rotates and at the same time descends as the hole is drilled; the two movements may be quite independent, the one proceeding from a driving shaft, the other operated by the workman.

A similar combination is employed when a train is varied by shifting one of its links. Fig. 5, Plate III. (p. 136), represents a case of this kind. The wheel *C* is mounted on a shaft which can be shifted endways by an independent mechanism. The shifting of belts (Art. 61, p. 129 is another example.

Again, the movements of the working pieces may be connected by a transmitting train connecting the chains which produce them. In the self-acting feeds of planing and shaping machines the connection is intermittent, but it may also be continuous, and we then have a fertile means of producing complex movements variable at pleasure. In a screw-cutting lathe the tool is mounted on a slide-rest moved by a screw, and the work is attached to a rotating mandrel. Connecting these independent chains by a train of "change" wheels, the tool cuts a screw of any pitch.

The principle of all combinations of this kind is the closure of an incomplete or disconnected pair by independent chains. We may describe them as Multiple Chains.

(3.) If two closed chains have two or more pairs common, they must be of the same kind, for otherwise the pairs would not have the same relative motion, and the chains would lock each other. The differential mechanisms, examples of which have been already given, are cases of this kind. Thus in the differential pulley (Fig. 62, p. 127), if *A* and *C* be disconnected we have two simple pulley chains with common moveable pulley *B* and separate axles. Either of these might be operated independently. In the actual mechanism *A* and *C* are united, and the movement of *B* is the difference of the movements due to each separate chain.

Complex examples of similar combinations occur in the epicyclic mechanisms. Fig. 82 (p. 170) shows a combination of two of the differential trains described on p. 135. *C*, *C'* are wheels turning about the same axis in the frame-link *A* and united; *E*, *E'* are also united, but have a different frame-link *A'*. Both gear with the wheels *B*, *D*, which are disconnected, but turn on an axis common to *A* and *A'*. On comparing this with Fig. 65 it will be seen that two trains have been compounded by uniting the wheels *B*, *D*, which are common to both. If now one of the frame-links, say *A'*, is fixed, and *EE'* be rotated, the other frame-link *A* will rotate with a velocity which can be found on the principles of the article cited. For simplicity, *EE'* have been supposed to gear directly with *B*, *D*, but they may also gear with wheels of other diameters fixed to *B*, *D*, or the wheels may be replaced by a different train of mechanism, all that is necessary being that the motions of the pairs *BA'*, *DA'* should be connected.

Many examples of this mechanism may be found—especially in the case where *C'*, *C*, are equal and the train reduces to three bevel wheels (p. 136). In traction engines and tricycles, for instance, a mechanism of this kind is sometimes employed to facilitate turning. *A'* is then the frame of the machine, *B* and *D* are equal bevel wheels attached to the axle, which is divided into halves, each connected with one of the driving wheels. If now the motive power be applied to *A*, *B* and *D* will rotate, but not necessarily with the same velocity, and the machine may therefore be guided in a curve by the front wheel without the slipping which would occur if the driving wheels were fixed to an undivided axle.

Combinations of this class are not essentially different from multiple chains in which the elementary chains are connected by a train, as described above. They may be called Compound Trains ; all consisting of simple trains compounded in various ways, either for non-kinematical reasons or to enable the train to be varied at pleasure.

(4.) All the preceding combinations are formed of simple closed chains united together in various ways ; no new chain is obtained, but merely an aggregation of forms already known. Certain mechanisms, however, occur which if taken to pieces by separation of united links are found to contain one or more chains which are not closed.

Take for simplicity a common slider-crank mechanism, and imagine the crank pin, instead of being fixed to the crank, to be mounted on a slide so as to be free to move to and from the centre. The chain is now augmented by an additional sliding pair, and is no longer closed, so that it cannot be used as a mechanism. If, however, we introduce a screw, which moves the slide, we may lock the sliding pair in any position and thus obtain a closed chain, one link of which can be varied at pleasure.

Fig.82.

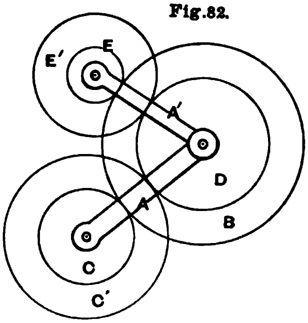
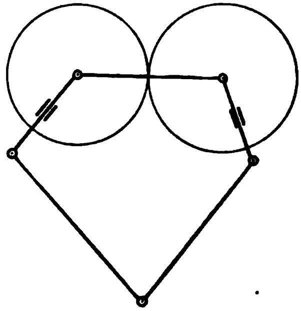


Fig.83.



This mechanism is used in practice to obtain a varying stroke in a sliding piece. It is often required to make the stroke increase or diminish at each revolution of the crank. A wheel attached to the screw then comes in contact with a projecting piece and moves through a small space, the screw chain being locked by friction during the rest of the revolution. The mechanism thus varies at intervals its own law of motion.

By a suitable transmitting train however a continuous variation may be produced, and the combination then furnishes us with an entirely new mechanism. An important example is the wheel crank chain (Fig. 83), formed by combining a simple wheel chain with an open crank chain of five links. A number of mechanisms may be derived from this chain by inversion, but for particulars the reader is referred to Reuleaux's work already cited.

Another example is shown in Fig. 6, Plate II. (p. 110), which represents a mechanism employed in sewing machines to give two strokes to a sliding piece for one revolution of a shaft. We have here a closed double slider chain combined with a single slider rendered incomplete by omission of the crank pin. Combinations of this class are called by Reuleaux "true" compound chains to distinguish them from the preceding classes, in which no new mechanism results from the combination. Perhaps the words "higher" and "lower" would more clearly express the meaning.

NOTE.—In Fig. 77, page 159, the blocks A, C revolve in the same direction, and the centrodes are circles of infinite size. To represent them equal circles of finite size are employed, which give the same motion but in opposite directions.

PART III.

DYNAMICS OF MACHINES.

PART III.—DYNAMICS OF MACHINES.

CHAPTER VIII.

PRINCIPLE OF WORK.

SECTION I.—BALANCED FORCES (STATICS).

88. *Preliminary Explanations. Definition of Work.* If the principal object of a piece of mechanism be to do some kind of work it becomes a machine. Many mechanisms—as for example clocks and watches— are not, properly speaking, machines; for though work is done during their action, yet the object of the mechanism is not the doing of the work but the measurement of time or some similar operation. Even in these cases, however, the forces in action cannot in general be excluded from consideration, and therefore in all mechanism a study of the manner in which forces are transmitted and modified is essential. This part of the subject is called the Dynamics of Machines.

A body can in general only be moved into a different position or be changed in form or size by overcoming resistances which oppose the change. This process is called doing WORK, and the amount of work is measured by the resistance multiplied by the space through which it is overcome. If there be many resistances, the total work done is the sum of that done in overcoming each resistance separately.

Consider the case of a weight raised vertically. Here the resistance is due to the action of gravity which is overcome by some external force, and the work done is simply the product of the weight and the height through which it is raised. The weight is measured by comparing it with that of a certain quantity of matter called a pound, the weight of which is taken as a unit for measuring forces. This mode of measurement has the disadvantage of giving a different unit for different points on the earth's surface, because the force of gravity varies according to the position of the point, and, for scientific purposes therefore, force is measured by the velocity which, when unbalanced, it produces in a given quantity of matter. In practical applications, however, gravitation measure is preferable, especially as the variation is very small, and the measure may be made precise when necessary by specifying the place on the earth's surface at which our operations are taking place. The unit of space is generally 1 ft., so that the unit of work is 1 lb. raised through 1 ft., or, as it is generally called, 1 foot-pound. Other units, however, such as, for example, "foot-tons," may also be employed for special purposes.

89. *Oblique Resistance.*—In the case just considered, the resistance is directly opposed to the movement which is taking place; if this be not so, it must be resolved into two components, one along and the other perpendicular to the direction of motion. The second of these is balanced by a constraint to which the motion is subject or by the opposition which the inertia of the body offers to a change in its direction at any finite rate; it is the first alone in overcoming which work is done. In Fig. 84 let R be a resistance applied at a point A which moves through a distance AB in a direction inclined at an angle θ to the direction of the resistance, then the work done is $R.\cos\theta.AB$, but if BN be drawn perpendicular to the direction of R to meet that direction in N,

$$AN = AB.\cos\theta,$$

and therefore the work done is $R.AN$.

Now AN is the distance through which A has moved in the direction of the resistance, so we obtain another rule for estimating the work done against an oblique resistance. It is equal to the product of the resistance into the distance moved in the direction of the resistance.

Fig.84.

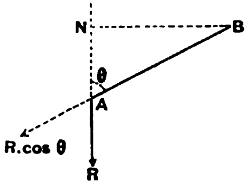
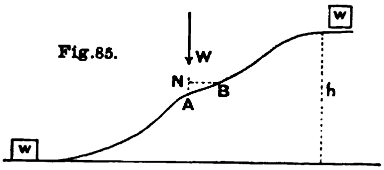


Fig.85.



Suppose for example that a weight is raised, but that, instead of being lifted vertically, it is moved in any curved path—there being no friction or other resistance than that due to gravity.

Considering any small portion AB of the path (Fig. 85), the resistance being always vertical, the work done is $W.AN$. So the total work of raising the weight is $W.\Sigma AN$ or $W.h$, which is independent of the path described by the lifted weight, but depends simply on the height through which the weight is raised.

If there are a number of weights each of them raised through different heights, the total work done in raising all the weights is the sum of the works done in raising each weight separately; and the direct method of finding the total work is to add the separate results for each weight. But it may be determined by another method thus—

Let W_1, W_2, W_3 &c. be a number of weights which are at heights y_1, y_2, y_3 &c. above a given datum plane. Now suppose they are raised so that they are at heights Y_1, Y_2, Y_3 &c. above the same plane. The total work done in raising the weights will be the sum of the products,

$$W_1(Y_1 - y_1) + W_2(Y_2 - y_2) + W_3(Y_3 - y_3) + \&c.$$

Now suppose the centres of gravity g and G for the initial and final positions of the weights to be at heights $\bar{y}$ and $\bar{Y}$ above the datum plane.

The centres of gravity g and G are such that if all the weights were collected at either centre, the moment of the collected weights about the plane is equal to the sum of the moments of each separate weight, before being collected, about the same plane. This is mathematically expressed thus

$$\bar{y} = \frac{W_1y_1 + W_2y_2 + W_3y_3 + \&c.}{W_1 + W_2 + W_3 + \&c.}$$

$$\text{and } \bar{Y} = \frac{W_1Y_1 + W_2Y_2 + W_3Y_3 + \&c.}{W_1 + W_2 + W_3 + \&c.}.$$

By subtracting we have

$$\bar{Y} - \bar{y} = \frac{W_1(Y_1 - y_1) + W_2(Y_2 - y_2) + W_3(Y_3 - y_3) + \&c.}{W_1 + W_2 + W_3 + \&c.};$$

hence the total work done in raising the weights may be expressed

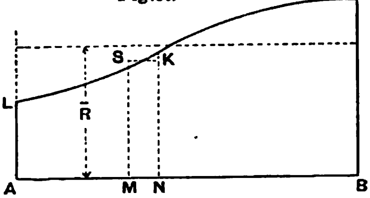
$$= (W_1 + W_2 + W_3 + \&c.) \times (\bar{Y} - \bar{y})$$

$$\text{or } \Sigma W\,(\bar{Y} - \bar{y}).$$

That is to say, the total work of raising a number of weights is equal to the product of the sum of the weights by the vertical displacement of the centre of gravity of the weights.

90. *Variable Resistance.*—Let us next consider the work required to be done to overcome a variable resistance. The whole distance through which the resistance is overcome must then be divided into a number of parts, each being so small that, for that small space, the magnitude of the resistance may be treated as sensibly uniform. The work of overcoming the resistance through each of the small spaces being thus found, the total work will be the sum. The estimation can generally be most conveniently performed by a graphical construction. We will, for simplicity, take the case in which the direction of action of the resistance is that of the line of motion. Suppose a body moved from A to B against a resistance the magnitude of which varies from point to point in such a way that it is represented by the ordinates of the curve standing above AB. (Fig. 86.) For the small distance MN the resistance will vary slightly, but will have a mean value represented by SM or KN suppose, and the work of overcoming the resistance through the small space MN is $MN \times SM$ or is exactly represented by the area of the curve standing above MN; and so for any other small portion of the displacement of the body. Thus the total work of overcoming the resistance through AB is represented by the whole area $ALTB$ = mean resistance $\bar{R} + AB$.

Fig.86.



The curve LST is called a curve of resistance. Two important special cases may be mentioned, both of which frequently occur.

(1) Let the resistance vary uniformly. This is the case of a perfectly elastic spring which is compressed, as will be further explained hereafter. The curve of resistance is a straight line AST (Fig. 87*a*) where AB is the compression of the spring, BT the corresponding compressing force R_1. During the compression R is at first zero and gradually increases to R_1, its value at any intermediate point being graphically represented by the ordinate SN corresponding to the compression AN. The work done is the area of the triangle, that is $\frac{1}{2}R_1.AB$, and the mean resistance $\frac{1}{2}R_1$.

(2) Let the resistance be inversely proportional to the distance of the point of application from a given point O. (Fig. 87*b*.)

Fig.87a.

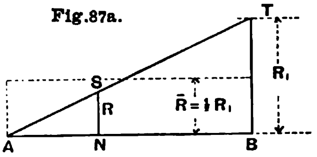
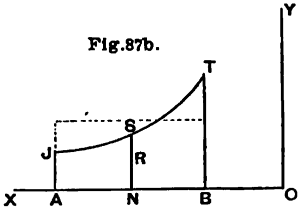


Fig.87b.



This applies to many cases of the compression of air and other elastic fluids. In the figure $NS = R$ is the resistance and $ON.NS$ is constant, so that the curve of resistance JST is an hyberbola. Let the ratio $OA : OB$ be called r, this is called the ratio of compression; then from the geometry of the hyperbola we know that the area of the curve is equal to the constant rectangle $ON.NS$ multiplied by $\log_e r$, the logarithm being Napierian, or as it is often called "hyperbolic" from this property of the hyperbola. If ON be denoted by V this gives a formula in frequent use for the work done in this kind of compression.

$$\text{Work done} = RV \log_e r.$$

91. *Resistance to Rotation. Stability of a Vessel.*—It often happens that we have to consider the resistance of a body to rotation about an axis. Let A (Fig. 88) be the point of application of a force P which resists the rotation of a body about an axis C perpendicular to the plane of the paper. If the resistance at A be not in the plane of rotation P must be supposed to be the component in that plane; the other component will be parallel to the axis of rotation and need not be considered. Let θ be the angle it makes with the direction of A's motion then $R = P.\cos\theta$ is the effective resistance, the other component of P merely producing pressure on the axis. As the body turns through an angle i the resistance R will be overcome through the arc AA', and, assuming in the first instance R constant, the work done will be—

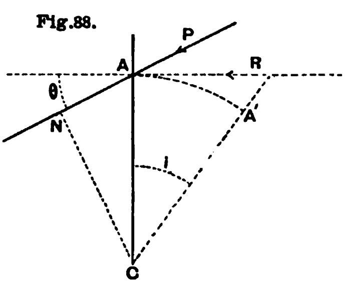


Fig.88.

$$\text{Work done} = R\,.\,AA' = R\,.\,CA\,.\,i.$$

But, dropping a perpendicular CN on P's direction,

$$CN = CA\,.\cos\theta$$

$$\therefore\ \text{Work done} = P\,.\,CN\,.\,i = Mi,$$

where M is the moment of the resistance about the axis of rotation. If there be many resistances then the same formula will hold if M be understood to mean the total moment of resistance.

We can readily extend this to the case of a variable moment by the graphical process already described for a linear resistance, the base of the diagram now representing the angles turned through and the ordinates the corresponding moments. As an example take the case of a heavy pendulum swinging about an axis O (Fig. 89a), let g be the centre of gravity, $Og = l$, and let it be swung through the angle i from the vertical, then the moment of resistance is

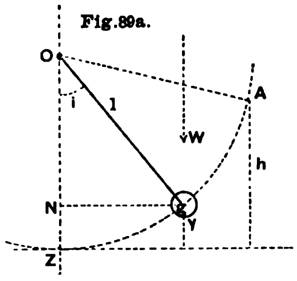
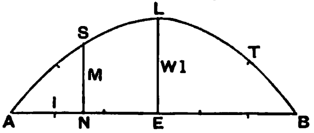


Fig.89a.

$$M = W\,.\,gN = W\,.\,l\sin i.$$

Fig. 89b.



In Fig. 89*b* draw a curve on the base *AB* such that the horizontal ordinate *AN* at every point represents the angle *i* on the same scale that *AB* represents two right angles, while the vertical ordinate represents *M*. This curve will be the curve of resistance, and in the present case is a curve of sines of which the maximum ordinate *LE* is *Wl*. The angles being supposed reckoned in circular measure so that $AB = \pi$, the area of the diagram from *A* up to any point *S* will represent the work done. We can, however, in his example find this work otherwise, for *g* rises through the height *NZ*, and therefore if *U* be the work

$$U = Wl(1 - \cos i).$$

By use of the integral calculus it can be verified that this is also the value of the area *ASN*.

It is not necessary that the axis of rotation should be fixed in estimating the work done during rotation, provided that the resistance be a couple, for then there is no pressure on the axis. An important example is that of a vessel floating in the water and steadily heeled over by the action of a couple *M* produced by external agency, or more frequently by shifting the weights on board in such a way that the displacement and trim remain constant. Then for each angle of heel this couple has a certain definite value which can be found either by calculation or by observation of the shift of the weights. The moment of resistance which is equal and opposite to *M* is called the Statical Stability of the vessel, and the curve of resistance drawn as above described is called the Curve of Stability. The construction of this curve is an important part of the design of the vessel. Such curves, though usually unsymmetrical, often bear a general resemblance to a curve of sines (Fig. 89*b*), the ordinate increases to a maximum which gives the maximum stability and then diminishes to zero at an angle of heel called the "Angle of Vanishing Stability." If the vessel be heeled beyond this angle it capsizes.

According to the principles of this article the area *ANS* of the curve represents the work done in heeling the vessel over. This is called the Dynamical Stability, and as is shown elsewhere (see the chapter on Impact in Part IV.) represents the resistance to heeling over to that angle by a sudden gust.

An important typical case is when the curve of stability is derived from a curve of sines. In this case suppose the angle of vanishing stability to be π/k, where *k* is some given number, then the ordinate *S* for any angle *i* is given by the equation

$$S = S_1 . \sin ki,$$

and the stability is the same as that of a heavy pendulum swinging through k times the angle. The dynamical stability is easily shown to be $1/k^{\text{th}}$ part of that of the pendulum, and is therefore

$$U = \frac{S_1}{k}(1 - \cos ki).$$

92. *Internal and External Work.*—In all that precedes, the position of a body has been changed by overcoming external resistances. All forces, however, arise from the mutual action between two bodies or between two parts of the same body, and every change of position must be with reference to some other body which is regarded as fixed. Work, then, consists in a change of relative position of two bodies notwithstanding a mutual action between the two which opposes the change. In raising weights the second body is the earth, but the pair of bodies may be such as occur in mechanism, and the mutual action between the two may be due to springs or an elastic fluid, or to the resistance of some body to separation into parts. In scissors, nutcrackers, bellows, and other similar instruments, the elements of the pair are exactly alike and their existence is recognised in popular language.

In reckoning the work done either body may be regarded as fixed, the result must be the same and will be unaffected by any movement of the pieces common to both; thus when air is compressed in a cylinder the work done depends on the pressure of the air and the amount of compression, not on the movements of the cylinder within which the air is contained. In other words the motion to be considered is the motion of the pair as defined in Art. 46, p. 94.

In every case where we have to do with a number of pieces connected in any way, we may distinguish between the resistances due to the mutual action between the pieces themselves and those due to the mutual action between the pieces and external bodies. The internal resistances require work to be done in changing the relative position of the pieces themselves, while the external resistances require work to be done in changing the position of each piece relatively to external bodies. These two kinds of work are called Internal Work and External Work respectively. In two cases we can at once foresee that the internal work will be zero, first when the pieces are disconnected, secondly when they are rigidly connected. Thus for example if a heavy mass of matter be raised, we need only consider the rise of the centre of gravity (Art. 89) if the mass be rigid; but if not, any change of form which occurs ought to be taken into account. In raising ordinary solid bodies and masses of earth the internal work may usually be disregarded.

93. *Energy. Principle of Work.*—Hitherto we have been speaking of the *resistance* which is being overcome during the process of doing work, let us now fix our attention on the *effort* which overcomes the resistance.

The forces arising from the mutual action between a pair of bodies, when not purely passive like the normal pressure between two surfaces in contact, are of two kinds. The first always oppose the motion of the pair, in other words they are always resistances. Friction between two surfaces is the simplest example of this, and hence such actions are called Frictional Resistances. The second on the other hand promote or oppose the motion of the pair according to the direction in which the motion is taking place, so that a resistance becomes an effort when the direction of motion is reversed. Such actions are conveniently described as Reversible; and systems of bodies, in which they occur, possess, when the parts are suitably disposed, the power of doing work. This power is called ENERGY. As examples of bodies possessing energy may be taken a raised weight, a compressed spring, or steam of high pressure. Change of velocity in a moving body likewise gives rise to efforts and resistances, but this is a matter for subsequent consideration. For the present we suppose all bodies with which we have to do to be in a state of uniform motion, or to move so slowly and steadily that no sensible action of this kind can arise.

Energy is measured by the quantity of work which it is capable of doing, and the process called doing work may also be described as the exertion or expenditure of energy, so that we write

$$\text{Energy exerted} = \text{Work done.}$$

If the effort which is being exerted and the resistance which is being overcome be applied to the elements of the same lower pair, as when a weight is lifted vertically or a spring wound up, the effort and the resistance are equal, and the equation shows that the energy exerted by an effort is the product of the effort and the space through which it is exerted. Thus all the examples given above of the doing of work will also serve as examples of the exertion of energy simply by supposing the direction of motion reversed. In short the exertion of energy and the doing of work are merely different aspects of the same process.

In this case the effort and the resistance may be regarded as applied at the same point, but the equation has a much wider application than this, for it is equally true if the points of application be different, provided only that they are rigidly connected. Thus, for example, if we dig in the ground, the energy we exert at the handle of the spade is—if the spade be perfectly rigid—exactly equal to the work done at the blade. This can be shown to be a necessary consequence of the forces we are considering being balanced, and the equation may be regarded as a concise statement of the conditions of equilibrium of forces applied to a rigid body. It is preferable, however, for our purposes to regard it as the simplest case of a fundamental mechanical principle continually verified by experience. This principle may be called the PRINCIPLE OF WORK.

We have now a means of transferring the power of doing work, that is to say energy, from one place to another: evidently we are not restricted to one piece, as in the case of the spade. We may make use of a series of pieces through which energy may be transferred from piece to piece in succession; and if there were no frictional resistances to the relative motion of the pieces, there would be no loss of energy in the process. Thus the principle of work is true when the points of application of the effort and the resistance are mechanically connected in any way. Frictional resistances however absorb a portion of the energy whenever any relative motion occurs which they tend to prevent, and therefore a certain loss always accompanies the transmission of energy. Nevertheless the principle of work still holds good if overcoming friction be reckoned as part of the work done.

It may here be remarked that though frictional resistances are never a source of energy, yet friction may, like normal pressure between surfaces, transmit energy, and hence, in cases where one only of the bodies between which it is exerted belong to the set of bodies we are considering, may be an effort by means of which work is done on the set. Thus, for example, in the case of a shaft driven by a belt, the whole power of the engine is transmitted by friction closure between the belt and the pulleys; and if we consider the shaft alone apart from the rest of the mechanism, the friction may be regarded as the effort which drives the shaft. We cannot however in such cases properly speak of the friction as exerting energy; the source of energy is the steam, or other motive power, and the friction merely transmits it in the same way as the pressure between a connecting rod head and the crank pin transmits energy to the crank shaft. Nevertheless in both of these cases the phrase "energy exerted" may be used conveniently, though "energy transmitted" would be more precise.

If a piece of material through which energy is transmitted yield under stress applied to it, as in fact it always does, the energy exerted will not be equal to the work done. Either the change of relative position of the several parts of the piece will require work to be done in order to overcome the mutual actions between the parts which resist the change, or, conversely, those mutual actions exert energy during the change. In the first case the work is done at the expense of the energy transmitted; in the second the piece of material is a source of energy which increases the energy transmitted. In perfectly elastic material the mutual actions are reversible, and any energy exerted in overcoming them is stored up in the piece and recovered when the piece resumes its original form, as in the case of a watch spring. (Compare Art. 98.)

94. *Machines.*—A mechanism becomes a machine if we connect together two of its elements by a link capable of changing its form or dimensions, and so moving the mechanism, notwithstanding a resistance applied by a similar link connecting two other elements.

The elements connected may be called the "driving pair" and the "working pair," and these pairs often, though by no means always, have one element common, namely the frame-link of the mechanism. The driving link is the source of energy. As examples, we may take steam which connects the piston and cylinder which form the driving pair in a steam engine, or gravity which, as in Art. 62, is to be conceived replaced by a link exerting the same effort. The working link is gravity in cranes and other hoisting machines, or a piece of material the deformation of which is the object of the machine, as in the case of machine tools.

In addition to the driving and the working links, the force of gravity acts on all the parts of the machine, and frictional resistances have to be overcome; but these are matters for snbsequent consideration.

The driving and working pairs are very frequently kinematic pairs of the lower class. Let us suppose them in the first instance sliding pairs. Let the driving pair move through a space x, then the working pair will move through a space y, which is in a certain definite proportion to x depending on the nature of the mechanism. Let P be the driving effort, which, by taking x small enough, can be made as nearly uniform as we please; and let R be the resistance opposing the motion of the working pair, then

$$\text{Energy exerted} = Px;\ \text{Work done} = Ry,$$

and these must be equal, therefore

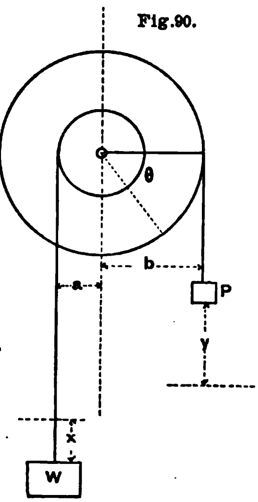
$$\frac{P}{R}=\frac{y}{x}=\frac{\text{Velocity of Working Pair}}{\text{Velocity of Driving Pair}};$$

from which it appears that the ratio of the effort to the resistance, or as we may briefly call it, the "force ratio" is the reciprocal of the velocity ratio of the driving and working pairs. In works on mechanics this is also known as the Principle of Virtual Velocities.

If the pairs be turning instead of sliding pairs, then the effort and resistance are moments, and the velocities will be angular; and if one pair be sliding, the other turning, a suitable "radius of reference" must be selected (p. 94) to compare the motions and the forces, but the same principle holds good.

In the simplest machines, known frequently as the "mechanical powers," we have a 2 or 3-linked chain, so that the driving pair and working pair are identical or very closely connected. But they may be separated by a long train of mechanism and have no common link. In all cases it must be carefully remembered that the effort and the resistance arise from the mutual action between the elements, each consisting of two equal and opposite forces, just as in the straining actions considered in Chapter II. and elsewhere. Either of these as before measures the magnitude of the action.

Fig. 90.



95. *Verification of the Principle of Work in Special Examples.*—We will now take some examples to illustrate and verify the principle of work, neglecting friction.

(1.) Take the common wheel and axle. Suppose P to be just sufficient to lift the weight W, so that the two forces exactly balance one another. Now let P descend through the distance y (Fig. 90), and W rise through the corresponding distance x.

As P falls it is said to exert energy. Energy exerted $= Py$. This is employed in overcoming the resistance to the rise of the weight W. Work done $= Wx$. The principle of work asserts that Energy exerted = Work done, that is $Py = Wx$.

Suppose the wheel and axle to turn through the angle θ, then $y = b\theta$ and $x = a\theta$. Then in order that the weights P and W may statically balance one another, $Pb = Wa$; from which it follows that $Py = Wx$, verifying the principle of work.

Also, we may write,

$$\frac{P}{W} = \frac{a}{b} = \frac{x}{y} = \frac{V}{v},$$

where v, V are the velocities of P, W respectively, thus showing that the force ratio is the reciprocal of the velocity ratio.

In this simple example both the force ratio and the velocity ratio remain constant throughout the movement. In general this will not happen.

(2.) Take the case of the mechanism of the steam engine for an example. Neglect friction and let the driving pressure on the piston be P. A thrust which we will call S will be produced along the connecting rod and transmitted to the crank pin as shown in Fig. 91. At the crank pin this force S may be resolved into two components, one acting along the crank arm and the other, R, perpendicularly to it. The last alone will tend to turn the crank, the other component producing only a pressure on the shaft immediately balanced by the pressure of the bearings on the journals of the shaft.

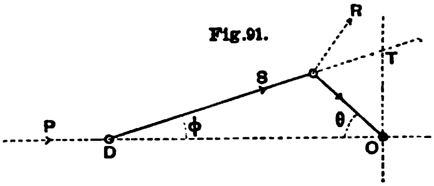


Fig. 91.

This component R which tends to turn the shaft is called the *crank effort*. If the turning effort on the crank is perfectly balanced at all points of its revolution by some suitable resistance, then the resisting force which must be applied at the crank pin at right angles to the crank arm in order to balance perfectly the pressure of the steam on the piston must be equal and opposite to the component R previously referred to. The force ratio will be P/R. We have, with the notation employed in Chap. V., $S \cos \phi = P$ and $S \sin (\theta + \phi) = R$.

$$\text{Thus } \frac{R}{P} = \frac{\sin (\theta + \phi)}{\cos \phi} = \frac{\sin OBT}{\sin OTB} = \frac{OT}{OB}.$$

That is, the crank effort is to the steam pressure as the intercept OT is to the crank arm OB.

But we have previously shown (see p. 101) that this fraction expresses the velocity ratio of piston to crank pin; hence we have again found in this case that the force ratio is the reciprocal of velocity ratio, and the curve which we previously drew to represent the varying velocity of the piston, the crank pin moving uniformly, will represent also the varying crank effort, the pressure of the steam on the piston being uniform throughout the stroke. So we may call it the Curve of Crank Effort.

(3.) The same thing may be proved to be true for every mechanism, the forces acting on which balance one another. In some cases it may be easier to determine the force ratio than the velocity ratio or *vice versa*. In any case, either may be inferred by taking the reciprocal of the other. As an additional example take the case of two pieces driving one another by simple contact (Fig. 92). We have already found the velocity ratio by a direct process (p. 148), but we may also determine it in the following way. When A presses on B there is a resistance R

equal and opposite to the pressure, and normal to the portions of the surfaces in contact, if we suppose no friction to exist. Drop perpendiculars p_A and p_B on the common normal. Then the moment of the driving pressure R which A exerts on B or the turning moment due to $A = M_A = Rp_A$. Similarly the moment of the resisting force which B exerts on A or the moment of resistance to turning which B opposes to $A = M_B = Rp_B$. Hence it appears that

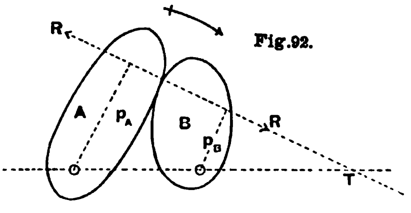


Fig. 92.

$$\frac{\text{Driving moment}}{\text{Resisting moment}} = \frac{M_A}{M_B} = \frac{p_A}{p_B}.$$

But we have previously proved that this fraction is the angular velocity ratio of the piece B to the piece A, and thus we show that the moment ratio is the reciprocal of the angular velocity ratio.

96. *Periodic Motion of Machines.*—One of the most essential characteristics of a machine is the periodic character of its motion. Each part goes through a cycle of changes of position and velocity and returns periodically to its original place. When moving steadily the periods are equal and the velocity of each piece is the same at the beginning and end of each period. That this may be the case it is not necessary that the driving effort should balance the working resistance in every position; on the contrary, this seldom happens; it is sufficient if the mean effort be equivalent to the mean resistance, or as we may otherwise express it

Energy exerted during a period = Work done in the period;

a condition which always governs the action of a machine in steady motion. In reckoning the energy and work the action of gravity on any piece of the machine may be omitted, for, if the piece rise through any height during one part of the period, it will fall through an equal height during another part. The work done consists partly of the work which the machine is designed to do, and partly of frictional resistance to the relative motion of the parts of the machine, or in other words of Useful Work and Waste Work. The ratio of the useful work to the energy exerted is called the Efficiency of the machine and its reciprocal the Counter-Efficiency. The efficiency of a machine depends partly on the kind of machine and partly on the speed, as will be explained in the chapter devoted to frictional resistances (Chap. X.). In estimating the power required to drive a machine a value is assumed for the efficiency derived from experience of machines of the same or nearly the same type. Examples will be given hereafter.

97. *Power. Sources of Energy.*—The sources of energy are—

(1.) Living agents ;

(2.) Gravity acting usually by means of falling water ;

(3.) Springs and elastic fluids ;

(4.) Gunpowder and other explosive agents.

The energy thus derived may be traced further back to the action of heat and chemical affinity, and we may add to the list electric and magnetic forces, but the foregoing is a sufficient statement for our present purpose. A machine which employs such agents directly is called a Prime Mover, or, more briefly, a Motor, but a number of machines may be driven from one prime mover which serves as their source of energy. In general, each source of energy has a motion and an effort peculiar to itself while the work is required to be done at a different place and under different circumstances. A machine, then, is a mechanism which transmits energy and converts it into a form suitable to the work to be done.

The rate at which energy is exerted is called Power ; it is this which measures the value of a source of energy and the expense of the work which is being done. The ordinary unit of measurement is the conventional horse-power of 33,000 foot-pounds per minute, or 550 per second, a quantity much greater than the working power of an ordinary draught horse on the average of a day's work. The unit of power employed universally on the Continent is somewhat less, being 75 kilogrammetres per second or 32,550 foot-pounds per minute.

In prime movers the effort may generally be regarded as applied at a point which moves with a known mean velocity ; then the horse-power is given by the equation

$$\text{H.P.} = \frac{PV}{33{,}000},$$

where P is the mean value of the effort in lbs. and V the mean velocity in feet per minute.

In machines driven from a prime mover the effort is generally a moment M which exerts the energy $M.2\pi$ in every revolution of a driving shaft. We then have

$$\text{H.P.} = \frac{M.2\pi n}{33{,}000},$$

where M is the mean moment and n the revolutions per minute.

98. *Reversibility. Conservation and Storage of Energy.*—The resistance overcome at the working point may be either frictional as in machine tools or reversible as in machines for raising weights. In the second case, if the machine were stopped and set in motion in the reverse direction it would, if friction could be neglected, work equally well, the driving effort and working resistance would be interchanged, and constructive modifications might be required, but otherwise the action is unaltered. This may be described by saying that the machine is Reversible. Many machines actually occur in both their direct and their reversed forms; thus a pump is a reversed hydraulic motor. Hence it appears that in reversible machines the power of doing work, that is to say, energy, is not lost after being exerted, for by reversing the machine it may be employed a second time. Thus it is that we describe the action of reversible machines as a transfer of energy, and are led to conceive of energy as indestructible and independent of the bodies through which it is manifested. No machine, indeed, is completely reversible, for in all cases frictional resistances occur to a greater or less extent, while many machines are completely non-reversible; but we shall see as we proceed that even then energy is not lost but only converted into another form, so that we have in reversible machines the first and most simple example of the great natural law called the Conservation of Energy. The importance of reversibility as a test of maximum efficiency will be seen more fully hereafter.

Again, we can store up energy and use it as required when it is inconvenient to resort to any of the usual sources. For example, by a few turns of the watch key we store energy in the mainspring which is supplied at a regular rate to the watch throughout the day. So the hydraulic accumulator (Part V.) receives energy from the pumping engines and supplies it at irregular intervals to the hydraulic machines which lift weights and move gates in a dockyard or work the guns in a ship of war.

A large part of what follows in the present work is merely a development of what has been said here: in the succeeding chapters of the present division we consider machines comprising solid elements only, while in a future division we shall consider the transmission and conversion of energy by means of fluids.

EXAMPLES.

1. A waggon weighs 2 tons and its draught is $\frac{1}{50}$th of its weight. Find the work done in drawing it up a hill 1 in 20, half a mile long. Find also how long three horses will take to do it supposing each horse to work at the rate of 16,000 foot-pounds per minute.

Work done = 370 ft.-tons. Time occupied = 17′ 15″.

2. A force of 10 lbs. stretches a spiral spring 2″, find the work done in stretching it successively 1″, 2″, 3″, &c., up to 6″. *Ans.* 2½, 10, 22½, 40, 62½ and 90 inch-lbs.

3. Find the H.P. required to draw a train weighing 200 tons at the speed of 40 miles an hour on a level, the resistance being estimated at 20 lbs. per ton. Find also the speed of the train up a gradient of 1 in 100, the engine exerting the same power. H.P. required = 426⅔. *Ans.* Speed up the incline = 18·87 miles per hour.

4. The resistance of H.M.S. "Iris" at 17 knots is estimated at 40,000 lbs., what will be the H.P. required simply to propel the ship. Find also in inch-tons the moment, on each of the twin screw shafts, equivalent to this power, the revolutions being 80 per minute. *Ans.* H.P. required = 2,088. Moment on each shaft = 367 inch-tons.

5. The curve of stability of a vessel is a common parabola, the angle of vanishing stability 70°, and the maximum moment of stability 4,000 ft.-tons. Find the statical and dynamical stabilities at 30°. *Ans.* Statical stability = 3,918 ft.-tons. Dynamical stability = 1,283 ft.-tons.

6. Verify the principle of work, neglecting friction, in :—(*a*) The differential pulley (Art. 59). (*b*) A pair of 3-sheaved blocks. (*c*) The hydraulic press (Art. 62).

7. From the results in question 6, p. 103, deduce the crank efforts for the given positions of the piston and the mean crank effort, supposing the effective steam pressure on the piston 20 tons and neglecting friction.

Crank effort at quarter stroke in the $\left\{\begin{array}{l}\text{forward stroke} = 18{\cdot}4 \text{ tons.}\\ \text{backward ,,} = 16{\cdot}6 \text{ tons.}\end{array}\right.$ Mean = 12·74 tons.

8. Show that the efficiency of a machine is equal to the velocity ratio multiplied by the force ratio.

SECTION II.—UNBALANCED FORCES (KINETICS).

99. *Kinetic Energy of a Particle.*—We now proceed to consider the cases in which efforts or resistances arise from the changes of velocity of the parts of a system, which changes thus become a source of energy or require energy in order to produce them. The commonest observation is sufficient to show the importance of such cases: a cannon ball possesses a great power of doing work, and a railway train requires energy to be exerted by the steam to obtain the requisite speed, quite irrespectively of that necessary to maintain the speed when once produced.

First, suppose a weight under the action of gravity only. Unless it be supported by a vertical force exactly equal to the weight it will fall with a gradually increasing velocity. Let it be wholly unresisted, let it start from rest and fall through a height h, then we know that it will acquire a velocity v given by the formula

$$v^2 = 2gh,$$

where g is a number which for velocities in feet per second ranges from 32·1 at the equator to 32·25 at the pole, and having intermediate values at other points on the earth's surface according to the intensity of gravity at the point. The average value 32·2 is usually adopted for this important constant, and the height h is called the "height due to the velocity."

During the whole fall, the weight W of the body has been exerting an effort upon it which overcomes an equal resistance occasioned by the change of velocity which is taking place; thus an amount of energy has been exerted, and an amount of work done equal to Wh. Resistance of this kind is of the reversible kind, for if we imagine the weight, after reaching the ground, projected up again with the same velocity, it will, if wholly unresisted, attain the height from which it originally fell.

Hence we describe the weight as possessing energy, and the amount it possesses when moving with velocity v is

$$Wh = \frac{Wv^2}{2g}.$$

Energy due to motion is called Kinetic Energy, to distinguish it from that kind of energy considered previously, which is a consequence of the relative position of the parts of a system, and which is called Potential Energy. The kinetic energy of a body depends on its velocity only, not on the direction of its motion nor on the way in which its motion has been produced; and the energy exerted in changing the motion of a body is always represented by an exactly equivalent increase of kinetic energy, whether the effort be uniform or variable, or whether its direction coincide with the direction of motion or not. To illustrate this, consider the following cases.

(1) Let the body move in a straight line under the action of a force P, in that line let it start with velocity V, and after moving through a space x let its velocity be v, then, it is shown in works on elementary dynamics, that v is given by a formula which may be written

$$Px = \frac{Wv^2}{2g} - \frac{WV^2}{2g}.$$

Now, the left-hand side of the equation is the energy exerted by P, and the right-hand side is the increase of kinetic energy of the body.

If P be a resistance instead of an effort, then work is done at the expense of the kinetic energy which is now diminished. If P be variable we must represent it graphically by a curve as in Art. 90, and it should be especially remarked that the ordinate of the curve of areas deduced in Art. 31 will, on affixing a suitable scale and measuring the ordinates from a suitable base line, represent the height due to the velocity of the body.

(2) Let the body be constrained by means of a smooth guiding curve to move along a given path by a force P in any direction, then the energy exerted by P is the same as that exerted by the resolved part of P in the direction of motion. But this resolved part accelerates the motion just as if the body moved in a straight line, so that this case is reduced to the last.

(3) The pressure on the guiding curve will be the difference between the normal component of P and the force necessary to change the direction of P's motion. If the two are equal the guiding curve may be removed, and we obtain the case where the body moves freely, as in the case of a projectile in vacuo.

100. *Partially Unbalanced Forces. Principle of Work.*—Again, the effect which is changing the motion of the body may be partly balanced by an external resistance to which the body is subject. If this be the case we can imagine it separated into two parts, a part which is, and a part which is not, balanced. The energy exerted by the first is employed in overcoming the external resistance, while that exerted by the second is employed in increasing the kinetic energy of the body. Or the resistance may be greater than the effort, then the excess is overcome at the expense of the kinetic energy of the body, the velocity of which now diminishes.

In the present treatise we shall use the phrases "energy exerted" and "work done" only in reference to efforts and resistances other than those due to inertia, subject to which convention, we may state the principle of work as applied to cases where the forces are partially unbalanced, as follows—

$$\text{Energy exerted} = \text{Work done} + \text{Change of Kinetic Energy}.$$

In this statement the work done may be greater or less than the energy exerted. In the first case the change of kinetic energy is a decrease, in the second an increase.

Not only does this principle apply to a single body, but—subject to the observations of the preceding section—to a set of bodies mechanically connected in any way, provided that one of them be fixed to the earth; or, in other words, that a body of great mass like the earth be one of the set. When no one of the set predominates over the rest it is necessary to consider further how the kinetic energy should be reckoned: for the present, however, we shall suppose this condition satisfied.

A simple case is that of Atwood's machine. Let the descending weight P be greater than the rising one Q. Neglecting friction, the excess sets the two weights in motion. Let P descend through a distance y, then Q rises through the same distance, and therefore

$$\begin{aligned} \text{Energy exerted} &= Py. \\ \text{Work done} &= Qy. \end{aligned}$$

Let v be the velocity of the two weights; then supposing them to start from rest,

$$\text{Kinetic energy acquired} = (P+Q)\frac{v^2}{2g}.$$

From principle of work

$$Py = Qy + \frac{(P+Q)v^2}{2g}; \quad \therefore\ v^2 = \frac{P-Q}{P+Q}2gy.$$

The law of increase of velocity is, therefore, the same as that of a body falling freely, but the rate of increase is less. This formula is the same as that obtained by other methods, and we have therefore here a verification of the principle of work.

101. *Kinetic Energy of the Moving Parts of a Machine.*—Instead of a single body, suppose we have a system of bodies, and we require to know the total kinetic energy of the system. The direct method is to find the energy of each separate particle of the system and add the results. In the particular case of a rotating rigid body we are able to express the result of the summation in a convenient and simple form. First consider a ring of small section rotating about an axis in the centre perpendicular to its plane. Every portion of the ring will move with the same velocity, v say, and the kinetic energy of the ring may, as before, be written $Wv^2/2g$.

We may express this another way, as follows:—If n be the revolutions per second, and a the radius, $v = 2\pi an$,

$$\therefore \frac{Wv^2}{2g} = W \cdot \frac{4\pi^2 n^2}{2g} a^2.$$

If the ring is not complete, but W is the weight of a portion which has the same centre of rotation, the expression will still hold.

Now, suppose we have a body consisting of a number of particles rigidly connected together, rotating about a centre O, at n revolutions per second.

Let the weights of the particles be w_1, w_2, w_3, w_4, etc.,
rotating about O at distances y_1, y_2, y_3, y_4, etc.

By adding together the results for each particle, we obtain for the kinetic energy of the system,

$$\frac{4\pi^2 n^2}{2g}(w_1 y_1^2 + w_2 y_2^2 + w_3 y_3^2 + \text{etc.})$$

Now suppose a is such a radius that

$$a^2 = \frac{w_1 y_1^2 + w_2 y_2^2 + w_3 y_3^2 + \text{etc.}}{w_1 + w_2 + w_3 + \text{etc.}},$$

then substituting, we may write

$$\text{Kinetic energy} = \frac{4\pi^2 n^2}{2g}(w_1 + w_2 + w_3 + \text{etc.})a^2 = \frac{4\pi^2 n^2}{2g} Wa^2.$$

By this method we are always able to reduce any system of bodies to a ring, which ring is often called the *Equivalent Fly Wheel*, and the radius a is called the *Radius of Gyration*. The quantity Wa^2/g is usually called the *Moment of Inertia*, and denoted by the symbol I.

However numerous the particles are, the expression obtained above will hold, and so will be true if they are sufficient in number to make up a solid body. In a continuous body, the separate weights w_1, w_2, w_3, etc., must be taken indefinitely small and close together to get accurate results, and the results of the summation may be most conveniently arrived at by the use of the calculus. The quantity W/g is called the mass of the body, and but for the introduction of this factor the symbol I would have the same meaning as in Chap. XII. Hence all the results there given may be used here for thin plates simply by multiplication by the mass of a unit of area. In addition, the following simple cases will be sufficient. The fourth is a particular case of the second.

1. Solid cylinder rotating about its axis. Radius $= r$. $\quad a^2 = \frac{r^2}{2}$
2. Rectangular parallelopiped rotating about an axis. Diagonal of either end $= 2d$. $\quad a^2 = \frac{d^2}{3}$
3. Sphere rotating about a diameter. Radius $= r$. $\quad a^2 = \frac{2r^2}{5}$
4. Rod rotating about an axis perpendicular to it through one end. Length $= l$. $\quad a^2 = \frac{l^2}{3}$

In other cases such as occur in practice, the body is generally too irregular and complex in form to render mathematical formulæ useful; we then apply the rule given in Chap. XII. for plane areas, which by a similar process can readily be extended to solids. That is to say, if I be the moment of inertia of a body about any axis, I_0 that about a parallel axis through the centre of gravity at a distance h,

$$I = I_0 + mh^2,$$

where m is the mass of the body. In applying this rule the body is cut up into portions to which the values just given apply exactly or with sufficient approximation, just as in the chapter cited.

In estimating the kinetic energy of a fly-wheel, which consists of rim, arms, and boss, since the rim is by far the most important part for storing energy, it is generally sufficient to consider it alone. If it be desired to take the remaining parts into account, an addition of about one-third the weight of the arms may be made to the weight of the rim. The combined effect of arms and boss is said to amount to an addition of, on the average, about 8 per cent. to the weight of the rim.

If the body have a motion of translation, combined with a motion of rotation about its centre of gravity, it will be shown hereafter that its total kinetic energy is the sum of that due to the translation and the rotation taken separately, so that the whole can be found by preceding rules. As an example of the use of this principle, consider the case of a ball rolling down an inclined plane, the ball and plane being sufficiently rough that slipping does not take place between them; and suppose the resistance to rolling, called the rolling friction, is insensible. In this case the whole energy due to the descent of the ball is employed in generating kinetic energy in the ball, which will be stored in it by virtue of its two motions of translation and rotation. Let V be the velocity of translation, A the angular velocity, r the radius of sphere; then since no slipping occurs $V = Ar$.

Let the ball descend through a vertical height h, then the energy exerted is Wh, equating which to the kinetic energy stored we obtain

$$Wh = \frac{WV^2}{2g} + \frac{WA^2a^2}{2g},$$

where a = radius of gyration is given by $a^2 = \frac{2}{5}r^2$.

$$\therefore\ Wh = \frac{WV^2}{2g} + \frac{WA^2}{2g} \cdot \frac{2}{5}r^2 = \frac{7}{5}W\frac{V^2}{2g}.$$

$$\therefore\ V^2 = \frac{5}{7}2gh.$$

Thus the velocity of the ball will be less than if it simply slid down the plane without rotating in the proportion $\sqrt{5} : \sqrt{7}$.

The total kinetic energy of the moving parts of a machine in any position may be found by drawing a diagram of velocity for that position in the manner explained in Chaps. V. and VI. Each part may be divided into a number of small portions, and the centre of each portion may be laid down on the diagram, as explained on page 112. If now the diagram be imagined to represent a set of particles rigidly connected, of masses equal to those of the particles in question, the moment of inertia of those particles about the pole of the diagram must be the total kinetic energy required; the radius vector of each particle representing the velocity of the corresponding portion.

102. *Conservation of Energy.*—The principle of work may also be stated in another form, which, though not so convenient in practical applications, is much employed by scientific writers. It has already been explained that, when there are no frictional resistances, the power of doing work (energy) exerted in doing a given amount of work is not lost but merely transferred from one place to another (Art. 98), while it appears from the present section that any energy exerted in changing the motion of a body is represented by an exactly equivalent amount of kinetic energy stored up in the moving body; hence it follows that in any dynamical system, which receives no energy from without and supplies none to external bodies, the total amount of energy is always the same if there be no frictional resistances. We express this by the equation

Kinetic Energy + Potential Energy = Total Energy = Constant, and call it the principle of the Conservation of Energy. In all actual motions frictional resistances occur which gradually absorb the energy, but we shall find hereafter that this process is accompanied by the generation of heat which is equivalent to the energy absorbed, a fact which leads us to conclude that heat is a form of energy, so that the principle still holds good.

103. *Examples.*—Let us now illustrate and verify the principle by some examples.

(1) Suppose a weight suspended by a string and oscillating under the action of gravity, forming the simple pendulum Og (Fig. 89*a*, p. 179), of length l.

Let the pendulum start from the position OA, and when it reaches the position Og let its velocity be v. Let the height of g above the tangent at the lowest point be y, and that of A, h, then we know that

$$v^2 = 2g(h - y),$$

which may be written, if W be the weight,

$$W\frac{v^2}{2g} + Wy = Wh.$$

Here the first term on the left-hand side is the kinetic energy of the weight and the second term Wy the potential energy, that is to say, the power of doing work which the weight possesses, in virtue of its height y above the lowest position it is capable of occupying. The sum of the two is the total energy Wh, and the motion consists in a continual interchange between the kinetic and potential energies. It is, of course, supposed that the resistance of the air is neglected; this is a resistance of the frictional kind, and continually absorbs energy from the weight which is thus at last reduced to rest.

The time of an unresisted double oscillation is shown in works on dynamics to be

$$T_0 = 2\pi\sqrt{\frac{l}{g}}$$

when the oscillations are small enough to be sensibly isochronous. Larger oscillations are sensibly slower, as shown by the approximate formula.

$$T = T_0\left\{1 + \frac{\theta^2}{16}\right\} = T_0\left\{1 + \frac{n^2}{52521}\right\},$$

where θ is the angle of swing in circular measure, and n is the same angle in degrees.

(2) The pendulum has been here supposed to be merely a heavy particle attached to the end of a string without weight. Let us next

suppose a rigid body, the centre of gravity of which is g, oscillating about a centre O. Let v be the velocity of g, then

$$\text{Kinetic Energy} = W\frac{v^2}{2g} + W\frac{k^2A^2}{2g} \text{ (p. 193)},$$

where k is the radius of gyration about the centre of gravity, and A the angular velocity. If L be the length Og of the compound pendulum, this may be written

$$\text{Kinetic Energy} = \frac{Wv^2}{2g}\left\{1 + \frac{k^2}{L^2}\right\}.$$

The potential energy is the same as if the whole weight were concentrated at g; therefore, assuming the pendulum to start from the position OA, as before,

$$\frac{Wv^2}{2g}\left\{1 + \frac{k^2}{L^2}\right\} + Wy = Wh.$$

Comparing this with the result previously obtained for the simple pendulum, it is not difficult to see that the motion is identical if

$$l = L + \frac{k^2}{L}$$

which is the length of the simple equivalent pendulum.

(3) Take the case of a projectile unresisted by the air. Let A be the point from which the projectile starts with velocity V. If we draw through A a horizontal line AL, from this set up an ordinate $AH = h = V^2/2g$, and then draw a horizontal line HK, this line will be the directrix of the parabola in which the projectile moves. When the projectile has reached any point in its path, which is at a height y from the ground and at which it has the velocity v, the total energy possessed by the projectile $= W\left(y + \frac{v^2}{2g}\right)$. This being equal to that which it had at starting $= W\frac{V^2}{2g} = Wh$, $\frac{v^2}{2g} = h - y$, and so the projectile will, at every point of its path, have a velocity due to its having fallen from the directrix.

EXAMPLES.

1. The energy of 1 lb. of pebble powder is 70 foot-tons. Find the weight of charge necessary to produce an initial velocity of 1300 feet per second in a projectile weighing 700 lbs., neglecting the recoil of the gun and the rotation of the shot.

Wt. of powder required = 117 lbs.

2. In Example 1 suppose the gun fired at an elevation of 30°, and resistance of the atmosphere neglected, find the kinetic and potential energies of the shot at its greatest elevation. Also deduce the greatest elevation.

Horizontal velocity = velocity at highest point $= 1300\frac{\sqrt{3}}{2}$,

Kinetic energy at highest point = 6150 ft.-tons,

Potential " " = 2050 "

$$\frac{\text{Potential energy}}{\text{Wt. of shot}} = 6560{\cdot}6 \text{ feet} = \text{maximum elevation.}$$

3. A train is running at 40 miles an hour, find the resistance in pounds per ton necessary to stop the train in 1000 yards on a level. Also find the distance in which the train would be brought up by the same brake power on a gradient of 1 in 100, both when going up and when going down.

Resistance = 39·9 lbs. per ton.

Distance required to bring up the train when ascending
the gradient ... ... = 640 yards.
When descending ... ... = 2280 „

4. The reciprocating parts of an engine running at 75 revolutions per minute weigh 25 tons, of which parts weighing 20 tons have a stroke of 4 feet and parts weighing 5 tons a stroke of 2 feet. Find the energy stored in the parts, assuming a pair of cranks, OP, OQ at right angles and neglecting obliquity of connecting rod.

$$\text{Velocity of parts attached to crank } P = PN\frac{V}{OP}.$$

$$\text{,, \quad ,, \quad ,, \quad } Q = QM\frac{V}{OP}.$$

Where V is the velocity of the crank pin and PN, QM are perpendiculars on the line of centres.

Assuming weights attached to these cranks each equal W. Then energy stored in these weights together $= \frac{WV^2}{2g}(PN^2+QM^2)\frac{1}{OP^2} = \frac{WV^2}{2g}$.

In example, total kinetic energy = 40·7 ft.-tons.

5. One weight draws up another by means of a common wheel and axle. The force ratio is 1 to 8 and the velocity ratio is 9 to 1. Find the revolutions per minute after 10 complete revolutions have been performed, neglecting frictional resistances and the inertia of the wheel and axle. Diameter of axle 6 inches.

Revolutions per second = 2·14.

6. In Ex. 1 suppose the gun rifled so that the projectile makes 1 turn in 40 diameters, find the additional powder charge required to provide for the rotation of the shot, the diameter of shot being 12 inches and the radius of gyration $4\frac{1}{4}$ inches.

Additional powder required = ·407 lb.

7. A disc of iron rolls along a horizontal plane with velocity 15 feet per second, and comes to an incline of 1 in 40 on to which it passes without shock. Find how far it will ascend the incline, neglecting friction.

Distance along incline it will run = 209·6 feet.

8. In Ex. 5 suppose the weight of wheel = weight of axle, and the two together = sum of weights, obtain the result, taking account of the inertia of the wheel and axle.

After 10 revs. it will rotate at 1·22 revs. per second.

9. Assuming that when a vessel rolls her dynamical stability is the same as when steadily heeled over (Art. 91), and neglecting that part of her kinetic energy which is due to the motion of her centre of gravity (Art. 101), write down her equation of energy (Art. 103). If the curve of stability be a true curve of sines, show that the vessel will keep time with a pendulum of length l swinging through k times her angle of heel, where

$$k\theta_0 = \pi\,;\; l = \frac{r^2}{lh},$$

θ_0 being her angle of vanishing stability and r her radius of gyration, while wh is the maximum stability.

Note.—The rolling is here supposed unresisted. Observe that the deviation from isochronism is much greater than in a simple pendulum swinging through the same angle, k being always greater than unity.

REFERENCES.

Numerous elementary examples on the application of the Principle of Work will be found in Twisden's *Practical Mechanics*.

CHAPTER IX.

DYNAMICS OF THE STEAM ENGINE.

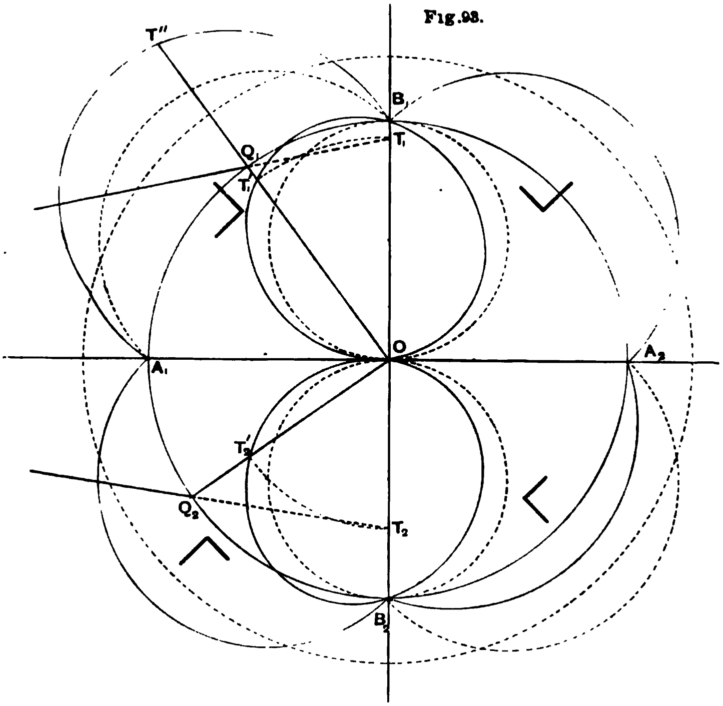
104. *Construction of Polar Curves of Crank Effort.*—One of the most common and important applications of the principles of the preceding chapter is to the working of steam engines, and we shall investigate this question, chiefly with reference to fluctuations of stress, energy and speed. Throughout, frictional resistances are neglected.

In Ch. V. a curve was constructed which shows the velocity ratio of piston and crank pin, and it has been proved (p. 186) that this curve must also give the ratio of the effort tending to turn the crank to the pressure of the steam on the piston, so that it may also be called a Curve of Crank Effort. If there are two or more cranks, the crank effort can be obtained by suitably combining the results for each taken separately, and a curve may then be drawn representing the combination. There are two kinds of such curves, the Polar and the Linear. First suppose two cranks at right angles, steam pressure uniform, and the same on both pistons. Let us commence with the polar curve.

Suppose $OT_1'B_1$, $OT_2'B_2$ (Fig. 93) to represent the polar curve of crank effort for an engine constructed as in Art. 49, and let the two cranks be in the positions OQ_1, OQ_2, each pointing towards the cylinder. Add together the corresponding crank efforts OT_1', OT_2', which are given by the curve, and set off their sum along OQ_1, we thus obtain a radius OT'', which represents the total crank effort for the two engines taken together. It may also be considered as the leverage at which the pressure on one piston must act to produce the same turning moment. Performing this construction for a number of positions of the cranks, we obtain a polar curve showing the crank effort in every position.

If the connecting rod is indefinitely long the single curve of crank effort consists of the pair of circles on OB_1, OB_2, shown dotted in the diagram. If we add together radii of these circles, the combined curve of crank effort will consist of four portions of circles passing the points $A_1B_1A_2B_2$; each of the circular arcs if produced would pass through the point O. These arcs are also dotted in the diagram. When the crank is in a quadrant lying towards the engine, the actual crank

Fig. 93.



effort is in excess of that due to a long connecting rod. So for the positions OQ_1, OQ_2, shown, for each the crank effort is in excess, and thus the curve of combined effort will for the quadrant A_1B_1 lie outside the circular arc. When the cranks are in the two upper quadrants the effort for the leading crank is less than when the connecting rod is long, whereas for the following crank it is greater; and the diminution of one is very approximately equal to the excess of the other; that is, the sum is the same as that obtained by neglecting the shortness of the rod. The true combined effort is then for the quadrant B_1A_2 represented by the circle. In the next quadrant both are diminished; and the true curve will lie inside the circle A_2B_2, while for the fourth quadrant it will again coincide with the circular arc.

We may, if we please, lay off the sum of the radii on the following crank instead of the leading; the same series of curves would be obtained, but would be turned backwards through an angle of 90°.

To add to this the circle of mean crank effort we equate the work done on the two pistons in the double strokes to the work due to the mean effort R_m exerted through a complete revolution.

$$P \times 2 \times 4a = R_m \times 2\pi a.$$

$$\therefore R_m = \frac{4}{\pi} P.$$

In these curves the steam pressure P is represented by the radius of the crank-pin circle, so the mean crank effort will be represented on the diagram by drawing a circle, shown dotted, with centre O and radius $= 4OQ/\pi$.

If there are three or more cranks inclined at any angles, the combined crank effort diagram can be constructed by adding together three or more radii vectores of the curve of single crank effort, and laying the sum off on either of the cranks.

105. *Construction of Linear Curves of Crank Effort.*—The linear curve of crank effort, which is more useful for most purposes, is constructed as follows:—

Take a base line A_1A_2 = semi-circumference of the crank-pin circle, and let the circle and this base line be divided into the same number of equal parts, and at the points of division of the base line set off ordinates such as SN, VM both above and below the base equal to lengths of the common ordinates of the single crank effort diagram such as OT'_1, OT'_2, and so we construct the linear crank effort diagram for a single crank. Neglecting the obliquity of the connecting rod, the diagram will consist of two curves of sines shown dotted, one above, the other below (Fig. 94). To get the combined crank effort diagram we have only to add together proper ordinates according to the angle between the cranks, just as we did in drawing the polar diagram. When the cranks are at right angles it will be seen that when the leading crank is, for example, at Q_1 or N the following crank is at Q_2 or M; and if the ordinate MV is laid off on the top of ordinate NS we obtain a point W on the curve of combined crank effort. If the same process be followed throughout we obtain the diagram shown in Fig. 94, consisting of four curves. If the connecting rod be taken as indefinitely long, and ordinates of the dotted curve be added together the combined diagram will consist of four curves, also curves of sines shown dotted in the diagram, all alike and all of the same height. But taking proper account of the shortness of the rod, we observe that for one quadrant of the revolution when both cranks lie towards the cylinder, each ordinate added is in excess of that, neglecting obliquity, and then we obtain the highest curve. In the next quadrant the height of the curve is less and is the same as if we neglected the shortness of the rod. In the next quadrant when both cranks are away from the cylinder the shortness of the rod makes the crank effort for each engine less, and we get a very low curve for the combination. This is followed in the last quadrant by a curve like the second.

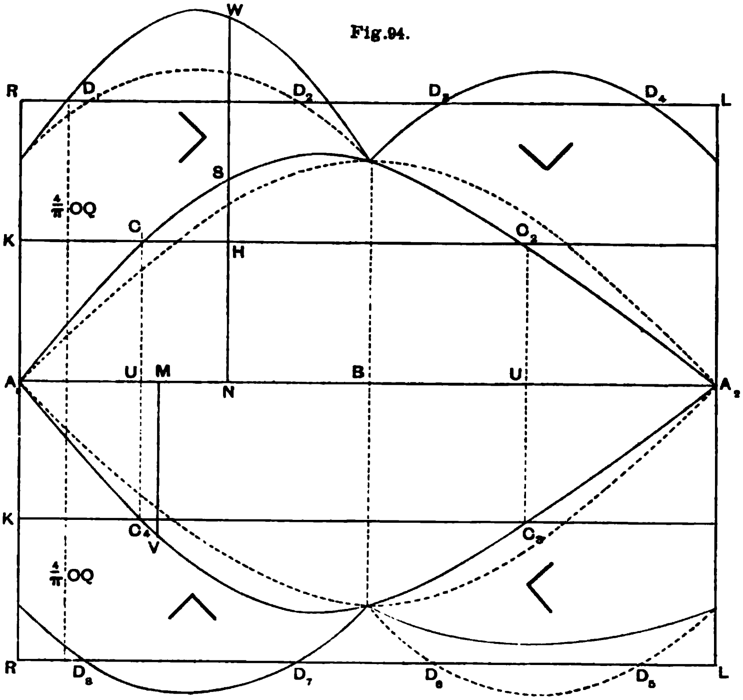


Fig. 94.

The mean crank effort will be represented by a horizontal line at a height $4OQ/\pi$, as before. Setting off this line we observe that unless the connecting rod is longer than is usual in ordinary practice, the actual crank effort will be less than the mean throughout the whole of one of the quadrants.

At the points where the straight line *RL* cuts the curves the actual crank effort is equal to the mean.

106. *Ratio of Maximum and Minimum Crank Effort to Mean.*—One of the principal objects in the construction of curves of crank effort is the determination of the ratio which the maximum and the minimum values of that quantity bear to its mean value as determined from the power of the engine. It is on these quantities that the strength required for the shaft depends, besides which, too great an inequality in the turning moment on the shaft is frequently injurious to the machine which is being driven by the engine, or to the work which the machine is doing.

Approximate mathematical formulæ, analogous to those given on p. 102 for piston velocity, may be used in simple cases, but in general it is preferable to construct a diagram. The annexed table gives some numerical results.

Fluctuation of Crank Effort with Uniform Steam Pressure.

Ratio to Mean for	One Crank.	Two Cranks at right angles.	Three Cylinders at 120°, driving the same Crank.	Connecting Rod.
Maximum.	1·57	1·112	1·047	Indefinitely long.
Minimum.	0	·785	·907	
Maximum.	1·62	1·31	1·077	Four Cranks.
Minimum.	0	·785	·794	

The great influence which the length of the connecting rod has on the results should be especially noticed; we shall return to this hereafter, but now go on to consider the motion of the engine under the action of the varying crank effort.

107. *Fluctuation of Energy.*—We have already referred to the periodic character of the motion of a machine, and explained that when the mean motion is uniform we have for a complete period

$$\text{Energy exerted} = \text{Work done.}$$

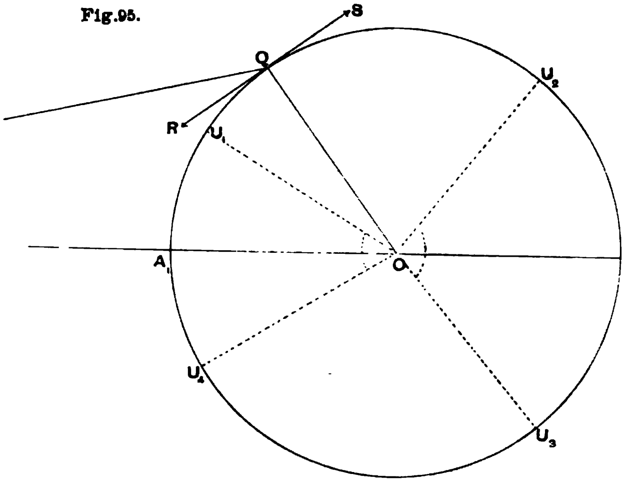
It will seldom happen however that this equation holds good for a portion of the period. In general, during some part of the period the work done will be greater, and in some part less, than the energy exerted.

In the first case some part of the kinetic energy of the moving parts is absorbed in doing a part of the work, and the speed of the machine diminishes; while in the second, a part of the energy exerted is employed in increasing the kinetic energy of the moving parts and the speed of the machine increases. Thus the kinetic energy of the moving parts alternately increases and diminishes, the increase exactly balancing the decrease. At some instant in its motion, the energy of the moving parts will be a minimum, and at some other point a maximum. The difference between the maximum and minimum energies is described as a "fluctuation" of energy of the machine. In general a number of these fluctuations occur in the course of a period, and the greatest of them is called the Fluctuation of Energy. It is most conveniently expressed as a fraction of the whole energy exerted during a complete period of the machine, and this fraction is called the Co-efficient of Fluctuation of Energy.

All this will apply to any machine taken as a whole, or to any part of that machine; for every piece of the machine has a driving point and a working point, and the equation of energy may be applied to it.

Take now the case of the mechanism of a direct-acting engine. Suppose the pressure P on the piston to be uniform. This through the connecting rod will produce a crank effort S, the magnitude of which for each position of the crank may be found as just now shown. To the crank and shaft S is the driving force and furnishes the energy exerted. At every point of the revolution of the shaft a certain resistance will be overcome, which resistance will tend to prevent the shaft from turning; it will not depend on the steam pressure, but on the sort of work that is being done. As the most simple ordinary case we will suppose the resistance overcome to be uniform, and we will neglect the inertia of the reciprocating parts (Art. 110). We may represent this constant resistance by a constant force R applied to the crank pin Q (Fig. 95), at right angles to the crank arm, resisting its motion. The magnitude of R is immediately determined by the application of the principle of work to a complete period, say one revolution. We have

Fig. 95.



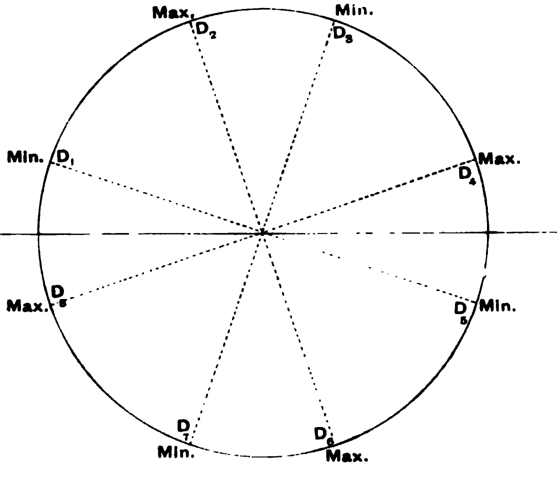
$$P4a = R \times 2\pi a. \quad \therefore R = \frac{2}{\pi}P.$$

This constant resisting force is the same as the mean crank effort. Then, so long as $S > R$ the speed of the crank shaft will increase, and when $S < R$ it will diminish.

Referring to the linear curve of crank effort (Fig. 94, p. 202) let

A_1N = the arc A_1Q (Fig. 95), then NS = crank effort S for this position of the crank. If an ordinate A_1K be set up to represent the constant resistance or mean crank effort, and a horizontal line parallel to base line be drawn, then NH being the representation of R the resistance overcome, the effort S will be greater for this position of the crank, and the difference HS will be employed in accelerating the motion of the machine. From the commencement of the revolution up to this position, the energy exerted is represented by the area A_1NS, whereas the work done is represented by the area A_1KHN. As the crank revolves from the position A_1 the crank effort increases until when at U_1 it is equal to the resistance. Up to this point the speed of rotation will have been diminishing. After passing the point U_1 the effort will be greater than the resistance and the speed of the engine will increase. Thus U_1 is a point of minimum speed at which the kinetic energy is a minimum. When the crank reaches the position U_2 the effort will again be equal to the resistance; and, since from U_1 to U_2 the effort has been greater than the resistance, during the whole of which time the engine has been increasing its speed, it follows that at the point U_2 the speed and the kinetic energy will have reached a maximum. The energy stored during this interval will be equal to the area C_1SC_2, and this will be the fluctuation of energy. During all the movement from U_2 to U_3 the speed of the engine will diminish, so that U_3 is another point of minimum kinetic energy. The kinetic energy stored from U_2 to U_3 is negative, and represented by $C_2A_2C_3$, which quantity also is the fluctuation of energy. Again at U_4 the kinetic energy is a maximum. If the resistance had not been uniform, but its varying

Fig. 96



magnitude represented by the ordinates of some curve of resistance, then where the curve of resistance intersected the curve of crank effort would be the points where the kinetic energies would be maximum and minimum, as just explained. By the graphical construction of such a curve of resistance the fluctuation of energy may be estimated by measuring the area of the crank effort curve cut off above or below the curve of resistance, which area will lie between consecutive points of maximum and minimum energies. If the energy be E, the fluctuation of energy may properly be denoted by ΔE. It is convenient to express this as a fraction of the total energy $4Pa$ exerted in a revolution. We have then for the co-efficient of fluctuation of energy $\dfrac{\Delta E}{4Pa} = k$.

The value of k does not depend on the size of the engine, but only on the length of the connecting rod and the way in which the steam pressure and resistance vary. If the connecting rod is indefinitely long, steam pressure and resistance uniform, $k = {\cdot}1052$. The shorter the connecting rod the greater will be the value of k.

An equally important case is that of two cranks at right angles also shown in Fig. 94. Neglecting the shortness of the connecting rod, then the line of resistance cuts each of the four curves in two points, the first of which is a point of minimum energy as shown in Fig. 96, on the preceding page. For this case $k = {\cdot}01055$ or one-tenth of its value for a single crank: eight fluctuations of equal magnitude occur in each revolution. When the connecting rod is short the curves of crank effort are not the same in each quadrant (see Fig. 94), and one of them lies wholly below the line of resistance. There are then six fluctuations in each revolution: four of these are nearly the same as before, but the other two are much greater, the values of k being ·037 and ·042, with a connecting rod of four cranks. The annexed table gives the maximum value of k for various cases, supposing steam pressure uniform and resistance uniform.

FLUCTUATION OF ENERGY.

Values of k supposing.			Length of Rod.
One Crank.	Two Cranks, at right angles.	Three Cylinders, at 120°, driving the same Crank.	
·1052	·01550	·00325	Infinite.
·1245	·0314	·0084	Six Cranks.
·1358	·0418	·0115	Four Cranks.

As before, the great influence of length of connecting rod on the results should be noticed. Frictional resistances, which are here neglected, generally increase the value of k.

In general the pressure of the steam in the cylinder of an engine varies throughout the stroke, and the construction of the curve of crank effort previously described must be modified on account of this. Suppose, instead of the steam being admitted throughout the stroke, it is cut off at a certain point and expanded so that the expansion curve is hyperbolic. For simplicity neglect the back pressure. At the point N in the stroke (Fig. 97) the pressure will have fallen to P, such that $\frac{P}{P_1} = \frac{OK}{ON}$. If we draw an ordinate P_m

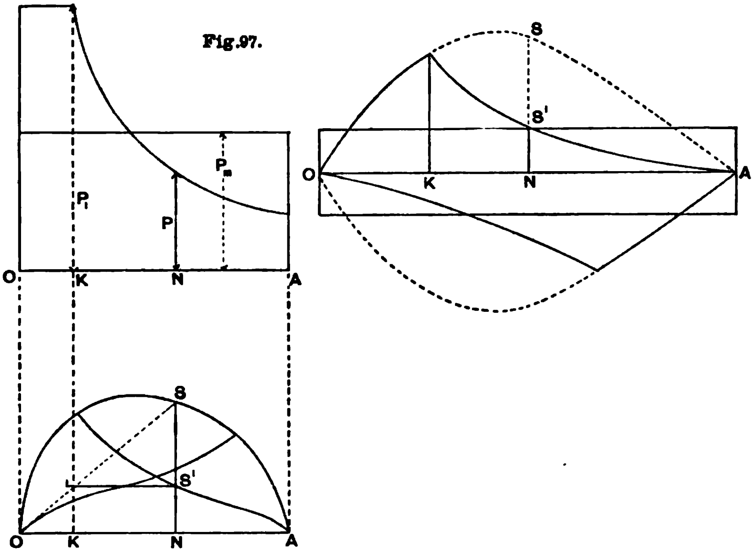


Fig. 97.

such that the area of the rectangle enclosed is equal to the area of indicator diagram, then $P_m = P_1 \frac{1 + \log_\epsilon r}{r}$ where $r = \frac{OA}{OK}$. Up to the point K the crank effort diagram will be the same as previously described, but after that point the crank effort will be less than that due to a uniform steam pressure. At the point N in the stroke, for example, the crank effort instead of being NS will be NS', found by drawing OS in the lower figure, to cut a vertical through the point K of cut-off and making $NS' = KL$, $\frac{NS'}{NS} = \frac{P}{P}$ In the expanded diagram, the base of which is taken equal to the circumference of the crank-pin circle, ordinates must be taken equal to NS', and a diagram so constructed, from which the fluctuation of energy may be calculated. Assuming the resistance to be uniform, it will have a value R such that

$$R\pi a = P_m 2a = 2aP_1 \frac{1 + \log_\epsilon r}{r},$$

$$R = \frac{2}{\pi} P_1 \frac{1 + \log_\epsilon r}{r};$$

and drawing a horizontal line above the base at a height to represent R, it will cut off an area above it which will be the fluctuation of energy. The diagram for the return stroke is shown below. It is not exactly the same as that for the forward stroke, because the

effect of obliquity is different. A general method of procedure applicable with any given indicator diagram is explained at the end of this chapter.

108. *Fluctuation of Speed. Fly-Wheels.*—Fluctuation of energy in an engine or any other machine is necessarily always accompanied by a fluctuation of speed; but the heavier the moving parts the less will be the fluctuation of speed. In most cases it is necessary that the fluctuation of speed should not exceed certain limits, as it would be injurious to the working parts of the machine and would sometimes impair the character of the work done; so it is a question of some importance to inquire as to what the weight of the moving parts must be to confine the fluctuation of speed within a given limit.

Consider the steam engine, and, first, take the case of a single crank. We have already for this case determined the points in the revolution at which the energy of the moving parts is a maximum and minimum, and also the fluctuation of energy. The energy of the moving parts consists of the energy of the rotating crank shaft and all its connections, as well as that of the reciprocating parts. If we imagine a case in which the shaft and all the parts which rotate with it are comparatively very light, then the points determined will be the points at which the piston and reciprocating parts move fastest and slowest, the motion would be very irregular, and, in fact, the engine would not get over the dead points. To avoid this the weight of the rotating parts is made considerable as compared with that of the reciprocating parts, and the heavier they are the more uniform the motion of the engine will be. To increase the uniformity, the weight must generally be artificially increased by the addition of a heavy fly-wheel to the shaft, and the inertia of this is predominant over that of the other moving parts of the engine. For the present we may neglect the inertia of the reciprocating parts and consider the fly-wheel alone.

On this supposition the energy and speed of the fly-wheel will be greatest and least at the points previously described, viz., where the curve of crank effort is cut by the line of uniform resistance. Let W be the weight, V the velocity of rim of fly-wheel; then

$$\frac{WV^2}{2g} = \text{Energy of Rim.}$$

The energy of the arms and boss may be estimated by the addition of a percentage to the weight of the rim, or be considered as furnishing a margin in favour of uniformity. On account of the danger of fracture the speed of periphery V should not exceed 80 feet per second. This is the limit of speed commonly stated, but the liability to fracture depends very much on the straining action on the arms of the wheel due to inequality between the crank effort and the resistance, and not merely on centrifugal forces. (See Ch. XI.) In large wheels the rim is in segments, and the speed is not more than from 40 to 50 feet per second.

Let V_1 and V_2 be the greatest and least speed of periphery due to the fluctuation of speed, then $\frac{W}{2g}(V_1^2 - V_2^2)$ is the fluctuation of energy of the wheel. By the graphical process previously described, we have been able to determine the fluctuation of energy in terms of the total energy E_0 expended in one revolution.

Equating these two we have

$$\frac{W}{2g}(V_1^2 - V_2^2) = kE_0,$$

where k is the co-efficient previously found.

Suppose now that it is required that the fluctuation of speed should not exceed a certain amount, then we may write

$$V_1 - V_2 = q \,.\, V_0,$$

where V_0 is the mean speed and q is a co efficient depending on the degree of uniformity which is considered desirable. In some cases q must not exceed ·02 or even less, whilst in others ·05 or even more may be sufficient.

We may generally assume with sufficient accuracy that

$$V_0 = \frac{V_1 + V_2}{2}$$

(see next Article), then we find by substitution that, at the mean speed,

$$\text{Energy of Wheel} = \frac{k}{2q} \,.\, E_0.$$

In a single crank non-expansive engine the value of k ranges, as we have seen, from ·1 to ·14 when the resistance is uniform. In expansive engines k may be ·25 even with a uniform resistance, and when an engine is doing very irregular work k may be unity.

If we have a pair of cranks at right angles, the kinetic energy of the reciprocating parts is the same, at the same speed, for all positions of the cranks. (Ex. 4, p. 198.) Consequently these parts may be considered as so much added to the weight of the fly-wheel. Besides this the value of k is much less, seldom reaching ·1 if the resistance is approximately uniform. Hence a lighter fly-wheel may be used. The difference however is not so great as it might appear, for in estimating the weight of wheel required, it is important to consider not merely the change of speed, but also the time in which the change takes place. A small change taking place rapidly may be as injurious as a much greater change taking place slowly. The values of the acceleration and retardation at any instant are proportional to the difference between the crank effort and resistance at that instant, which can be found from tables such as that on page 203, and some regard should be paid to these numbers in considering what value of q should be employed.

In any case then we may write

$$\text{Energy of Wheel} = K \,.\, E_0,$$

where K is a co-efficient, which will vary within much narrower limits than the two co-efficients of speed and energy on which it depends. In general, in the very cases in which the resistance is most irregular a greater variation in speed is admissible.

The old rule for fly-wheels, dating from the time of Watt, was that the energy of the wheel should be 3·75 times the energy exerted per stroke. This corresponds to $K = 1{\cdot}875$, and would be satisfied by $k = 1$, $q = {\cdot}267$, or by $k = {\cdot}125$, $q = \frac{1}{30}^{\text{th}}$. The first of these cases would be a very irregular resistance with a great variation in speed, and the second a moderately uniform resistance with a uniformity of speed which would be sufficient for most purposes. Heavier wheels are not unusual in modern practice, and it may be here remarked that the minimum weight necessary may depend partly on the rigidity of the shafting.

There is another method of obtaining the fluctuation of energy which, though not practically so convenient, is for some purposes advantageous. A curve representing the energy exerted may be constructed in this way: Suppose the steam pressure P constant, then in the movement of the crank pin from A to Q the piston moves from A to N and the energy exerted $= P \times AN$, which will be proportional to AN. Now in Fig. 98 take a base line AA' equal to the semi-circumference, and at the various points, such as Q, set up ordinates $QK = AN$, $A'A'' = AA'$, and so on; a curve $AKLA''$ will be obtained, which will represent by its ordinates the energy which has been exerted from the commencement up to the various points in the stroke. At the same time, the resistance being uniform, the work done will be proportional to the length of the arc AQ, since work done $= R \times AQ$. If from the base line AA' we set up ordinates to represent the work done, a straight sloping line will be obtained. If the work done = energy exerted in the complete stroke, they will both be represented by the same ordinate $A'A''$, and so the sloping line will meet the curve at the point A''. The intercept between the curve and line AA'' measured on the vertical ordinate will at any point be the difference between the energy exerted and the work done reckoned from the commencement of the stroke up to that point, and what we have called the fluctuation of energy will be the vertical intercept between two tangents to the curve $AKLA''$ drawn parallel to AA''.

From this we can derive a curve which will represent the varying angular velocity of the crank; but, in order to simplify the measurement and description, let the vertical intercepts of the curve just described be laid off from a horizontal base line, as shown below.

For suppose we know the moment of inertia of the equivalent fly-wheel of the engine and the angular velocity of the crank in some one position: the ordinate of the curve ALA'' at this point measured from a properly taken base line must represent the energy of the moving parts. Thus, if the base line be drawn in proper position, all ordinates measured from it will represent the square of the velocity of revolution of the crank shaft. If the speed of the machine is great, the base line will be some distance below the curve. On the other hand, if the speed is small, the base line will be close to the curve. There is manifestly a minimum speed at which the machine can be kept revolving; it is that which corresponds to the case in which the base line touches the curve. At one instant of the period of the machine the energy will then be zero.

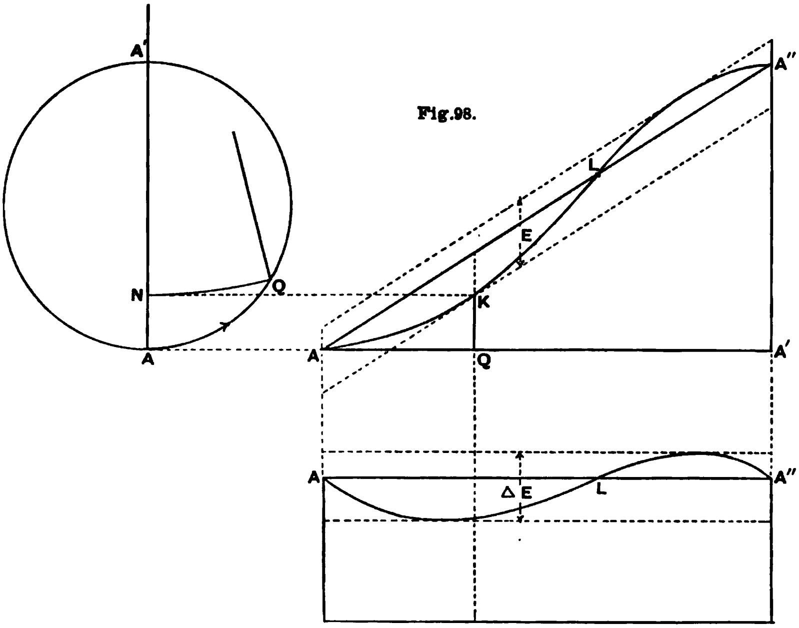


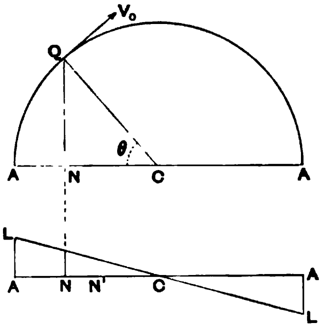
Fig. 98.

Drawing such a base line all the ordinates measured from it will represent the square of the angular velocity, and we can from this deduce a curve of angular velocity. It will be noticed that half the sum of the greatest and least angular velocities is not exactly, but only approximately, the mean angular velocity. The true mean may be determined by means of the curve of angular velocity, the construction of which has just been described.

109. *Correction of Indicator Diagram for Inertia of Reciprocating Parts.*—All that has been said respecting the fluctuation of energy and speed of a machine as a whole, applies to each of the several parts of which it is constructed. The energy supplied by the driving power is transmitted through each piece in succession from the driving pair to the working pair. For each piece the energy exerted is equal to the work done for the whole period; but for a part of the period the two are unequal, so that the kinetic energy of the piece varies. If the motion of the piece be known, the variation of its energy can be used to determine the difference between the driving force on the piece considered and on the piece immediately following it. Of this calculation an important example is the change in the crank effort caused by the inertia of the reciprocating parts of an engine.

In this calculation we neglect, in the first instance, the obliquity of the connecting rod, and suppose the crank to rotate uniformly. Let Q (Fig. 99) be the centre of the crank pin describing a circle AQA with velocity V_0, then the position of the piston is represented by N, and its velocity is

Fig. 99.



$$V = V_0 . \sin \theta,$$

from which it follows that the kinetic energy of the reciprocating parts must be given by

$$\text{Kinetic Energy} = \frac{WV_0^2 \sin^2 \theta}{2g} = \frac{WV_0^2}{2g}\left(1 - \frac{x^2}{a^2}\right),$$

where W is the weight of the piston, piston rod, and other reciprocating parts, and x is the distance of the piston from the centre of its stroke.

Take now two positions N, N', at distances x_1, x_2 from the centre and find by this formula the change of kinetic energy as the piston moves from N to N'. Evidently we shall have

$$\text{Change of Kinetic Energy} = \frac{WV_0^2}{2g} . \frac{x_1^2 - x_2^2}{a^2}.$$

Now this energy must have been obtained from the steam pressure which drives the piston and accelerates its motion. Let P be the mean value of that part of the whole steam pressure which is employed in this way between N and N', then $P . NN'$ is the energy exerted in this way, so that

$$P(x_1 - x_2) = \frac{WV_0^2}{2g} . \frac{x_1^2 - x_2^2}{a^2},$$

or dividing by $x_1 - x_2$,

$$P = \frac{WV_0^2}{2g} . \frac{x_1 + x_2}{a^2}.$$

This formula gives the mean value of the pressure in question between any two points N, N', and therefore, if we take the points near enough, we shall obtain the actual pressure at any point of the stroke. Putting $x_1 = x_2 = x$ we get

$$P = \frac{WV_0^2}{ga} . \frac{x}{a}.$$

It is convenient to express our result as a pressure in lbs. per square inch by dividing by the area of the piston in square inches, then

$$p = p_0 \cdot \frac{V_0^2}{ga} \cdot \frac{x}{a},$$

where p_0 is the weight of the reciprocating parts divided by the area of the piston, or, as we may call it, the "pressure equivalent to the weight of the reciprocating parts."

When $x = a$ we get the pressure at the commencement of the stroke required to start the piston: here the pressure is greatest, and elsewhere varies as the distance from the centre. At the centre the pressure is zero: the piston then for the moment moves with uniform velocity and requires no force to change its motion. When past the centre the pressure is so much addition to the steam pressure because the piston is at every instant being stopped: this is shown by the formula, since x is then negative. All this is shown graphically by drawing a straight line LCL through C such that

$$AL = p_0 \cdot \frac{V_0^2}{ga}.$$

The ordinate of that straight line represents the pressure due to inertia for each position of the piston. After subtracting this from the actual steam pressure the effective pressure is found, which is transmitted to the crank pin, and furnishes the crank effort.

The value of p_0, the pressure equivalent to the weight of the reciprocating parts, varies considerably according to the size and type of engine, but in ordinary cases ranges from 1½ to 3 lbs. per square inch. In return connecting rod engines, and in some other types where the reciprocating parts are exceptionally heavy, p_0 may reach 4½ or 5 lbs. per square inch. This being given, the pressure due to inertia will vary inversely as the stroke and directly as the square of the speed; in the short-stroke high-speed marine engines common in the present day, the correction for inertia is sometimes very considerable. It is hardly necessary to say that it is only the value of the crank effort at particular points of the stroke which is affected. The mean value must remain unaltered, for any energy employed in overcoming inertia at one part of the stroke must be given out again at another part, so that the total energy exerted by the steam remains the same. Further, when there are a pair of cranks at right angles the total crank effort is little altered. The effect is best seen by correcting an indicator diagram for the inertia of reciprocating parts in the following way. Consider, for simplicity, a theoretical indicator diagram (Fig. 100) $SQZA$, in which BB is the back-pressure line, QZ the expansion curve, then, but for inertia, the ordinates reckoned from BB of SQZ give the effective pressure of the steam. Set up *BL* equal to the pressure necessary to start the piston found above and draw the straight line *LCL*, then the actual effective pressure will be obtained by measuring the ordinates to the sloping base *LCL* instead of the original base *BB*. It will be seen that the general effect is to equalize the steam pressure throughout the stroke.

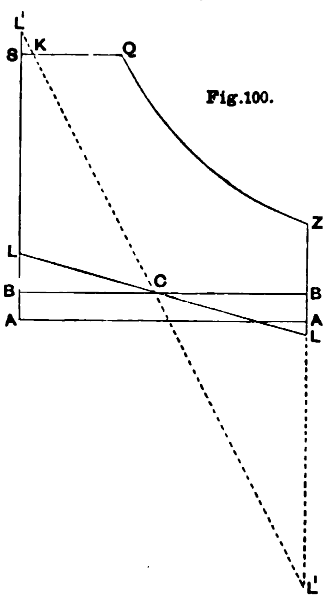


Fig. 100.

In engines running at a very high speed the pressure necessary to start the piston at the commencement of the stroke may be greater than the steam pressure (see Ex. 11, p. 219), which will be shown on the diagram by the point *L* rising above *S*, as shown by the dotted line *L′CL′* of the figure. The direction of stress on piston rod and connecting rod is then reversed, which will produce a shock if the brasses are at all loose. This gives a limit to the speed with which the engine can safely be driven.

In obtaining the preceding results it has been supposed, first, that the crank rotates uniformly and, secondly, that the connecting rod is indefinitely long. To take account of the variation in the velocity of the crank, it would be necessary to draw a curve representing that velocity, and deduce from it a curve showing the kinetic energy of the piston in every position. In general, however, the inertia of the rotating parts will be sufficient to reduce the variation in speed within narrow limits, and the error caused by neglecting it may be disregarded. The effect of obliquity is of more importance: to obtain it we may either use the formula for piston velocity given on p. 102, instead of the simpler formula employed above (Ex. 13, p. 220), or we may derive a curve of kinetic energy from the known curve of piston velocity and take the differences of equidistant ordinates. For the sake of variety, however, we will employ a method depending on a different principle.

Divide the crank-pin circle into a number of equal parts, and supposing the connecting rods drawn, let them cut the vertical through 0 in the points 1′, 2′, 3′ in Fig. 101. Also find and mark off the corresponding positions of the piston 1″, 2″, 3″, &c. Now, since the lengths, 01′, 02′, 03′, &c., represent the velocities of the piston and reciprocating parts when in positions 1″, 2″, 3″, &c., the difference between any two consecutive lengths, for example 1′, 2′, will represent the change of velocity that has taken place in the corresponding movement of the piston 1″, 2″. If we suppose the crank pin to revolve uniformly and divide the circle into equal parts, equal times will be occupied in the motions from point to point, and therefore equal times in the motions between consecutive positions 1'', 2'', 3'', 4'', &c., of the piston. Accordingly the differences 01', 1'2', 2'3', &c., will represent the force required to change the velocity of the reciprocating parts; and if we set them up as ordinates between the corresponding positions of the piston, we shall obtain the curve expressing the effect of inertia. The ordinate should be erected from the position of the piston when the crank-pin is at the middle of the intervals 1, 2, 3, &c.

It will be seen that the greater the number of parts into which we divide the crank-pin circle the less will be the ordinates representing the effect of inertia, though in all the curves the same character will be preserved. Accordingly it is possible to determine the number of parts into which the crank circle should be divided, or to determine the angle between consecutive radii, 01, 02, &c., such that the ordinates of the inertia curve be of such a length that they represent the pressure per square inch of piston area required for inertia on the same scale that the indicator diagram is drawn. The ordinates of the resulting inertia curve may then be directly employed to correct the indicator diagram.

Fig. 101.

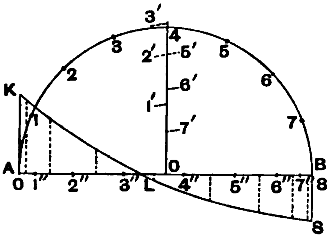
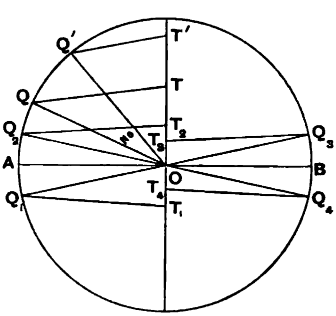


Fig. 102.



Let N be the number of revolutions per minute; Q, Q' consecutive points on the crank-pin circle; and let $QOQ' = n°$. Further suppose that the crank-pin circle is drawn on a scale of x inches to the foot. Then

$$\frac{\text{change of velocity of piston}}{\text{velocity of crank-pin}} = \frac{TT'}{OQ},$$

$$\therefore \text{ change of velocity of piston } \Delta v, \text{ in feet per second} = \frac{2\pi N}{60} TT',$$

where TT' is to be measured in feet on the scale x inches = 1 foot.

$$\therefore \quad \Delta v = \frac{2\pi N}{60} \frac{TT'}{x}, \text{ where } TT' \text{ is to be measured in inches.}$$

Now this change of velocity takes place in the time occupied by the movement through QQ given by

$$\Delta t \text{ seconds} = \frac{60}{N} \cdot \frac{n°}{360} = \frac{n°}{6N}.$$

Dividing Δv by Δt we get the rate of change of velocity,

$$\frac{\Delta v}{\Delta t} = \frac{2\pi N}{60} \cdot \frac{(TT') \text{ inches}}{x} \cdot \frac{6N}{n°}.$$

Now the mass of the reciprocating parts $\times \frac{\Delta v}{\Delta t}$ will be the magnitude of the force due to inertia.

$$\therefore \text{ Force due to inertia} = \frac{W}{g} \frac{\Delta v}{\Delta t} = \frac{W}{g} \frac{2}{10} N^2 \frac{(TT' \text{ in inches})}{x n°}.$$

Divide both sides by the area of the piston, and let p = pressure per square inch due to inertia, which will be in lbs. if W is taken in lbs. Also let p_0 = pressure equivalent to weight of reciprocating parts in lbs. per square inch.

$$\therefore p = p_0 \frac{2\pi}{10g} N^2 \frac{(TT' \text{ inches})}{xn^\circ}.$$

If now the indicator diagram be drawn on a scale of y lbs. to the inch, the pressure p equivalent to inertia will be represented on the same scale by taking a length in inches—

$$\frac{p}{y} = p_0 \frac{2\pi}{10g} \frac{N^2}{xyn^\circ} (TT' \text{ inches}).$$

Now it is required that n° be so taken that p/y in inches shall be the same thing as TT' in inches. Consequently

$$p_0 \frac{2\pi}{10g} \frac{N^2}{xyn^\circ} = 1.$$

$$\therefore n^\circ = \frac{2\pi}{10g} p_0 \frac{N^2}{xy} = \cdot 0195\, p_0 \frac{N^2}{xy}.$$

If now we draw a number of cranks inclined to each other at an angle of n°, we may obtain as many points on the curve of inertia as we please.

In practice it will in general be sufficient if we determine the end ordinates AK, BS, and the point L (Fig. 101) and draw a fair curve through these points. The ordinates AK and BS will be determined if we take (Fig. 102) AOQ_1, AOQ_2, BOQ_3, BOQ_4, each equal $\frac{1}{2}n^\circ$, then Q_1OQ_2 and Q_3OQ_4 being each equal n°, T_1T_2 will equal AK and $T_3T_4 = BS$. They are, however, most simply determined by use of the formula given in Ex. 13, p. 220.

Further, the curve will cross the base line at the point L, at which the piston will have its maximum velocity, which will occur approximately when the crank is at right angles to the connecting rod.

$$\therefore OL = \sqrt{(\text{con. rod})^2 + (\text{crank})^2} - \text{connecting rod}.$$

110. *Construction of Curves of Crank Effort for any given Indicator Diagram.*—If the varying magnitude of the steam pressure is given by the actual indicator diagram of the engine we may deduce the true crank effort as follows:—Let Fig. 1, Plate V., be a pair of indicator diagrams. The examples chosen are from the low-pressure cylinder of H.M.S. "Nelson." * Before proceeding to make use of them they should be corrected for inertia, and, where the engines are vertical, for the weight of the reciprocating parts. The curve of pressure due to inertia is KLS in Fig. 1, which has been drawn, as just described, to the same scale as the indicator diagram. If we draw a line MN parallel to the base line of the inertia curve to represent p_0, the pressure due to the weight of the reciprocating parts, then the intercept between MN and KLS will be the necessary correction for inertia and weight combined. In applying the correction, the forward pressure in one of the pair of diagrams should be taken in conjunction with the back pressure of the other, for it is the difference between these which gives the true effective pressure on the piston. Let the dotted lower curves be the result of the correction, so that the virtual pressure which is transmitted to the crank pin is to be measured by the vertical intercept between the upper steam

---

* The author is indebted to Mr. T. Hearson for the example here given, and for the method of drawing the curve of inertia which has just been described.

Plate.V.

Scale 1 inch = 12 lbs

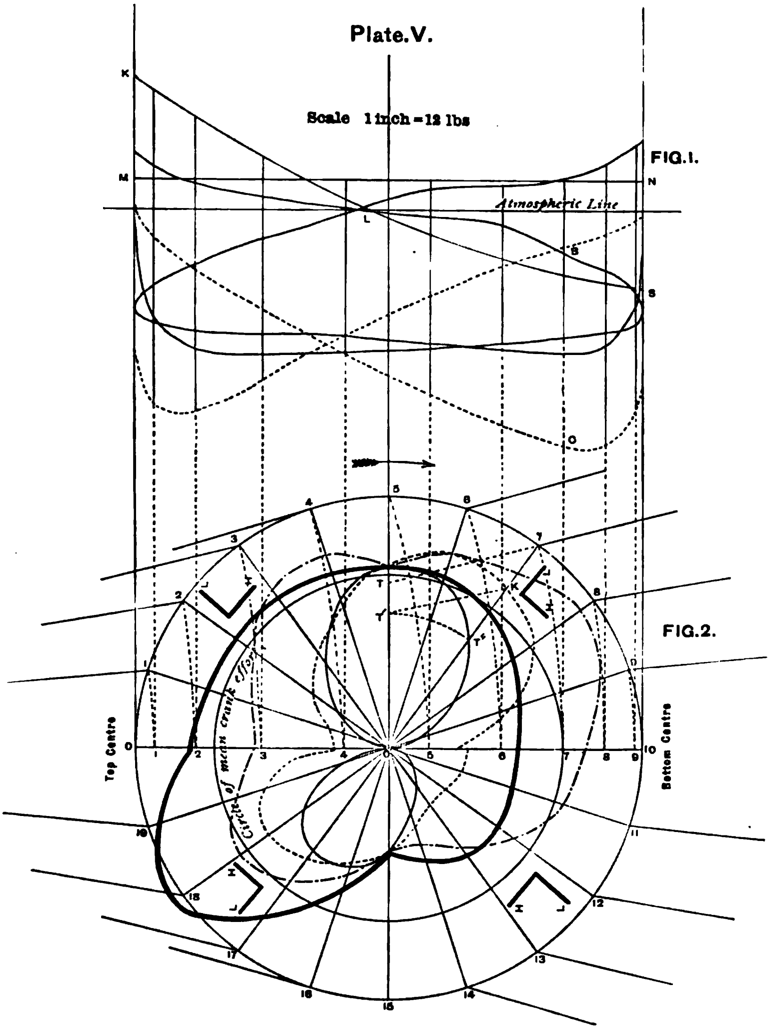


FIG. 1.

FIG. 2.

Co-efficients-

Of Crank Effort, 1·72. Of Fluctuation of Energy, ·108.

To face page 216.

curve and the dotted curve, such as BC for example. Immediately below the diagram draw a crank-pin circle with diameter equal to the length of the indicator diagrams. Divide the crank-pin circle into, say 20, equal parts, and suppose the crank pin to be successively at these points of division; determine the corresponding positions of the piston in its stroke. Whilst doing this, mark the directions in which the connecting rod lies when the crank pin is in these several positions. Let the positions of the piston in the line of stroke be set off along the diameter 0, 10. Through these points draw verticals to intersect the indicator diagrams. The intercepts of these verticals will give us the virtual steam pressure at each of the points of the stroke and corresponding to each position of the crank in its revolution. Next, having in Fig. 2 drawn a number of radii through the points 1, 2, 3, &c., lay off from the centre O along each, the respective intercepts of the indicator diagram which represent the virtual pressures of the steam when the cranks are in those positions. We thus draw what we may call a polar curve of virtual steam pressure. We have for example taken OK equal to BC in the figure, and similarly for all other radii.

Now, referring to page 186, we observe that if the connecting rod in any position be drawn to cut the vertical through O, in a point T, as for example in Fig. 2 when the crank is at 7, then the length OT will represent the crank effort on the same scale that the length of the crank arm $O7$ represents the magnitude of the steam pressure. If now through K we draw KT' parallel to $7T$, then by similar triangles $\frac{OT'}{OK} = \frac{OT}{O7}$, and thus on the same scale that OK represents the steam pressure OT' will represent the crank effort. Now along the crank $O7$ set off a length $OT'' = OT'$, and perform a similar operation for each of the positions of the crank. If through the points so obtained we draw a continuous curve it will be the polar curve of crank effort which we require, for it will represent by its radii in any position the actual crank effort when the crank is in that position; and we see that, in the construction, account is taken not only of the angular position of the crank, but also of the steam pressure which is available for turning the crank. Taking both indicator diagrams we thus draw the curve for the complete revolution of the engine. By transfer of the radii of the polar curve to the crank circle unrolled, we can construct a linear curve (Art. 105), and thus determine the fluctuation of energy.

In Fig. 2 the thick curve has been drawn to show the crank effort due to the high and low pressure cylinders combined, by adding to the radii of the original curve the corresponding radii of the high pressure curve (not shown in the figure) after correction for difference of scale. In this engine the high pressure crank is 90° in advance of the low: if it had been 90° behind the low the fluctuation of crank effort would have been less. This is shown by the large dotted curve in the figure. The circle of mean crank effort is added to facilitate comparison.

111. *Periodic Motion of Machines in General.*—The motion of a steam engine, which we have been describing in detail in this chapter, may be taken as a typical example of the transmission of energy by any machine whatever. Neglecting frictional resistances the energy is transmitted without alteration from a driving pair to a working pair—when the complete period of the machine is considered; but the rate of transmission varies from instant to instant during the period. The alternate excess and deficiency of energy is provided for by the moving parts of the machine, which serve as a store of energy or "kinetic accumulator," which can be drawn upon at pleasure. For equable motion it is necessary that they should be sufficiently heavy, and that the rotating pieces should greatly predominate over the reciprocating pieces. If the speed be very great reciprocating pieces are to be avoided altogether, especially in cases of higher pairing with force closure (Ex. 17, p. 220).

It has been supposed that the mean resistance at the working pair is exactly equivalent to the mean effort at the driving pair. If this be not the case the machine will rapidly alter its mean speed, till the balance is restored by alteration of the effort or the resistance or both. The balance seldom exists for long, and some means of controlling the machine is therefore generally indispensable, but this is a matter for subsequent consideration.

EXAMPLES.

1. In the case of a pair of cranks at right angles, draw the polar diagram of crank effort when the connecting rod is indefinitely long, and find the ratio of maximum crank effort to mean. Find also the position of the cranks when the actual crank effort is equal to the mean.

Maximum crank effort = 1·11 mean.

2. Draw the diagram and obtain the results as in the last question, when the length of connecting rod is equal to 4 cranks.

Maximum crank effort = 1·307 mean.

3. Draw the linear diagram of crank effort, assuming two cranks at right angles and connecting rod = 4 cranks.

4. What is the maximum length of connecting rod for which the crank effort is less than the mean throughout one quadrant?

Connecting rod = 7·1 cranks.

5. From the diagram of crank effort constructed in question 3, determine the co-efficient of fluctuation of energy, 1st. When the connecting rods are indefinitely long; 2nd. When the length equals 4 cranks.

Connecting rod indefinitely long. Co-efficient of fluctuation of energy = ·011.

Connecting rod = 4 cranks. Co-efficients are ·011, ·042, ·011, ·009, ·038, ·009.

6. A pair of engines of 500 H.P., working on cranks at right angles with connecting rods = 4 cranks, are running at 70 revolutions per minute. Find the maximum and minimum moments of crank effort, and the fluctuation of energy in ft.-lbs.; assuming the steam pressure and resistance uniform.

Maximum moment of crank effort = 49,125 ft.-lbs.
Minimum moment of crank effort = 29,465 ft.-lbs.
Mean moment of crank effort = 37,500 ft.-lbs.
Fluctuation of energy = 9,900 ft.-lbs. Co-efficient = ·042.

7. In the case of a single crank the steam is cut off at one-fourth of the stroke. Neglecting back pressure and inertia, find the ratio of maximum to mean crank effort, and also the ratio of the fluctuation of energy to the energy of one revolution.

Maximum = 2·9 mean crank effort.
Fluctuation of energy = ¼ energy of one revolution.

8. Construct a diagram of crank effort for three cranks at angles of 120°. The lines of stroke of the three pistons are parallel, the steam pressure constant, and the resistance uniform. Find the ratio of maximum to mean crank effort, and the co-efficient of fluctuation of energy for a connecting rod of 4 cranks.

Maximum = 1·077 mean crank effort.
$k =$ ·0115.

9. In a pair of cranks at right angles, connecting rod 4 cranks long, the reciprocating parts have a stroke of 4 feet and weigh 20 tons. The steam pressure is uniform, and equal to 50 tons on each piston, and the resistance moment is uniform. Find the least number of revolutions the engines can make without the aid of a fly-wheel, and draw a curve of angular velocity ratio for this case.

Ans. At the point of maximum speed the least number of revolutions will be 50 per 1′. To obtain the curve and the least number of *complete* revolutions, see p. 211.

10. The pressure equivalent to the weight of the reciprocating parts of an engine is 4 lbs. per square inch, the stroke is 4 feet. Find the pressure necessary to start the piston, when the engines are making 75 revolutions per minute. If the steam pressure be initially at 30 lbs. above the atmosphere, and the cut-off at ¼th the stroke, find the effective pressure at each eighth of the stroke, taking account of the inertia of the piston, and assuming a constant back pressure of 3 lbs.

Pressure equivalent to inertia at commencement of stroke = 15·3 lbs. per sq. in.
Effective pressure at commencement = 26·4
,, ,, 1st eighth = 30·3
,, ,, 2nd ,, = 34·0
,, ,, 3rd ,, = 23·
,, ,, 4th ,, = 19·4
,, ,, 5th ,, = 18·7
,, ,, 6th ,, = 19·5
,, ,, 7th ,, = 21·2
,, ,, 8th ,, = 23·5

11. In the last question find the number of revolutions per minute necessary to produce a shock near the commencement of the stroke. If the steam be cut off at ⅛th, or earlier, show that a shock occurs also at other points of the stroke. *Ans.* 124.

12. In question 10 construct a curve showing the kinetic energy of the piston at each point of the stroke, and deduce a curve showing the pressure due to inertia of the piston.

Take the curve of piston velocity previously constructed, and PN being any ordinate of it, the kinetic energy of the piston will be proportional to th square of PN, so we have only to draw a curve whose ordinates vary as $(PN)^2$.

Having drawn the curve of kinetic energy, take the difference between consecutive equidistant ordinates of that curve and set them as an or dinate from a new base line AB as Cd, and so construct a curve whose ordinates will be proportional to the pressure equivalent to inertia.

13. By use of the formula

$$V = V_0 \left(\sin\theta + \frac{1}{n} \cdot \sin\theta \cdot \cos\theta\right)$$

(page 102) for the velocity of the piston, prove that the pressure necessary to start and stop the piston at the ends of the stroke is given by

$$p' = p_0 \frac{V_0^2}{ga}\left(1 \pm \frac{1}{n}\right).$$

14. Draw a curve of kinetic energy of an oscillating cylinder, assuming a mean radius of gyration for the cylinder and piston, and deduce the bending moment on the piston rod.

NOTE.—The force of inertia in this case is so great that the speed of oscillating engines is limited.

15. If n be the revolutions per minute of a fly-wheel and d its diameter: show that the weight of wheel necessary for a given regularity in an engine of given indicated power is

$$W = C \cdot \frac{IHP}{n^3 d^2},$$

where C is constant.

NOTE.—The diameter is generally about 3½ times the stroke (S), and according to a well-known empirical rule for piston speed (V) employed in calculating nominal horse-power $V^3 \propto S$. If this be assumed $n^3 d^2$ is constant, and the weight of wheel is then proportional to the indicated horse-power, a rule sometimes employed, 100 lbs. being allowed for each horse-power.

16. The fluctuation of energy of an engine of 150 $I.H.P.$ is 13 per cent. of the energy exerted in one revolution. The revolutions are 35 per minute, find the weight of a fly-wheel 20 feet in diameter, that the fluctuation in speed may not exceed one-fortieth. *Ans.* 8 tons.

17. In the cam movement shown in Fig. 1, Plate IV., page 154, suppose the cam a circular disk of radius equal to the stroke of the sliding piece. Supposing the force of the spring twice the weight of sliding piece: find the greatest number of revolutions per 1′ the mechanism can make when the cam rotates uniformly.

Ans. If S be the stroke in inches, n the revolutions,

$$n = \frac{216}{\sqrt{S}}.$$

18. In a 3-cylinder Brotherhood engine, the stroke is S inches, the revolutions n per 1′, the total pressure on one piston P; show that, to avoid reversal of the stress on the piston rod, the weight of a piston and rod must not exceed

$$W = 70{,}500 \frac{P}{n^2 S},$$

NOTE.—In a double acting engine there is necessarily reversal at the ends of the stroke: in the Brotherhood this is avoided by the use of 3 cylinders at 120°, the inner ends of which communicate constantly with a central chamber containing steam at full pressure. These engines therefore may run at a very high speed if the cut off at the outer end be sufficient.

CHAPTER X.

FRICTIONAL RESISTANCES.

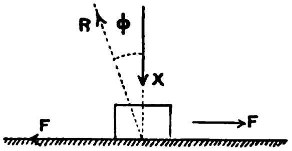
112. *Preliminary Remarks.*—The action of a machine consists, as we have seen, in a transmission of energy from a driving pair to a working pair, through a number of intermediate pairs, which change in a given way the motions proper to the source of energy. In the absence of friction, the energy transmitted from piece to piece in a complete period would be the same for all the pairs, but, in consequence of frictional resistances, a certain part of the energy is lost at each transmission. These frictional resistances are of two kinds, one due to the relative motion of the elements of the pairs one upon another, the other to the changes of form which the flexible parts of the machine undergo, for example to the bending of ropes and belts. It is to the first kind that the word "friction" is specially appropriated, although it is not essentially different from the second kind, which in some cases is also called "stiffness."

We commence with the case of linkwork mechanisms in which the friction is due simply to the sliding of one surface upon another. The pairing is in this case of the lower class.

SECTION I.—EFFICIENCY OF LOWER PAIRING.

113. *Ordinary Laws of Sliding Friction.*—If one body rests on another (Fig. 102) and is pressed against it with a force X, a mutual action takes place between the two which resists sliding. The magnitude of this mutual action or tangential stress (Ch. XII.) is measured by the force F which is necessary to produce sliding, and the ratio F/X is called the co-efficient of friction and will be denoted by f. The value of f depends on the nature and condition of the surfaces in contact, whether rough or smooth, dry or lubricated. Under certain circumstances and within certain limits it is

Fig. 102.



independent of the area of the surfaces in contact and of the velocity of sliding. These statements may be called the "ordinary" laws of friction. The evidence on which they rest and the limitations to their truth will be considered hereafter; for the present we assume them as applicable to all the cases we consider.

The work done in overcoming friction may be estimated just as in the case of any other resistance. If the body move through a space x the work done is Fx or $f.Xx$ if X be uniform, and if it be not, a curve is constructed giving X at every point, then the area under that curve multiplied by the co-efficient f is the work done (see Ex. 2). If R be the reaction of the surface upon which the body we are considering rests, ϕ the angle its direction makes with the normal to the plane,

$$R\,.\cos\phi = X:\ R\,.\sin\phi = F;$$

$$\therefore \tan\phi = f,$$

an equation which shows, that the total mutual action between two plane surfaces, which slide over one another, makes an angle with the normal to the plane, the tangent of which is the co-efficient of friction. The magnitude of this angle then is fixed, but its direction varies according to the direction of the sliding. It may therefore be called the "friction angle," but it is also often called the "angle of repose," because it is the greatest inclination of a plane on which the body can rest under the action of gravity without slipping. In the solution of questions respecting friction, graphically or otherwise, it is often convenient to suppose it known.

114. *Friction of Bearings.*—Next suppose the surfaces in contact cylindrical. In Fig. 103 ABA represents a cylinder pressed down into a semicircular bearing by a force S, the direction of which passes through the point O, which is the intersection of the axis of the cylinder with the plane of the paper. We may take this to represent the ordinary case of a shaft and its bearing from which the cap has

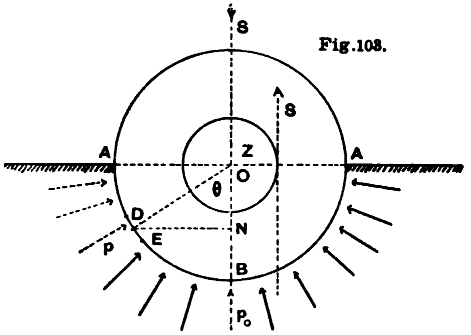


Fig.103.

been removed, S being the resultant of all the forces acting on the shaft which, for the moment are supposed to have no tendency to turn the shaft. The force S is balanced by the reaction of the bearing which, when the bearing is in good condition, consists of a pressure distributed over the whole semi-cylindrical surface. Let DE be a small element of the surface, p the pressure, θ the angle the radius of DE makes with the direction of S, then we must have

$$\Sigma p\, DE \cos\theta = S.$$

If now we knew the law according to which p varies from point to point, we could by use of this equation find the actual value of p and also find the total amount of the distributed pressure, that is to say, $\Sigma p . DE$ which we will call X. Evidently then we shall have

$$X = k . S,$$

where k is a co-efficient depending on the law of distribution and therefore to some extent uncertain. When a bearing is well worn and imperfectly lubricated it is probable that (see Art. 115) if p_0 be the pressure at B

$$p = p_0 . \cos\theta,$$

that is, that the intensity of the pressure at any point varies as ON the distance of the point below the centre. This is the same law as that which the pressure of a heavy fluid follows, supposed occupying the semicylinder ABA, and it is shown in books on hydrostatics that

$$\frac{\text{Total pressure}}{\text{Resultant pressure}} = \frac{4}{\pi} = k.$$

Next suppose the shaft to be turned by the action of a couple M applied to it, then if a be the radius

$$M = \Sigma f . p . DE . a = f . Xa = fk . Sa.$$

In this formula we have some doubt as to the value of k, and we are not sure that the co-efficient f would be the same for a curved as for a plane surface; we therefore replace fk by f', where f' is a special co-efficient of axle friction determined by experiment. If there is a cap on the bearing, which is screwed down, the value of S is increased by an amount about equal to the tension of the bolts.

The loss of energy per revolution in overcoming axle friction is evidently $M . 2\pi$, or if d be the diameter,

$$\text{Work lost} = \pi f' Sd.$$

The reaction of the bearing surface on the shaft is partly normal and partly tangential. The normal part balances S and the tangential part balances M, hence the two parts may be combined into a single force opposite and parallel to S at such a distance z from O that

$$Sz = M, \text{ or } 2z = f'd,$$

that is to say, the line of action of the mutual action between the shaft and its bearing always touches a circle, the diameter of which is f times the diameter of the shaft. This circle is called the Friction Circle of the shaft or pin considered. When the bearing has a cap on, the force S must be increased by the tension of the bolts in calculating M, but not for any other purpose, and the diameter of the friction circle is consequently increased, it may be very considerably. The utility of this rule will be seen presently.

The real pressure between a shaft and its bearing varies from point to point, as we have seen. What is conventionally called the "pressure on the bearing" is something different. Let l be the length of the bearing, then ld is the area of the diametral section, and

$$p = \frac{S}{ld}$$

is the quantity in question. It is a sort of mean value of the actual pressure, and will bear some definite relation to it depending on the law of pressure. For the particular law of pressure given above

$$p = p_0 \cdot \frac{\pi}{4}.$$

The work lost by friction per square inch of bearing surface per 1′ is evidently proportional to pv, where v is the rubbing velocity in feet per minute. An equivalent amount of heat is generated as we shall see hereafter, and it is upon the rate at which this heat can be abstracted by the cooling influences to which the bearing is exposed that the amount of bearing surface required depends. In marine engine bearings the value of pv is sometimes as much as 60,000, though at the expense of a considerable liability to heating, and in railway machinery it is not less. At lower speeds the value is smaller. According to a rule given by Rankine,

$$p\,(v + 20) = 44{,}800.$$

115. *Friction of Pivots.*—In pivots and other examples in which the revolving shaft is subject to an endways force the surfaces in contact are frequently conical. In Fig. 104 a conical surface AB is pressed against a corresponding conical seating by a force H, and revolves at a given rate. If the surface be divided into rings, one of which is seen in section at DE, the pressure on those rings may be resolved vertically upwards, and must

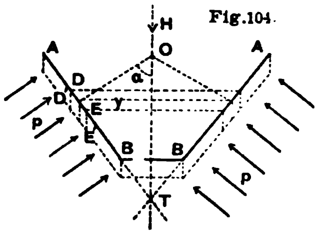


Fig.104.

then balance H. Hence if p be the pressure on DE a ring the radius of which is y,

$$\Sigma p \,.\, DE \,.\, 2\pi y \cos \alpha = H,$$

where α is the angle a normal to the conical surface makes with the axis.

When the bearing is somewhat worn the conical surface will have descended through a certain space, and it may be assumed that all points such as DE will descend through an equal space, so that the wear of the surface measured normal to itself is proportional to $\cos \alpha$. But if v be the velocity of rubbing of the ring DE, the wear will be proportional to pv, that is to py: hence

$$py \propto \cos \alpha.$$

This principle determines the most probable distribution of the pressure on worn surfaces in any case, and has already been used above for the case of a journal. In the present case α is constant, and we have

$$py = \text{constant} = p_1 y_1 = p_2 y_2,$$

where the suffixes 1 and 2 refer to the upper and lower edge; hence, by substitution, if l be the length AB of the conical surface,

$$py \,.\, 2\pi l \,.\, \cos \alpha = H,$$

a formula which determines the pressure at every point. The moment of friction is evidently

$$\begin{aligned} M &= f\Sigma p DE 2\pi y^2 \\ &= f \,.\, py \,.\, 2\pi \,.\, \Sigma y \,.\, DE = \frac{fpy 2\pi \Sigma y \Delta y}{\cos \alpha}, \end{aligned}$$

where Δy is written for the projection of DE on the transverse plane. By use of the integral calculus this is readily seen to be

$$M = fpy 2\pi \frac{y_1^2 - y_2^2}{2 \cos \alpha} = fpy 2\pi l \,.\, \frac{y_1 + y_2}{2},$$

$$\text{or } M = f \,.\, H \,.\, \frac{y_1 + y_2}{2 \cos \alpha},$$

a formula which shows that the friction is the same as that of a ring of small breadth, of diameter equal to the mean of the greatest and least diameters of the portion of a cone considered. In the case of a simple flat-ended pivot the equivalent ring is half the diameter of the pivot. If the pressure were uniform throughout, the diameter of the equivalent ring would be ⅔ instead of ½ the diameter of the pivot, and the actual diameter in practice will probably vary between these limits.

Pivots are sometimes used in which the surfaces in contact are not cones, but are curved, so that in wearing the pressure and wear are the same throughout (Schiele's pivots). That this may be the case we must have, since p is constant,

$$y \propto \cos \alpha,$$

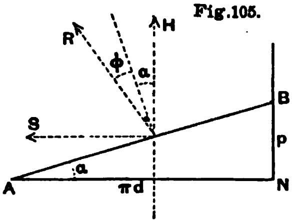
that is to say, if we draw a tangent *DET* to meet the axis in *T*, *ET* must be constant. The curve which possesses this geometric property is called the "tractrix." It is traced readily by stepping from point to point, keeping the tangent always of the same length. Pivots of this kind are very suitable for high speeds, as the wear is very smooth.

116. *Friction and Efficiency of Screws.*—In any case of a machine in steady motion the principle of work takes the form (Art. 96)

$$\left.\begin{matrix}\text{Energy exerted}\\ \text{in a period}\end{matrix}\right\} = \left\{\begin{matrix}\text{Useful work done + Work wasted}\\ \text{in overcoming frictional resistance.}\end{matrix}\right.$$

The simplest case is that of a screw which we will suppose to be square threaded and applied to a press, or to some similar purpose. The pressure between the nut and the thread is distributed uniformly along the thread, if the screw be accurately constructed and slightly worn. As shown in the last article in the similar case of a pivot, the friction may be regarded as concentrated on a spiral traced on a cylinder the diameter of which may be expected to be about the mean of the external and internal diameter of the screw. Fig. 105 shows one convolution of this spiral unrolled. *AB* is the thread, *BN* parallel to the axis of the screw, is the pitch *p*, and *AN* is the circumference πd. *H* is the thrust of the screw, being the force which the screw is overcoming by means of a couple applied to turn it about its axis. *R* is the action of the screw thread which (Art. 113) makes an angle ϕ with the normal, where ϕ is the angle of repose. The normal itself makes an angle α with the axis of the screw, where α is the pitch angle given by the formula

Fig. 105.



$$\tan \alpha = \frac{p}{\pi d}.$$

This force *R* arises from the turning forces applied to the screw, and must have the same moment *M* about the axis of the screw; its vertical component therefore must be *H* and its transverse component a force *S* such that

$$S \cdot \frac{d}{2} = M.$$

Hence the equations

$$M = \frac{Rd}{2} \cdot \sin(\alpha + \phi),$$

$$H = R \cdot \cos(\alpha + \phi).$$

Also considering a complete revolution of the screw,

$$\text{Energy exerted} = M \,.\, 2\pi = R\pi d \,.\, \sin(\alpha + \phi),$$
$$\text{Useful work done} = H \,.\, p = Rp \,.\, \cos(\alpha + \phi),$$

from which it follows that the efficiency of the screw is

$$\text{Efficiency} = \frac{\tan \alpha}{\tan(\alpha + \phi)}.$$

It is not difficult to show that this fraction is greatest when $\alpha = 45° - \frac{1}{2}\phi$, and its value is then

$$\text{Maximum efficiency} = \left(\frac{1 - \frac{1}{2}f^2}{1 + \frac{1}{2}f}\right) \text{approximately.}$$

For ordinary values of f then, the best pitch angle is approximately 45° and the efficiency is considerable.

In practice, however, the pitch angle is much smaller, its value in bolts and the screws used in presses ranging from ·035 in large screws to ·07 in smaller ones; the efficiency is then less, often much less, than one third, the object aimed at being not efficiency but a great mechanical advantage.

If the pitch be sufficiently coarse, it will be possible to reverse the action, the driving force being then H and the resistance a moment opposing the rotation of the screw. In a well known kind of hand drill and a few other cases this occurs in practice; the force R is then inclined on the other side of the normal, and the efficiency is in the same way as before found to be

$$\text{Efficiency} = \frac{\tan(\alpha - \phi)}{\tan \alpha}.$$

In most cases, however, α is less than ϕ, and the screw is then incapable of being reversed. Non-reversibility is often a most valuable property in practical applications, the friction then serving to hold together parts which require to be united or to lock a machine in any given position.

In estimating the efficiency of screw mechanisms the friction of the end of the screw acting like a pivot or of the nut upon its seat must be included; in screw bolts this item is generally as great as the friction of the threads. The friction due to lateral pressure of the screw on its nut may usually be neglected, but when necessary it may be estimated by the same formula as is used for shafts. The above investigation, strictly speaking, applies only to square-threaded screws; it has, however, been shown that the efficiency is only slightly diminished by the triangular or other form of thread usually adopted for the sake of strength.* The formulæ here given for screws may be applied to any case of a sliding pair in which the driving effort is at right angles to the useful resistance. A simpler case is that in which the driving effort is parallel to the direction of sliding. This is given in Example 1, page 244. In all cases observe that the efficiency diminishes rapidly when the velocity-ratio is increased. This, which is common to most mechanisms, limits the mechanical advantage practically attainable. The hydraulic press is an exception, as will be seen hereafter.

---

* *Cours de Mécanique Appliquée aux Machines*, par J. V. Poncelet, p. 386. Paris, 1874.

117. *Efficiency of Mechanism by Exact Method.*—In the preceding cases the efficiency is the same for any motion of the mechanism whether large or small. Generally, however, it will be different in each position of the mechanism, and by the "efficiency of the mechanism" is then to be understood the ratio of the useful work done in a period to the energy exerted in the period.

The exact calculation of the loss of work by frictional resistances in mechanism is generally very complicated, so that it is best to proceed by approximations the nature of which will be understood on considering an example with some degree of thoroughness. The case we select is that of the mechanism of the direct-acting vertical steam engine such as is represented in Plate I., p. 108.

The losses by friction are (1) the loss by piston friction, (2) friction of guide bars, (3) friction of crosshead pin, (4) friction of crank pin, (5) friction of crank-shaft bearings. Of these, the first two are considered separately (Ex. 2, p. 244), and for the present neglected, while the last three are treated by a graphical method as follows.

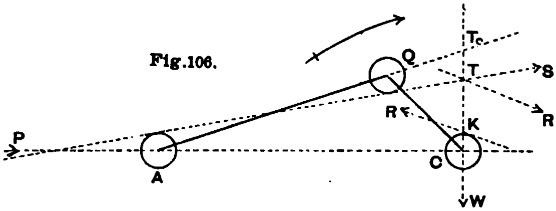


Fig.106.

In Fig. 106 *CQA* are the friction circles of the three parts in question, which for the sake of clearness are drawn on a very exaggerated scale while the bearings themselves are omitted. We will neglect the weight of the connecting rod and its inertia; of these the first is generally relatively inconsiderable, but in high speed engines the last is often very large and makes the friction very different at high speeds and low speeds (see Ch. XI.). The weight of the crank shaft and all the parts connected with it is supposed to act through the centre of the shaft; for simplicity we will call it *W*. The pressure on the piston after correction for piston and guide-bar friction is denoted by *P*. Then, in the absence of friction, the line of action of the thrust on the connecting rod is the line joining the centres of the friction circles, and the moment of crank effort is $P.CT_0$, where T_0 is the intersection of that line with the vertical through *C*. But the line of action in question must now touch the friction circles (Art. 114), and the true moment of crank effort on the same principle must be $P.CT$, where *T* is the intersection of this common tangent with the vertical *CT*. Thus $P.TT_0$ is the correction for friction of the crosshead and crank pins. Next observe that the forces acting on the crank shaft are *W* the weight and *S* the thrust of the connecting rod; these may be compounded into one force *R* passing through *T* as shown in the diagram. The reaction of the crank-shaft bearing is an equal and opposite force *R* which must touch the friction circle and cut *CT* in a certain point *K*. Now the horizontal component of *R* is the same as that of *S*, namely *P*; therefore the true moment of crank effort after allowing for friction is $P.TK$.

By performing this construction for a number of positions, as in the last chapter, we obtain a diagram of crank effort corrected for friction. The area of this curve will give us the useful work done in a revolution, the ratio of which to the energy exerted is the efficiency of the mechanism: and its intersections with the line of mean resistance will give the points of maximum and minimum energy and the fluctuation of energy as corrected for friction. When the crank makes a certain angle with the line of centres *TK* vanishes. Within this angle no steam pressure, however great, will move the crank, as is well known in practice. It may be called the "dead angle," all points within it being dead points.

118. *Efficiency of Mechanism by Approximate Method.*—The process just described is not too complicated for actual use in the foregoing example, but in many cases it would be otherwise, and it may therefore be frequently replaced with advantage by a calculation of the efficiency of each of the several pairs of which the mechanism is made up taken by itself.

Each pair consists of two elements, one of which transmits energy to the other, with a certain deduction caused by the friction between the elements. The ratio of the energy transmitted to the energy received may be called the efficiency of the pair. If c_1, c_2, c_3... be the efficiencies of all the pairs in the mechanism it is evident from the definition that the efficiency of the whole mechanism must be

$$c = c_1 \,.\, c_2 \,.\, c_3 \ldots$$

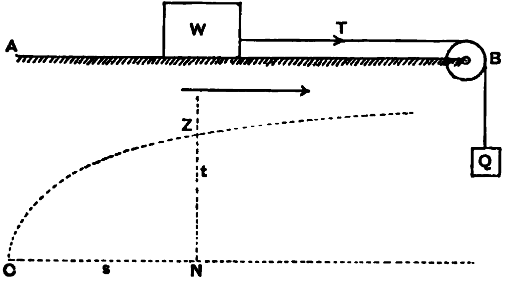
In some cases the efficiency of each pair will be independent of the frictional resistances of all the other pairs, and may be found separately.

In general this is approximately, but not exactly, true, a point which will be best understood by a consideration of the foregoing diagram. For example, the friction of the guide bars is diminished in consequence of the friction of the crank pin, because the obliquity of the connecting rod is virtually diminished. The supposition is, however, often sufficiently nearly true to enable a rough estimate to be made of the efficiency of the mechanism by finding the efficiencies of the several pairs taken alone, all the others being supposed smooth. In doing this mean values are taken for variable forces, if the amount of variation be not considerable. The uncertainty and variability of the co-efficients on which frictional efficiency depends are such as to render refined calculations of little practical value.

119. *Experiments on Sliding Friction* (*Morin*).—The ordinary laws of friction, which may be comprised in the single statement that the co-efficient of friction depends on the nature of the surfaces alone, and not on the intensity of the pressure or on the velocity of rubbing, were originally given by Coulomb in a memoir, published in 1785, although some facts of a similar kind were previously known. They are therefore often called Coulomb's laws. Yet Coulomb's experiments were scarcely sufficient to establish them, and the subject was reinvestigated by others, especially by the late General Morin, whose memoirs were presented to the French Academy in 1831-4. Morin's experiments were so elaborate and exact that they may be considered as conclusively proving the truth of Coulomb's laws within certain limits of pressure and velocity, and under the circumstances in which they were made: it will therefore be advisable to explain them briefly.

A sledge loaded with a given weight was caused to slide along a horizontal bed AB more than 12 feet long (Fig. 107), the rubbing surfaces being formed of the materials to be experimented on. The necessary force was supplied by a cord passing over a pulley at B to a

Fig.107.



descending weight Q. The tension of the cord T was measured by a spring dynamometer, and could likewise be inferred from the magnitude of the weight after correction for the stiffness of the cord and the friction of the pulley. In one form of experiment the weights were so arranged that the sledge moved nearly uniformly: the corresponding friction was measured and found to be constant. In a second form, the times occupied by the sledge in reaching given points were automatically measured and compared with the spaces traversed, by setting them up as ordinates of the curve CZ shown below. The curve proved to be a parabola, showing that the space varied as the square of the time, from which it was inferred that the acceleration of the sledge was constant.

From both methods it appeared that the co-efficient of friction was exactly the same, whatever the pressure and whatever the velocity, provided the nature and condition of the surfaces were the same. A few important results are given in the annexed table; they are taken from Morin's latest memoir,* containing, besides many new experiments, tables of the results of the whole series. The limits to their application will be considered presently.

NATURE OF SURFACES.	CONDITION OF SURFACES.	CO-EFFICIENT OF FRICTION.
Wood on Wood,	Perfectly dry and clean, - -	·25 to ·5
Metal or Wood on Metal or Wood,	Slightly oily, -	·15
Do. do.,	Well lubricated,	·07 to ·08
Do. do.,	Lubricant constantly renewed,	·05

Full tables of Morin's results will be found in Moseley's work cited on page 241. The friction between surfaces at rest is often greater than when they are in motion, especially when the surfaces have been some time in contact: the excess, however, cannot be relied on, as it is liable to be overcome by any slight vibration.

120. *Exceptions to the Ordinary Laws in Plane Surfaces.*—From the exactitude with which Coulomb's laws were verified by Morin's experiments the inference was naturally drawn that they were universally true, but this is probably erroneous. Although no complete and thorough investigation has been made, it can hardly now be a matter of doubt that there are cases in which the laws of friction are widely different.

---

* Nouvelles Experiences . . . . faites à Metz en 1834. Page 99.

The known cases of exception for plane surfaces may be grouped as follows :—

(1) At low pressures the co-efficient of friction increases when the pressure diminishes. This has been shown by various experimentalists, as, for example, by Dr. Ball.* The lowest pressure employed by Morin was about three fourths of a lb. per square inch, and this is about the pressure at which the deviation noticed by Ball becomes insensible. This effect may be due to a slight adhesion between the surfaces independent of friction proper.

(2) At high pressures, according to certain experiments by Rennie,† the co-efficient increases greatly with the pressure. The upper limit of pressure in Morin's experiments was from 114 to 128 lbs. per square inch. At 32·5 lbs. per square inch Rennie found for metallic surfaces at rest ·14 to ·17, nearly agreeing with Morin; but on increasing the pressure the co-efficient became gradually greater, ranging from ·35 to ·4 at pressures exceeding 500 lbs. per square inch. The metals tried were wrought iron on wrought and cast iron, and steel on cast iron. Tin on cast iron showed only a slight increase in the co-efficient. In fully lubricated surfaces in motion we shall see presently the results are exactly opposite. This increased friction at high pressures may be due to abrasion of the surfaces.

(3) At high velocities the co-efficient of friction, instead of being independent of the velocity, diminishes greatly as the velocity increases. This was shown by M. Bochet in 1858. Similar results have been obtained by others, especially by Capt. Galton in some important experiments on railway brakes. ‡ The limit of velocity in Morin's experiments was 10 feet per 1″, and at somewhat greater velocities than this the diminution becomes perceptible. Morin's results have been shown to be applicable at the very lowest velocities by Professor F. Jenkin and Mr. Ewing. §

It appears difficult to explain the diminution at high speeds merely by a change in the condition of the surfaces; it should, probably, be regarded as part of the law of friction. Professor Franke in the *Civil Ingenieur* for May, 1882, has proposed the formula

$$f=f_0 \cdot \epsilon^{-av},$$

where f_0 is about ·29, and a (for velocities in metres per 1″) ranges from ·02 to ·04, according to the nature and state of the surfaces.

---

* *Experimental Mechanics*, by R. S. Ball, page 78. Macmillan, 1871.

† *Phil. Trans.* for 1829.

‡ See *Engineering*, vol. 25, pages 469-472.

§ *Phil. Transactions*, vol. 167, part II.

121. *Axle Friction.*—It has already been pointed out that the co-efficient of axle friction is not necessarily the same as that for plane surfaces sliding on one another, and, besides, the continuous contact of a shaft and its bearing is very different from the brief contact occurring in sledge experiments. Morin however made special experiments on the friction of axles and showed that the co-efficients were constant and nearly the same in the two cases. The diameters employed however were 4 inches and under, while the revolutions did not exceed 30 per minute, so that the rubbing velocity was not more than 30 feet per minute. The pressures were not great, the value of *pv* not exceeding 5,000.

Much greater values of *pv* than this occur in modern machinery, and then it is tolerably certain that the value of the co-efficient is much less and diminishes with the pressure. Already in 1855 M. Hirn had made a long series of experiments on friction, especially of lubricated surfaces. The following summary of his results is given by M. Kretz, editor of the third edition of the *Mécanique Industrielle.**

(*a*) That a lubricant may give a regular and minimum value to the friction it must be "triturated" for some time between the rubbing surfaces.

(*b*) The friction of lubricated surfaces diminishes when the temperature is raised, other things being equal.

(*c*) With abundant lubrication and uniform temperature friction varies directly as the velocity. When the temperature is not maintained uniform, the relation between friction and velocity depends on the law of cooling of the special machine considered. In ordinary machinery friction varies as the square root of the velocity.

(*d*) The friction of lubricated surfaces is nearly proportional to the square root of the area and the pressure.

The last result is equivalent to saying that the co-efficient of friction varies inversely as the square root of the pressure per unit of area. It is remarkable that this law has also been deduced by Professor Thurston from experiments made apparently without any knowledge of what Hirn had done† with pressures from 100 to 750 lbs. per square inch and a velocity of 150 feet per 1′.

It may be open to question whether Hirn's experiments are sufficient to establish all the above statements, but it cannot be doubted that for values of *pv* exceeding 5,000 the co-efficient of friction of well-lubricated bearings of good construction diminishes with the pressure, and may be much less than the value at low speeds as determined by Morin. Some further remarks on this question will be found in the Appendix.

---

* *Introduction a là Mécanique Industrielle*, par J. V. Poncelet. Troisième édition. Paris, 1870. Page 516.

† *Friction and Lubrication*, by R. H. Thurston. New York, 1879.

SECTION II.—EFFICIENCY OF HIGHER PAIRING.

We now proceed to consider higher pairing, commencing with the case of rolling contact. The friction is then described as "rolling friction."

122. *Rolling Friction.*—When a wheel rolls on soft ground the resistance to rolling is due to the fact that the wheel makes a rut and depresses the ground as it advances over it. Thus the resistance to motion is proportioned to the product of the weight moved into the depth of the depression. The depth of the rut depends on the radius as well as the breadth of the wheel. It is found that the resistance may be expressed by

$$R=\frac{bW}{r},$$

where W = weight, r = radius of wheel, and b is approximately a constant length. This might have been anticipated, since the depth of the rut is the versed sine of the arc of contact, and therefore for a given small arc is inversely as the radius. If the wheel roll on hard ground over a succession of obstacles of small height the law of resistance will be expressed by the same formula.

When the surface rolled over is elastic, and the pressure on it is not sufficient to produce a permanent rut, the resistance to rolling is not so easily explained. If we consider an extreme case, as for instance a heavy roller rolling on india-rubber, we shall be able to see to what action the resistance is due. The wheel will sink into the rubber, which will close up around it both in advance and behind, as shown in Fig. 108. At C the rubber will be most compressed. As the wheel advances and commences to crush the rubber in advance of it the rubber moves away to avoid the compression, heaping itself up continually in advance of the wheel. In this movement it rubs itself over

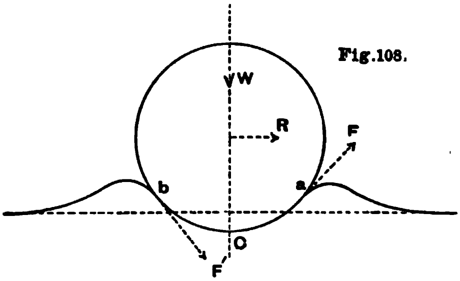


Fig. 108.

the surface *Ca* of the wheel, exerting on it a frictional force in the direction shown by the arrow *F*, which opposes the onward motion of the wheel. Again, the rubber in the rear is continually tending to recover its normal position and form of flatness, and in doing so rubs itself over the surface *bC* of the wheel in the direction shown by the arrow *F'*, which also tends to oppose the onward motion of the wheel. The effect of this creeping action of the rubber over the surface of the wheel is to cause the onward advance of the centre of the wheel to be different from that due to the circumference rolled out.* Moreover the vertical component of the reaction of the surface no longer passes through the centre of the wheel as it must do in the absence of friction, but is in advance by a small quantity b such that Wb is the moment of resistance to rolling.

Experiments on rolling resistance present considerable discrepancies, but within the limits of dimension of rollers which have been tried it appears that b is independent of the radius; this leads to a formula of the same form as before for the force necessary to draw the roller, namely

$$R=\frac{Wb}{r},$$

where b is a constant which for dimensions in inches is from ·02 to ·09 according to the nature of the surfaces. With very hard and smooth surfaces of wood or metal, the lower value ·02 may be employed. Rolling friction is not sensibly diminished by lubricants, but depends mainly on smoothness and hardness of the surfaces. It is probably influenced by the speed of rolling, but this does not appear to have been proved by experiment unless in cases where the resistance of the atmosphere and other causes make the question more complicated.

In many cases of rolling the surfaces are partly elastic and partly soft, so that the resistance to rolling is partly due to surface friction and partly to permanent deformation. The value of the constant b is then much increased. For wagon wheels on macadamized roads in good condition the value of b is about ·5", and on soft ground four to six times greater. The draught of carts is said to be increased by the absence of springs.

123. *Friction of Ropes and Belts.*—Frictional resistances are also produced by the changes of form and dimension of the parts of a machine occasioned either by the stresses necessarily accompanying transmission of energy or by shocks. In the present chapter we

---

* See a paper by Professor Osborne Reynolds, *Phil. Trans.*, vol. 166, to whom the true explanation of resistance to rolling in perfectly elastic bodies is due.

consider tension elements only, that is to say, chiefly ropes and belts.

In Fig. 109 *AB* is a pulley, the centre of which is *O*, over which a rope passes embracing the arc *AKB* and acted on by forces $T_1 T_2$ at its ends. If there be sufficient difference between T_1 and T_2 the rope will slip over the pulley notwithstanding the friction which tends to prevent it. Let the rope be just on the point of slipping, then its tension will gradually diminish from T_1 at *A* to T_2 at *B*. Let *T*, *T'* be the tensions at the intermediate points *K*, *L*, then the portion *KL* of the rope is kept in equilibrium by the forces *T*, *T'* at its ends, and a third force *S* due to the reaction of the pulley, the three forces meeting in a point *E*. On *OL* set off to *Ol* to represent *T*, and draw *lk* perpendicular to *S* to meet *OK* in *k*, then the sides and the triangle *Okl* will be proportioned by the three forces, so that *Ok* represents *T'* and *lk* *S*. The angle *S* makes with the radius will be the same for all arcs of the same length, and if *KL* be taken small enough will be the angle of friction (Art. 113).

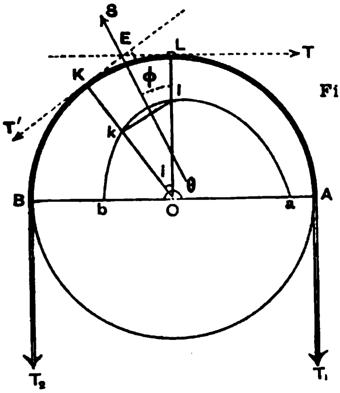


Fig. 109.

This construction can, if we please, be commenced at *A* and repeated for a number of small portions of the rope till we arrive at *B*; we shall obtain a spiral curve *alkb*, the last radius *Ob* of which represents T_2 on the same scale as the first *Oa* represents T_1. It is convenient however to have an algebraical formula to calculate T_2. Let the angle *KOL* be *i* and the angle *S* makes with the radius *φ*, then

$$\frac{T}{T'} = \frac{Ol}{Ok} = \frac{\sin Okl}{\sin Olk} = \frac{\cos (i - \phi)}{\cos \phi} = \cos i + \sin i \tan \phi.$$

If now the angle *i* be diminished indefinitely we may write $\cos i = 1$ and $\sin i = i$, so that

$$\frac{T - T'}{T'} = i \,.\, \tan \phi.$$

Replacing *i* by $\Delta\theta$, $T - T'$ by ΔT, and proceeding to the limit

$$\frac{1}{T}\frac{dT}{d\theta} = \tan \phi = f,$$

which being integrated gives

$$\frac{T_1}{T_2} = \epsilon^{f\theta},$$

where *f* is the co-efficient of friction, *θ* the angle subtended by the part of the pulley embraced by the rope, and ϵ the number 2·718 being the base of the Napierian system of logarithms. The formula is applicable even if the pulley be not circular. For a circular pulley the spiral curve, representing graphically the tension at every point, is the equiangular or logarithmic spiral of which the formula may be regarded as the equation. In constructing it graphically, the value of ϕ, for a small yet finite angle i, is found by replacing T/T by ϵ^{fi} and expanding the exponential: we thus get approximately

$$1 + fi = \cos i + \sin i \, . \tan \phi = 1 - \tfrac{1}{2} i^2 + i \, . \tan \phi,$$

$$\therefore \tan \phi = f + \tfrac{1}{2} i.$$

With small values of the co-efficient $2f$ may be a sufficiently small angular interval, but in general it will be advisable to take the angular interval equal to the angle of friction, then the value of ϕ is $1\frac{1}{2}$ times that angle. The construction being one in which errors accumulate, the formula is preferable when great accuracy is desired.

124. *Driving Belts.*—When a belt is stretched over a pulley by equal weights, the tension of the belt is not necessarily the same everywhere in the first instance; but if the pulley move steadily and the stiffness of the belt be disregarded, it must be so. Assuming this, let one of the weights be increased by a certain quantity Q and the pulley be held fast, then the tension of that side of the belt will be increased by an amount equal to Q at A, but diminishing to zero at L_1, a point determined by the intersection of the friction spiral $a_1 \, l_1$ (Fig. 110) with the circle alb, the radius of which represents the weight W. Similarly, if the other weight be diminished by Q', the tension will be diminished by an amount equal to Q' at B, but diminishing to zero at L_2. The portion L_1L_2 will remain at the original tension W. If QQ' be increased sufficiently, L_1, L_2 will coincide in one point L, the position of which will depend on the proportion between Q and Q'. While these changes take place in tension, corresponding changes of length must occur in the parts of the belt exposed to them, AL increases and BL_2 diminishes in length. Hence both these parts slip over the pulley and work is lost by friction, while L_1L_2 remains fixed. If now, instead of

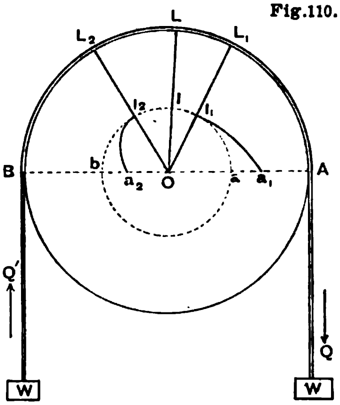


Fig.110.

altering the weights W, we imagine these weights held fast and the pulley forcibly rotated so as to increase A's tension by Q, and diminish B's tension by Q', L_1L_2 will rotate with the pulley, and the total increase of length of the one side must be equal to the total diminution on the other, from which consideration it is possible to calculate the ratio Q bears to Q'. In practical cases, however, the difference between Q and Q' is so small that it may be neglected without sensible error, and therefore, in all questions relating to the working of belts, it may be assumed that the mean tension of the two sides of the belt is independent of the power which is being transmitted. The difference of tensions, however, is directly proportional to the power, and may at once be calculated if the speed be known, while the ratio of tensions may be determined, so that the belt shall just not slip, by means of the formula above obtained. The value of the co-efficient of friction of leather on iron ranges from ·15 to ·46 according to the degree of lubrication: under ordinary circumstances ·25 may be considered an average value. This, however, is often greatly exceeded in practice, and one reason why large values are admissible is said by some to be the effect of atmospheric pressure. The sectional area of belts is fixed by consideration of strengths, and as their thickness varies little, this is equivalent to saying that a certain breadth of belt is required for each horse-power transmitted. (See Ex. 11, page 245.)

125. *Slip of Belts.*—When a belt is stretched over a pair of pulleys, one of which drives the other, notwithstanding a resistance not so great as to cause slipping of the belt as a whole, it appears from what has been said that a certain arc exists on each pulley on which the belt does not slip. The length of these arcs has already been found, but in the present cases the movement of the pulleys causes them to place themselves where the belt winds on to the pulleys, so that the driving pulley has the speed of the tight side of the belt and the driven pulley that of the slack side. The two sides have different speeds, because the same weight of belt must pass a given point in a unit of time, wherever that point be situated, and therefore the speed must be greater the greater the elongation, that is to say the greater the tension. Hence the driving pulley moves quicker than the driven pulley by an amount which can be calculated when the tensions and the elasticity of the leather are known, and this "slip" measures the loss of work due to the creeping of the belt over the pulleys described above. In ordinary belting this loss is small, not exceeding 2 per cent. The length of belts, however, must not be too great, or its extensibility will be inconvenient, especially if the motion of the machine be not sufficiently uniform.* Within moderate limits extensibility is favourable to smooth working.

126. *Stiffness of Ropes.*—When a rope is bent it is found that a certain moment is required to do it depending on the dimensions of the rope and, besides, on its tension. The reason of this is best understood by referring to the corresponding case in a chain with flat links united by pin joints. If d be the diameter of the pin, T the tension of the chain, there will be a certain moment of friction resisting bending which, if the pin be an easy fit, will be simply $\frac{1}{2} fTd$, but if it be tight will be

$$M = \tfrac{1}{2}fTd + \tfrac{1}{2}fT_0d,$$

where T_0 is a constant depending on the tightness. If the chain pass over a rotating pulley without slipping, this frictional moment has to be overcome both when bending on and when bending off the pulley. The effect shows itself by a shift outwards on the advancing and inwards on the retiring side of the chain, so as to increase the leverage of the resistance and diminish that of the effort. In the present case the two shifts are equal, being each given by the formula

$$x = \tfrac{1}{2}fd\left\{1 + \frac{T_0}{T}\right\}.$$

The case of a rope differs from this only in being more complex: in the act of bending, the fibres move over each other, and the relative motion is resisted by friction due to pressures which are partly constant and partly proportional to the tension. The shift of the centre line of the rope is visible on the side of the resistance, but hardly perceptible on the side of the hauling force, showing that most of the loss of work is due to the bending on the pulley. The magnitude of the shift varies so much according to the mode of manufacture and the condition of the rope that it is useless to attempt more than a very rough estimate. According to a formula given by Eytelwein, if d be the diameter of the rope,

$$x = c \, . \, d^2,$$

where c is a constant, which for dimensions in inches is taken as ·47 for hemp ropes; but this value is too large, except for light loads, and small diameters of pulley. The loss of work per revolution is $T \, . \, 2\pi x$, and if D be the effective diameter of the pulley,

$$\text{Efficiency} = \frac{D}{D + 2x}.$$

There is a loss of work by the stiffness of belts of a similar kind, but of uncertain amount. By most authorities it is considered so small as to be negligible.

---

* See a footnote by M. Kretz, *Cours de Mécanique Appliquée aux Machines par Poncelet*, page 264.

The shift of the line of action of the tension of a rope due to its stiffness has the effect of diminishing its strength.

127. *Friction of Toothed Wheels and Cams.*—The friction of toothed wheels is partly rolling and partly sliding, but the first is relatively small and may be neglected. To determine the sliding friction, let $PT = z$ (see Fig. 71, page 144), then (page 149) the velocity of rubbing is given by the formula

$$v = (A + A')z,$$

which may be written, if V be the speed of periphery of the pitch circles, R, R' the radii,

$$v = z\left\{\frac{1}{R} + \frac{1}{R'}\right\}V.$$

If, therefore, the wheels be supposed to turn through a small space δx measured on the pitch circles, the pair of teeth will slide on one another through the small space δy, given by the formula

$$\delta y = \left(\frac{1}{R} + \frac{1}{R'}\right)z\delta x.$$

This enables us to find the work done in overcoming friction, for if P be the pressure between the pairs of teeth,

$$\text{Work done} = f\int Pdy = f.\left(\frac{1}{R} + \frac{1}{R'}\right)\int Pzdx.$$

The pressure between the teeth will vary as the wheels turn according to some unknown law, depending on the way the teeth wear, but the variation is probably not great. Assuming it constant, and further, supposing that the chord PT (Fig. 71) is equal to the arc PT, and therefore to x the arc turned through by the wheels after the teeth pass the line of centres,

$$\text{Work done} = f.P.\left(\frac{1}{R} + \frac{1}{R'}\right)\frac{x^2}{2}.$$

The same formula applies before the line of centres, and if we assume the arcs of approach and recess each eqnal to the pitch p, we shall have for the whole work lost by the friction of a pair of teeth,

$$\text{Whole Work lost} = fP\left(\frac{1}{R} + \frac{1}{R'}\right)p^2.$$

The energy transmitted during the action of a pair of teeth is $2\ Pp$, therefore the counter efficiency is

$$1 + e = 1 + f\left(\frac{1}{R} + \frac{1}{R'}\right)\frac{p}{2} = 1 + f\pi\left(\frac{1}{n} + \frac{1}{n'}\right),$$

where n, n' are the numbers of teeth in the wheels. A smaller arc of action is sometimes employed in practice, and the friction will then be less. This is also the case in bevel gear. The formula shows that the friction is diminished by increasing the number of teeth.

A more exact solution of this question* can be obtained on the assumption that P varies as it would do if there were only one pair of teeth; but as this is uncertain it is not practically useful.

In all cam and wheel mechanisms the efficiency for a small movement in any position can be determined exactly by a graphical or other process. For the velocity-ratio can be found, as shown in Part II., and the force-ratio is determinate by the principles of statics, therefore the quotient which gives the efficiency can also be found. In the case of toothed wheels this method shows at once † that the friction of the teeth before the line of centres is greater than the friction after the line of centres. The difference appears insufficient to account for the injurious effects generally ascribed to friction before the line of centres, which however may be due to other causes. In cam mechanisms the efficiency in one position is little guide to the efficiency in a complete period, which can only be found by a process too intricate to be useful, or by making some supposition as the mean value of the pressure between the rubbing surfaces.

The counter efficiency of a train of m equal pairs of wheels is

$$1 + e = 1 + mf\pi\left(\frac{1}{n} + \frac{1}{n'}\right).$$

Assume now that a given velocity-ratio is to be provided by the train, and that the number of teeth in one wheel is given, then it is possible to find the value of m that the friction may be least. The solution of this problem is the same as that of finding the least possible number of teeth, and it was shown by Young that, for this, we ought to take m, so that the velocity-ratio for each pair of wheels is, as nearly as possible, 3·59. For example, if the train is to give a total velocity-ratio of 46, there should be three pair of wheels. The gain over a single pair in this case is one third, but will be much greater for higher velocity-ratios. The solution (first given by Mr. Gilbert) takes no account of axle friction, a circumstance which would greatly modify the result.

SECTION III.—FRICTIONAL RESISTANCES IN GENERAL.

128. *Efficiency of Mechanism in general.*—It appears from what has been said that an exact calculation of the frictional resistances is impracticable, partly because the process is too complex to be useful, but chiefly because the co-efficients to be employed are variable according to circumstances, and within limits, which are not precisely known. Hence when possible the efficiency of a machine is estimated, not by considering each particular element, but by direct experiment on the machine as a whole, and we conclude this chapter with some general principles which bear on this question.

---

* See Moseley's *Mechanical Principles of Engineering.*

† Ibid, page 286.

The effort employed to drive a machine may be greater or less, according to the resistance which is being overcome, and therefore the stress between each element will also vary according to this effort. As, however, these stresses depend also on other forces, such as weight and elasticity, which have no connection with the effort, but are always the same, they will not increase so fast, and the frictional resistances will accordingly be proportionally less the greater the effort. Some resistances are absolutely constant, for example, the friction of bearings, the load on which is simply the weight of a fly-wheel or other moving part: or the friction of a piston rod in its stuffing box. Others are sensibly proportional to the driving effort or the useful resistance, in which case, when the ordinary laws of friction apply, the loss of work increases in direct proportion to these quantities. The greater number depend on both variable and constant forces, but these may be in great measure separated into two parts, one of which is approximately constant and the other approximately proportional either to the driving effort or to the useful resistance. Hence, if U be the useful work done and E the energy exerted in a period of the machine,

$$E = U + kU + k' . E + B,$$

where k, k' are numerical co-efficients and B the work done in overcoming the constant resistances. In hydraulic and other machines, where fluid resistances occur, terms depending on the speed of the machine must be added, indeed this is so in all machines when driven at a high speed; because forces due to inertia increase the friction, and besides shocks and the resistance of the atmosphere have to be considered. Such cases, however, are not considered here.

If we transfer the term $k' E$ to the other side of the equation and divide by $1 - k'$, we get

$$E = (1 + e) U + E_0,$$

where e, E_0 are two new constants derived from the former ones, of which E_0 is the work done in driving the machine when unloaded, and $1 + e$ the counter-efficiency when the load is very great.

The same formula may also be written in a way which is sometimes more convenient. Let P be the mean value of the driving effort and R that of the useful resistance during a complete period, r

the mean value of the velocity-ratio of the working and driving pairs, then

$$P = (1 + e)Rr + P_0,$$

where P_0 is now the effort required to drive the machine when unloaded. In hoisting machines R is the weight lifted and P the hauling force usually called the power, R/P is the mechanical advantage or purchase.

In the steam engine, if p_m be the actual mean effective pressure, p'_m the part of that pressure employed in overcoming the useful resistance, p_0 the pressure necessary to drive the engine when unloaded,

$$p_m = (1 + e)\, p'_m + p_0.$$

The value of e may be taken as ·15 or in large engines somewhat less. The constant p_0, often called the "friction pressure," is from 1 to 1½ lbs. or in marine engines 2 lbs. or more per square inch. At high speeds and pressures the ordinary laws of friction fail and e is diminished, the constant friction is then relatively of more importance.

If the direction of motion of the machine be reversed so that the original resistance becomes the driving effort and the effort the resistance, the same general formula is approximately true, but the constants k, k' are interchanged. Unless under special conditions the efficiency is not the same in the two cases, and in fact is generally very different. Let us suppose that in a machine working against a known reversible resistance, the driving effort is gradually diminished until the machine reverses, and let E' be the work done when reversing, we have the equations

$$E = U + kU + k'E + B,$$
$$U = E' + k'E' + kU + B,$$

from which by subtraction and dividing by U we find

$$\frac{E'}{U} = \frac{2}{1 + k'} - \frac{1 - k'}{1 + k'} \cdot \frac{E}{U},$$

a formula which gives the efficiency when reversing. If the original efficiency be less than $\frac{1}{2}(1 - k')$, the machine will not reverse even when the driving force is entirely removed. In most forms of hoisting machines k' is small enough to be neglected, and we have the important principle that a machine will not reverse if its efficiency is less than ·5. It will not reverse under any circumstances if $k > 1$. As previously explained in the case of a screw, non-reversibility is a property so valuable in practical applications as to be worth obtaining at the sacrifice of efficiency. The differential pulley block is a common example.

129. *Friction Brakes.*—Frictional resistances are not only a source of loss, they are also usefully employed in machines for various purposes

In screws and driving belts we have already found them employed for the purpose of locking a pair or closing a kinematic chain, and many instances of the same kind might be referred to. Another application of equal importance is for the purpose of absorbing surplus energy, which might otherwise produce dangerous effects, or which requires to be disposed of in order to stop a machine. An apparatus for this purpose is called a "brake."

The most powerful brakes are those in which fluid resistances are used, but when the amount of energy is small as compared with the surfaces available, the friction of solids may be employed. The energy thus absorbed is converted into heat, and is dissipated by radiation and conduction. Sufficient surface must be provided to prevent the temperature rising too high.

A brake is generally applied to a rotating wheel or drum, and consists either of a solid block of wood or metal pressed against the wheel by some suitable mechanism; or else of a strap of metal, often lined with small blocks of wood, embracing the drum and tightened by a lever or otherwise. Three common forms are shown in Plate VII.; two of these (Figs. 1 and 2) are used as dynamometers, and will be referred to as such in the next chapter.

EXAMPLES.

1. A weight is moved up a plane inclined at 1 vertical to n horizontal by an effort parallel to the plane: show that the counter-efficiency is $1+nf$, where f is the co-efficient of friction. Find the value of n for a mechanical advantage of 10:1 and a co-efficient ·05. *Ans.* $n=20$.

2. Show that the pressure on the guide bars of a direct-acting engine is approximately proportional to the ordinates of an ellipse, and deduce the work lost per stroke.

Referring to Fig. 91 let X be that pressure, then

$$X=S\,.\sin\phi=P\,.\tan\phi=\frac{P}{n}\sin\theta \text{ approximately.}$$

If the radius of the crank circle represent P, and an ellipse be drawn with the same major axis, and minor axis $=P/n$, X will be the ordinate of the ellipse at a point representing position of piston.

$$\text{Loss of work per stroke}=f\times\text{Area of semi-ellipse}$$
$$=\tfrac{1}{2}f\,.\,\pi\,.\,a\frac{P}{n}=\frac{\pi fsP}{4n},$$

where s is the stroke and f the co-efficient of friction.

3. *A* bearing 16″ diameter is acted on by a horizontal force of 50 tons and a vertical force of 10 tons. Find the work lost by friction per revolution, using a co-efficient of one-eighteenth. Find also the horse-power lost by friction at 70 revolutions per minute. *Ans.* Loss of work = 11·87 foot-tons. H.P. = 56·4.

4. The thrust of a screw propeller is 20 tons, the pitch 20 feet. The thrust block is 18″ diameter at the centre of the rings. Find the efficiency with a co-efficient of friction of ·06. *Ans.* Efficiency = ·986.

5. Find the efficiency of a common screw and nut with pitch angle 45° and co-efficient ·16. *Ans.* Efficiency = ·72.

6. A screw bolt is ½″ diameter outside and ·393″ at the base of the thread. The effective diameter of the nut is ¾″, the pitch angle ·07, and the co-efficient of friction ·16; supposing it screwed up by a spanner two feet long, find the mechanical advantage.

Tension of bolt = 218 × pull on spanner.

7. Find the efficiency of a pair of wheels, the numbers of teeth being 10 and 75, and the co-efficient of friction ·15. *Ans.* ·954.

8. The stroke of a direct-acting engine is 4 feet, piston load 50 tons, load on crank-shaft bearings 10 tons, connecting rod 4 cranks: trace the curve of crank effort when friction is taken into account, assuming all bearings 16″ diameter and co-efficient one-eighteenth. Find the "dead angle."

9. In the last question, if the engine drive the screw propeller of question 4, find the efficiency of the mechanism, including thrust block, by the approximate method. The connecting rod may be supposed indefinitely long except for the purpose of estimating the efficiency of the guide bars.

$$\text{Efficiency} = \cdot 989 \times (\cdot 97)^2 \times \cdot 986 = \cdot 92.$$

10. A rope is wound thrice round a post, and one end is held tight by a force not exceeding 10 lbs. What pull at the other end would be necessary to make the rope slip, the co-efficient of friction being supposed ·366? *Ans.* 10,000 lbs.

11. Find the necessary width of belt three sixteenths inch thick to transmit 1 H.P., the belt embracing 40 per cent. of the circumference of the smaller pulley and running at 300 feet per 1′. Co-efficient = ·25. Strength 285 lbs. per sq. inch. *Ans.* Breadth = 4½″.

12. In question 10 construct the friction spiral showing the tension of the rope at every point.

13. The axles of a tramway car are 2½″ diameter, and the wheels 2′ 6″: find the resistance, being given, that the co-efficient of axle friction is ·08 and that for rolling ·09. *Ans.* Resistance = 28½ lbs. per ton.

14. Find the efficiency of a pulley 6″ diameter, over which a rope ½″ diameter passes, the axis of the pulley being ½″ diameter, and the load on it twice the tension of the rope. Co-efficient of axle friction ·08. Co-efficient for stiffness of rope ·47. *Ans.* Efficiency = 94 per cent.

15. From the result of the preceding question deduce the efficiency of a pair of three-sheaved blocks. *Ans.* Efficiency = 71 per cent.

16. A wheel weighing 20 lbs., radius of gyration 1′, is revolving at 1 revolution per second on axles 1″ diameter. It is observed to make 40 revolutions before stopping: find the co-efficient of axle friction. *Ans.* Co-efficient = ·059.

17. In a pair of three-sheaved blocks it is found by experiment that a weight of 40 lbs. can be raised by a force of 10 lbs., and a weight of 200 lbs. by a force of 40 lbs. Find the general relation between P and W, and the efficiency when raising 100 lbs.

$$P = \tfrac{3}{16} W + \tfrac{5}{2}. \quad \text{Efficiency} = \cdot 784 \text{ when raising 100 lbs.} \quad e = \tfrac{1}{8}.$$

18. Find the distance to which power can be transmitted by shafting of uniform diameter, with a loss by friction due to its weight of n per cent, assuming that the angle of torsion is immaterial, and co-efficient for strength 9,000 lbs. per square inch.

If f be co-efficient of friction, then the length of shafting is $13\tfrac{1}{3} \cdot \dfrac{n}{f}$.

REFERENCES.

On the graphical determination of the efficiency of mechanism the reader is referred to two papers by Prof. F. Jenkin in the *Transactions* of the Royal Society of Edinburgh. On the stiffness of ropes, see Weisbach, *Ingenieur-Mechanik*, vol. I., 3rd German edition, p. 300.

CHAPTER XI.

INCOMPLETE CONSTRAINT. STRAINING ACTIONS ON MACHINES.

130. *Preliminary Remarks.*—In the motion of a machine the relative movements of the several parts are completely defined by the nature of the machine, and the principal action consists in a transmission and conversion of energy. Hence it is that the principle of work is of such importance in all mechanical operations that it is desirable to consider it as an independent fundamental law verified by daily experience. Even in applied mechanics, however, we have sometimes to do with sets of bodies, the relative movements of which are not completely defined by the constraint to which they are subject, but partly depend on given mutual actions between them. When this is the case, the principle of work, though still of great importance, is not by itself sufficient to determine the motions.

Again, if we wish to study the forces which arise when the direction of a body's motion is changed, the principle of work does not help us, for no work is done by such forces. For example, the position of the arms of a governor, revolving at a given speed, cannot be found, except, perhaps, indirectly, by the methods hitherto employed. We then resort to the ordinary laws connecting matter and motion, which form the base of the science of mechanics, and of which the principle of work itself is often considered as a consequence.

The present chapter will be devoted in the first place to a brief summary of elementary dynamical principles, and afterwards to various questions relating to machines and the forces to which they are subject.

SECTION I.—ELEMENTARY PRINCIPLES OF DYNAMICS.*

131. *Impulse and Momentum.*—The effect of an unbalanced force P, acting during a certain time t, on a piece of matter, is to generate a

---

* The brief statement here made of principles assumed in subsequent articles of this treatise is not intended as a substitute for a treatise on elementary dynamics.

velocity v, which is proportional to P and t directly and the quantity of matter inversely. When the force P is equal to the weight W, as in the case of a body falling freely, the velocity generated in 1″ is known to be g, where g is a number which varies slightly for different positions on the earth's surface (Art. 99), but is precisely the same for all sorts of matter. We may express this by the equation

$$Pt = \frac{W}{g}v.$$

In this formula we may take W to mean the weight of the piece of matter as compared with that of a unit piece at a given point on the earth's surface. As formerly stated (Art. 88) this is called "gravitation measure," and has the defect of giving a varying unit of force, so that considerations of convenience alone induce us to employ it. If, instead of measuring W in units of weight, we compare it with the force P, which produces unit velocity in unit time, we have

$$W = Pg,$$

that is, the weight of the unit piece of matter is g units of force. Such units depend only on the unit piece of matter, and are hence called "absolute" units. For scientific purposes, and especially in electrical measurements, they are much employed.

Quantity of matter is called MASS, and, when absolute measure is used, is simply measured by comparing it with that of a standard piece, for example, in Britain, with a certain piece of platinum called a pound. The unit of force is then that which is necessary to produce in each second a velocity of 1 foot per 1″ in this piece, a quantity for which the name "Poundal" was suggested by the late Professor Clerk Maxwell; the weight of a piece is then g poundals, so that what is called a pound-weight in the common gravitation measure is about 32·2 of these units. When absolute measure is used, however, the Continental system of units depending on the mètre and gramme is likely to be universally employed. No more need be said on this point, as gravitation measure is exclusively used in this treatise.

When gravitation measure is used the term "mass" becomes simply a name for the ratio W/g and varies according to the place. If m be the mass, W the weight,

$$W = mg,$$

where g is taken equal to 32·2.

This explanation being premised we have

$$Pt = mv.$$

The products Pt, mv are called IMPULSE and MOMENTUM respectively, and the equation may be written

Impulse exerted = Momentum generated.

A unit of impulse is unit force exerted for unit time, usually 1 lb. for 1″, a quantity for which the expression "second-pound" may conveniently be used. If P be variable, then impulse is calculated in the same way as the energy exerted by a variable force (Art. 90), the abscissæ of the diagram now representing time instead of space.

The body we are considering may have a velocity at the commencement of the time t, and the force may be partially balanced; if so, v must be understood to be the *change* of velocity, and P the unbalanced part of the force.

132. *Centrifugal Force.*—So far, the equation of momentum is analogous to the equation of work, impulse representing the time-effect of force as energy represents its space-effect. There are, however, two important differences.

Change of kinetic energy arises from a change in the magnitude of the velocity irrespectively of direction, whereas change of momentum must be estimated in the direction of the force producing it, and includes change of direction. Hence the equation is applicable when the direction of the force is perpendicular to the direction of motion, so that the only effect produced is change of direction. The rate of change of velocity, taken in the most general sense, is called Acceleration, and the equation of momentum may also be written

$$P = mf,$$

where f is the acceleration estimated in the direction of the force. By taking the force perpendicular to the direction of motion we get the equation which connects the curvature of the path of a moving body with the force R, which compels it to deviate from the straight line, namely,

$$R = \frac{mv^2}{r},$$

where v is the velocity and r the radius of the circle in which it is moving at the instant considered. Like other forces this arises from the mutual action between two bodies: one of these is the moving body; the other, the fixed body which furnishes the necessary constraint. If we are thinking of the fixed body instead of the moving body, we call the force R the Centrifugal Force, being the equal and opposite force with which the moving body acts on the body which constrains it. The two forces together constitute what we have already called a Stress (Art. 1). To determine a stress of this kind it is necessary to refer the direction of motion to some body which we know may be regarded as fixed, and we are not at liberty to choose any body we please for this purpose, as in kinematical questions. What constitutes a fixed body is a question of abstract dynamics, into which we need not enter. For practical purposes the earth is taken as fixed.

If a body rotate about a fixed axis the centrifugal forces, arising from the motion of each particle, will not balance one another unless the axis be one of three lines, passing through the centre of gravity, which are called the "principal axes of inertia" at that point. In most cases occurring in practical applications the position of these lines can be at once foreseen as being axes of symmetry. This is the case, for example, in homogeneous ellipsoids and parallelopipeds. In the common case of a homogeneous solid of revolution, the axis of revolution, and any line at right angles to it through the centre of gravity, are principal axes. If the axis of rotation be parallel to one of these axes, but do not pass through the centre of gravity, the centrifugal forces reduce to a single force, which is the same as if the whole mass were concentrated at the centre of gravity. In all other cases there is a couple depending on the direction of the axis of rotation, as well as the force just mentioned. (Ex. 16, p. 267.)

133. *Principle of Momentum.*—Again, every force arises from the mutual action between two bodies, consisting in an action on one accompanied by an equal and opposite reaction on the other. Hence, if we understand by the total momentum of two bodies in any direction, the sum or the difference of the momenta of each, according as the bodies move in the same or in the opposite direction, it appears that the total momentum will not be affected by the mutual action between the two. And more generally, if there be any number of bodies we shall have

Total impulse exerted = Change of total momentum,

where, in reckoning the impulse, we are to take into account external forces alone, and not the internal forces arising from the mutual action of the parts of the set of bodies we are considering. This equation expresses one form of what we may call the Principle of Momentum; other forms will be explained hereafter in connection with questions relating to fluid motion (Part V.).

The total momentum of a number of bodies may be reckoned by direct summation, with due regard to sign, but it may also be expressed in terms of the velocity of the centre of gravity; for, let m be the mass of any particle of the system, the ordinate of which, reckoned from a given origin parallel to a given line, is x; also, let Σmx denote the sum of all the separate products mx, for all the particles of the system, and let M be the total mass, then we know that the ordinate of the centre of gravity* is given by the formula

$$\bar{x} = \frac{\Sigma mx}{M}.$$

---

* Called more correctly by modern writers on mechanics the "centre of mass."

Let the velocity of a particle parallel to the given line be u, then if x_1, x_2, be the ordinates at the beginning and end of 1″ we shall have

$$u = x_2 - x_1.$$

Hence, if $\bar{u}$ be the velocity of the centre of gravity parallel to the same line,

$$\bar{u} = \bar{x}_2 - \bar{x}_1 = \frac{\Sigma m(x_2 - x_1)}{M} = \frac{\Sigma mu}{M},$$

which equation may be written

$$M\bar{u} = \Sigma mu,$$

showing that the total momentum of the system is the same as if its total mass were concentrated in its centre of gravity. We conclude from this that the motion of the centre of gravity can only be influenced by external forces and not by any action between the parts of the system.

134. *Internal and External Kinetic Energy.*—If we multiply the equation just obtained by $2\bar{u}$ and remember that $\bar{u}$ being constant may be placed within the sign of summation, we obtain

$$2M\bar{u}^2 = \Sigma m \,.\, 2u\bar{u}$$

which, adding Σmu^2 to each side and re-arranging the terms, may be written

$$M\bar{u}^2 + \Sigma m(\bar{u} - u)^2 = \Sigma mu^2.$$

This is true in whatever direction the velocities are estimated, and we can therefore write down two similar equations for the velocities in two directions at right angles to the first. Now the resultant of three velocities at right angles is the square root of the sum of the squares of the components, also $\bar{u} - u$ is the velocity parallel to x relatively to the centre of gravity; hence if U be the resultant velocity of the centre of gravity, v, $\bar{v}$ the velocities of any particle relatively to the body regarded as fixed and relatively to the centre of gravity respectively, we have, adding the three equations together, and dividing by 2,

$$\tfrac{1}{2}MU^2 + \Sigma\tfrac{1}{2}m\bar{v}^2 = \Sigma\tfrac{1}{2}mv^2.$$

The first term on the left-hand side of this equation is what the energy would be, if the whole mass were concentrated at its centre of gravity, a quantity which may be described as the External Energy, or otherwise as the Energy of Translation of the system. The second term is the energy relatively to the centre of gravity considered as fixed, which may be called the Internal Energy. The right-hand side is the total energy of motion, and we see therefore that this is the sum of the internal and external energies. In the case of a single rigid body the motion relatively to the centre of gravity is always a rotation about some axis, and therefore

Energy of Motion = Energy of Translation + Energy of Rotation,

a principle already employed in a preceding chapter (p. 195).

In the case of a set of rigid bodies the internal energy is the sum of the energies of rotation of each together with the internal energy of a set of particles of the same mass occupying the centres of gravity of the bodies and moving in the same way.

135. *Examples of Incomplete Constraint.*—In the cases which occur in applications to machines and structures we usually have to consider two bodies moving in straight lines without rotation.

CASE I. *Recoil of a Gun.*—When a cannon is fired the shot is projected and the cannon recoils with velocities dependent on the relative weights of the shot, the cannon, and the charge of powder.

Here, the motion is due to the pressure of the gases generated by the combustion of the powder one way on the shot, the other way on the cannon. If the inertia of these gases could be neglected these pressures would be exactly equal at each instant and would cease as soon as the shot left the bore. The impulse exerted on shot and cannon would then be equal. In fact, the inertia of the powder gases causes the pressure to be greater and to last longer on the cannon than on the shot, so that the impulses on the two are not nearly equal. For the present we shall neglect this, and shall further suppose that the material of both shot and gun is sensibly rigid.

In general, recoil is checked by an apparatus called a "compressor," which supplies a gradually increasing resistance to the backward movement of the gun, while friction and the resistance to rotation of the shot resist the forward movement of the shot. In the first instance suppose there are no such resistances, let V be the velocity of recoil and M the mass of the gun, v the velocity and m the mass of the shot; then, since the impulse exerted is the same for both,

$$MV = mv.$$

Further, if the weight of the charge and the amount of work 1 lb. of it is capable of doing be known, the explosion will develop a definite amount of energy (E) which will be all spent in giving motion to the shot and the cannon.

$$\text{Energy of Explosion} = \tfrac{1}{2}MV^2 + \tfrac{1}{2}mv^2.$$

Here E is the sum of two parts—

$$\text{Energy of Shot} = \frac{M}{M+m}E,$$

$$\text{Energy of Recoil} = \frac{m}{M+m}E,$$

The energy of recoil has to be absorbed by the compressor, usually an hydraulic brake, which will be considered hereafter (see Part V.).

CASE II. *Collision of Vessels.*—When two vessels come into collision an amount of damage is done depending on the size and velocities of the vessels.

Here we may suppose the vessels moving in given directions with given velocities; let the velocities parallel to a given line be u_1, u_2, and the masses m_1, m_2, then, as in Art. 133, the velocity of the centre of gravity parallel to the same line is

$$\bar{u} = \frac{m_1u_1 + m_2u_2}{m_1 + m_2},$$

and therefore the velocities of the vessels relatively to their common centre of gravity must be

$$u_1 - \bar{u} = \frac{m_2(u_1 - u_2)}{m_1 + m_2}; \quad u_2 - \bar{u} = \frac{m_1(u_2 - u_1)}{m_1 + m_2}.$$

Two similar equations may be written down for the velocities in a direction at right angles to the first. Square and add corresponding equations, multiply by $\frac{1}{2}m_1$, $\frac{1}{2}m_2$, and add the pair of products, then (Art. 134)

$$\text{Internal Energy} = \tfrac{1}{2} \cdot \frac{m_1m_2}{m_1 + m_2}V^2,$$

where V is the velocity of either vessel relatively to the other, a quantity found immediately from the given velocities of the vessels by means of a triangle of velocities.

The total kinetic energy of the vessels is found by adding the energy of translation. As, however, this quantity cannot be altered by the collision, it is clear that the amount of work done must depend on the internal energy alone: we may properly call it therefore the "energy of collision." If the displacements in tons of the vessels be W_1, W_2, we shall have, in foot-tons,

$$\text{Energy of Collision} = \frac{W_1W_2}{W_1 + W_2} \cdot \frac{V^2}{2g}.$$

It is not, however, to be supposed that the whole of this is necessarily expended in damage to the vessels; if the circumstances of the collision be such that the vessels, even though completely devoid of elasticity, would have a motion of rotation or a velocity of separation of their centres of gravity, then the corresponding internal energy must be deducted. Also the influence of the water surrounding the vessels has been left out of account; this somewhat augments the effect by increasing the virtual mass of the vessels.

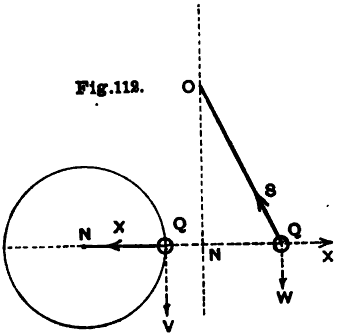
The same formula may be used for other cases of impact, but the effects of impact depend so much on the strength and stiffness of the colliding bodies that the subject must be postponed (Ch. XVI.).

SECTION II.—CENTRIFUGAL REGULATORS.

136. *Preliminary Remarks.*—Centrifugal forces may be employed in machines to do the work which is the object of the machine, as in certain drying machines where the substance to be dried is caused to rotate with great rapidity so that the fluid is expelled at the outer circumference: or, partially, in centrifugal pumps. More frequently they serve to move a kinematic chain connected with a shifting piece which regulates the speed of the machine. Such mechanisms are called Centrifugal Regulators or, more briefly, Governors.

137. *Simple Revolving Pendulum.*—In Fig. 112 Q is a heavy particle attached by a string to a fixed point O and revolving in a horizontal circle the centre of which is N vertically below O. This will be possible if the centrifugal force due to the motion of the particle just balances the horizontal component of the tension of the string. Let S be that tension, W the weight of the particle, and let the string make an angle θ with the vertical, then the horizontal and vertical components of S are

Fig. 112.



$$X = S \,.\, \sin\theta;\quad W = S \,.\, \cos\theta.$$

Let A be the angular velocity of the revolving particle, then it is shown in works on elementary dynamics that the centrifugal force is

$$X = \frac{W}{g} \,.\, A^2 \,.\, QN.$$

Equating these values of X and eliminating S,

$$W \,.\, \tan\theta = \frac{W}{g} \,.\, A^2 \,.\, QN.$$

Since $QN = ON \,.\, \tan\theta$, this reduces to the simple formula

$$ON = \frac{g}{A^2},$$

which shows that the vertical distance of Q below the point of suspension depends on the speed, not on the length of the string or the magnitude of the weight.

This distance is called the "height" of the revolving pendulum, and will be denoted by h. If t be the period, that is the time of a complete revolution, we find, since $At = 2\pi$,

$$t = 2\pi\sqrt{\frac{h}{g}},$$

showing that the period is the same as that of a double oscillation of a simple pendulum of length h (see Art. 103). The height of a simple revolving pendulum may often be conveniently adopted as a measure of a speed of revolution, and will then be spoken of as the "height due to the revolutions." Its value in inches for n revolutions per minute is given by the formula

$$n^2h = 35{,}232.$$

Instead of supposing the string attached to a point O in the axis of revolution, we may suppose it attached to a point K, rigidly connected by a cross-piece KE, with a revolving spindle ON. The same reasoning applies, O being now an ideal point, found by prolonging the string to meet the axis. The height of the pendulum is still ON, and is found by the same formula.

138. *Speed of a Governor to overcome given Frictional Resistances. Loaded Governors.*—In the simplest centrifugal governors two heavy balls are attached to arms, which are jointed either directly to a revolving spindle, or to the ends of a cross-piece attached to a spindle. Motion is communicated by links from the arms to a piece sliding on the spindle, the movement of which is communicated by a train of linkwork, either to a throttle valve directly controlling the supply of steam, or to an expansion valve which regulates the cut-off. In either case an upward movement of the arms has the effect of diminishing the mean effective pressure, and a downward movement of increasing it. Two forms of this mechanism are shown in the figures of Plate VI.: in one of these (Fig. 1) the weight of the sliding piece is increased by a large additional weight, the governor is then said to be loaded; while in the other (Fig. 2) the arms cross each other, the spindle being slotted, or the arms bent to permit this. The object of these arrangements we shall see presently.

If now the speed of revolution be increased or diminished, the arms move outwards or inwards, and so adapt the mean effective pressure to the work which is being done. If there were no frictional resistances the smallest variation of speed would produce a corresponding motion in the arms; but, as the linkwork mechanism necessarily offers a certain resistance, motion cannot take place until the change of speed has reached a certain magnitude, which is smaller the more sensitive the governor is. These frictional resistances are measured by a certain

Plate. VI.

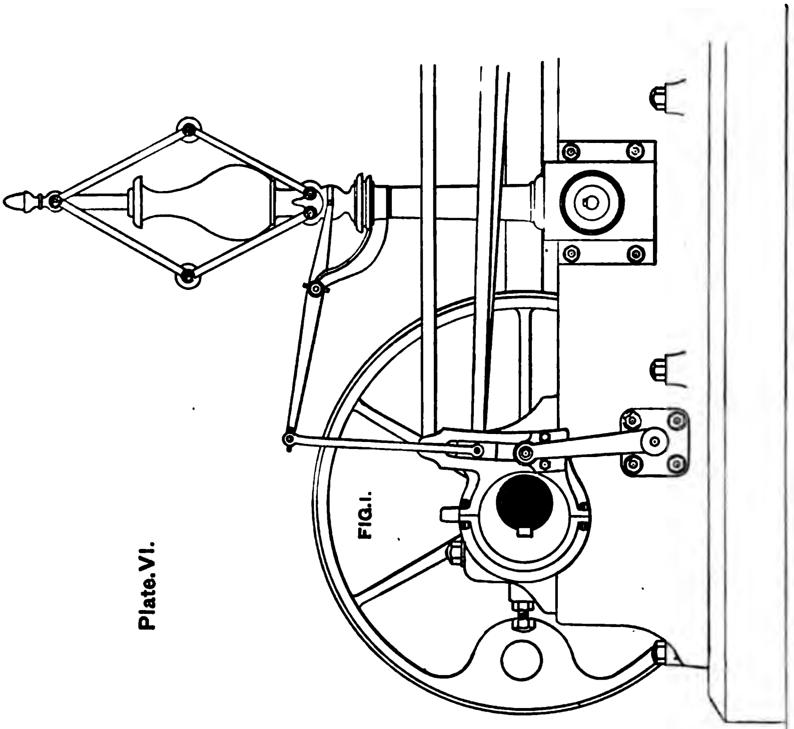


FIG. 1.

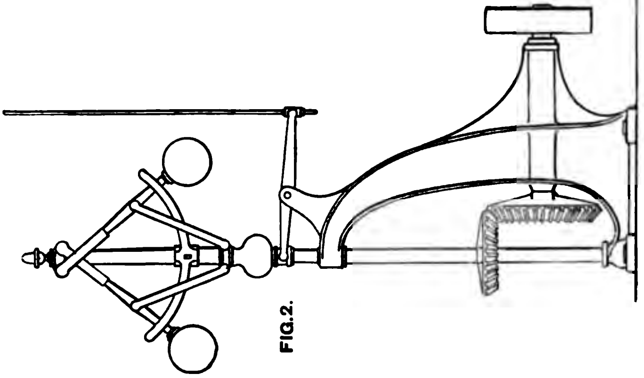


FIG. 2.

To face page 254.

addition to, or subtraction from, the weight of the sliding-piece, which might be determined experimentally, and therefore will be supposed a known quantity *F*. We first investigate what change of speed will be necessary to overcome them.

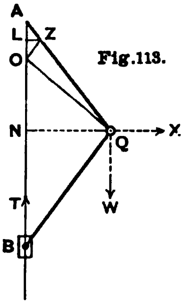


Fig. 113.

In Fig. 113 *AQB* is a triangle revolving about *AB*, which is vertical, a heavy particle is placed at *Q*, and the weights of the bars *AQ*, *BQ* are small enough to be neglected. If the triangle revolve at a speed corresponding to the height *AN* of a simple revolving pendulum *AQ*, there will be no stress on *BQ*, but if it be greater or less there will be a pull or thrust, the magnitude of which is determined thus:—

Set up *NO* equal to the height due to the revolutions, and join *QO*. Then it appears from what was said in the last article that if *NO* be taken to represent the weight *W* of the particle, *NQ* will represent *X* the centrifugal force, and therefore the resultant force on *Q* must be represented by *QO*. Through *O* draw *OZ* parallel to *BQ*, then *QOZ* is a triangle of forces for the joint *Q* of the triangular frame *AQB*, so that *QZ*, *OZ* must represent the stresses on *AQ*, *BQ* respectively. For our purposes we require the vertical component of the stress on the link *BQ*, which is obtained by drawing *ZL* horizontal: *OL* must be the force in question which we call *T*. In the figure *T* is an upward force, *O* being below *A*, and the speed of revolution therefore great In this construction the links need not be actually jointed to the spindle *AB*; they may, as in the simple pendulum, be attached to the extremities of cross-pieces fixed to *AB*. *A* and *B* are then ideal points of intersection of the links with the axis of rotation.

In general *AQ* and *BQ* are equal; we may then obtain a simple formula for *T*. Let $NO = h$, a quantity given by the same formula as before for a given speed, and let *AN*, the actual height of the governor, be denoted by *H*, then $OA = H - h$; but in the case supposed, $OA = 2OL$, therefore

$$2T = W \cdot \frac{H - h}{h}; \; h = H \cdot \frac{W}{W + 2T}:$$

formulæ which give the pull for any speed, and conversely the speed for which the pull will have a given value. In practical applications there are always two balls, so that if *W* be the weight of one, $2T$ will be the pull due to both.

We can now find within what limits of speed the mechanism can be in equilibrium. Let *w* be the weight of the sliding-piece *B*, inclusive of any load which may be added to it, *h* the height due to the speed at which there is no tendency to move the arms, h_1, h_2 the heights due to the speeds at which they are on the point of moving upwards or downwards respectively, then

$$h_1 = H\frac{W}{W + w + F}; \quad h = H\frac{W}{W + w}; \quad h_2 = H\frac{W}{W + w - F}.$$

In general F will be small compared with $W + w$, and then we have very approximately,

$$h_2 - h = h - h_1 = h\frac{F}{W + w}.$$

These results show that loading a governor gives it a power of overcoming frictional resistances, which would otherwise require a weight of ball equal to the sum of the load and the actual weight. Light balls may therefore be used as in the figure (Plate VI.) without sacrificing power, as the load may be made great without inconvenience. The speed of a loaded governor is greater than that of a simple governor of the same actual height, as appears from the formula for h. It may be altered at pleasure by altering the load. This arrangement is known as Porter's governor, from the name of the inventor.

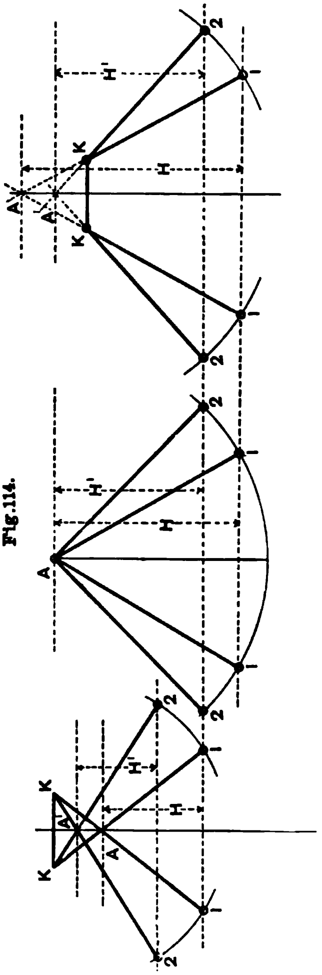


Fig. 114.

139. *Variation of Height of a Pendulum Governor by a Change of Position of the Arms.*—Next suppose the speed to alter so much that the arms actually change their position, then if H remained the same, the tendency to move would also be the same, and the movement must therefore continue until the speed is brought back within the limits for which rest is possible. In the ordinary pendulum governor, however, H alters in a way which depends on the mode of attachment and arrangement of the arms, as will appear from the annexed diagram (Fig. 114) which shows three cases.

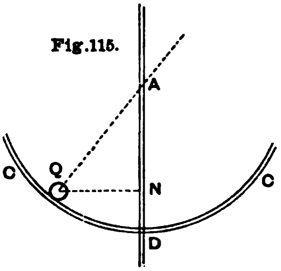
In the centre figure the arms are jointed to the spindle so that their centres of rotation are in the axis, in the two others they are jointed to a cross-piece *KK*, but differently arranged in the two cases. In all three, as explained in the preceding article, the height *H* is measured to *A*, the real or ideal intersection of the arms with the axis of rotation.

Suppose the arms to move from position 1 to position 2 in the figure; *H* diminishes to *H′*, but the amount of diminution is different in the three cases: in the right-hand figure it is greatest, and in the left-hand least. Indeed in the latter case where the arms are crossed it is possible, by making *KK* long enough, to change the diminution into an increase. (Ex. 4, p. 266.) If then, by an increase in the speed, the arms move into a new position, the speed required for equilibrium does not remain the same but increases, so that, when the adjustment has been effected between the energy and the work, the speed is increased, instead of being the same as before. Conversely, after adjustment to suit a diminished speed, the speed actually attained is diminished. Thus the effect of the variation in *H* is to widen the limits within which the speed can vary.

140. *Parabolic Governors.*—A governor may be constructed in which *H* does not vary at all.

In Fig. 115 *Q* is a ball resting on a curve *CC* attached to a vertical spindle. The curve lies in a vertical plane, and *D* is the lowest point. When at rest the ball can only be in equilibrium at *D*, but, if the spindle revolve, it may rest at another point, the position of which depends on the speed of revolution. If the curve be a circle we have only the pendulum governor in a different form, for, drawing the normal *QA* and the perpendicular *QN*, *A* will be a point to which *Q* might be attached by a string and the curve removed. Hence, *AN* must be equal to *h*, the height due to the speed of revolution. But if the curve be not a circle the same thing must be true, only *A* is now not a fixed point; hence in every case the sub-normal *AN* of the curve at the point of equilibrium must be equal to *h*. In general this geometrical condition determines one, and only one, position for a given speed; but if the curve be a parabola with vertex at *D*, *AN* will be constant, and therefore *Q* will rest in any position for one particular speed, but for lower speeds will roll down to *D*, and for higher speeds will move upwards indefinitely. We have here a governor for which,

Fig. 115.



neglecting frictional resistances, only one speed is possible. Such a governor is said to be "isochronous."

The curve arrangement is inconvenient for constructive reasons, but if it be replaced by a linkwork mechanism the ball still moves in a parabola. An isochronous governor is therefore often said to be "parabolic." The term is preferable, for no governor is actually isochronous on account of frictional resistances. A pendulum governor is much more nearly parabolic when the arms are crossed, and by properly taking the length of the cross-piece (Ex. 4, p. 266) it may be made exactly parabolic for small displacements. This arrangement is called Farcot's governor from the name of the inventor.

141. *Stability of Governors.*—If the curve CC be not a parabola H will diminish or increase as the ball Q moves outwards. Take the first case and suppose Q in equilibrium at a certain point when the speed of revolution has a given value. Let Q now be moved up or down, then, if released, it will not remain at rest, but will return towards its original position and oscillate about it, or in other words the equilibrium of Q is stable. A governor possessing this property is described as "stable," and its stability is greater the quicker H diminishes. Similarly when H increases for an outward movement of the balls the governor is "unstable," and a parabolic governor may properly be described as "neutral."

A certain degree of stability is necessary for the proper working of a governor, and the amount required is greater the greater the frictional resistances. For assuming the revolutions at which the machine is intended to work to be n, the balls commence to move outward at the speed $n + x$, where x is a small quantity depending on the frictional resistance. After starting, the frictional resistances are not increased, but on the contrary will somewhat diminish; and, in a neutral governor, the balls therefore move outwards with increasing speed until by alteration of the regulating valve the supply of energy is diminished and the speed of the machine lessened. This change however requires time, and besides the balls are in motion and have to be stopped. The consequence is that they move outwards too far, and the supply of energy being too small the revolutions diminish to $n - x$, the speed necessary to move the balls inwards, notwithstanding the frictional resistance. Thus the motion is unsteady, the balls oscillating, and the speed fluctuating, between limits wider than $n \pm x$ without ever settling down to a permanent regime.

The oscillation of the balls may be checked by a suitable brake, but it is preferable to employ a governor possessing a moderate degree of stability; the tendency to move the balls then diminishes as soon as the balls move, and they stop before moving far. The greater the frictional resistances the greater is the change required to enable the balls to return at once if they have moved too far for equilibrium. An important characteristic therefore of a good centrifugal governor is that the stability be capable of adjustment to suit the frictional resistances. Certain forms of compound governors, as for example that known as the "cosine," fulfil this condition and can, probably, be made more perfect than the simple pendulum governor.

All such mechanisms are however imperfect in principle, for they cannot come into operation till a certain change of speed has actually existed for a perceptible length of time. Where the changes of resistance are sudden and violent the best governor will scarcely prevent violent fluctuations in speed. In screw vessels, where this difficulty is much felt, it has been proposed to employ an auxiliary engine rotating against a uniform resistance; any difference of speed of which and the screw shaft immediately shifts the regulating valve.

In the "cup governor," invented by Dr. Siemens,* a regulator and an hydraulic brake are combined. A cup containing water rotates within a cylindrical casing; at low speeds the water remains within the cup, but as soon as the speed exceeds a certain limit centrifugal action causes it to pour over the edge of the cup into the space between the cup and the casing. A set of vanes attached to the cup rotate between fixed vanes attached to the casing, and break up the descending water, which re-enters the cup by an orifice in the bottom. There is then a great resistance to the motion of the cup which absorbs surplus energy. Some other forms of governor will be considered hereafter.

SECTION III.—STRAINING ACTIONS ON THE PARTS OF A MACHINE.

142. *Transmission of Stress in Machines.*—We have seen (Art. 94, p. 184) that a mechanism becomes a machine if certain links are added capable of changing their form or size, and so producing forces which tend to move the mechanism combined with other forces which resist the motion. Each link so added exerts equal and opposite forces on the elements it connects, and for the pair of forces the general word "Stress," may be used, which has been already employed in Article 1 in the case of the bars of a framework structure.

When the machine is at rest the forces, being all in pairs, balance each other, and have no tendency to move the machine as a whole. For example in the direct-acting vertical engine represented in Fig. 1, Plate I., p. 108, the driving link is the steam, pressing with equal force, one way on the cylinder cover, and the other way on the piston; the working link is the resistance to turning of the crank shaft, which exerts equal and opposite forces, one way on the crank, the other way on the frame which carries the crank-shaft bearings. The steam pressure and the working resistance may each be described as a "Stress." The forces which make up the stress are transmitted from the piston through the connecting rod to the crank, and, in the opposite direction, from the cylinder cover through the frame to the crank shaft. The horizontal pressure of the cross-head on the guide bars is in like manner balanced by the equal horizontal thrust of the connecting rod on the crank pin, combined with the moment of the working resistance.

---

* *Phil. Trans.*, 1866.

So in every machine, when at rest, or moving slowly and steadily, the stress is transmitted from the driving pair to the working pair, not only through the moveable parts of the machine, but in the opposite direction, through the framing; and this is a circumstance which must be always borne in mind in designing the framing. Thus, in our example, the steam cylinder and crank-shaft bearing must be rigidly connected by a frame strong enough to withstand the total steam pressure, and, in addition, the bending due to the lateral pressure on the guide bars.

We have here one of the simplest examples of the transmission of stress; whether in a machine or in a structure it always takes place in a closed circuit.

If the driving pair and the working pair are the same, and acted on by the same stress, the whole state of stress is the same for all the mechanisms which are derived by inversion from the same kinematic chain. All such mechanisms are therefore statically as well as kinematically identical; it is only when we consider machines in motion, or the straining actions due to gravity, that it is necessary to consider which link (if any) is fixed to the earth. For example, the only difference between the direct-acting engine of Fig. 1, and the oscillating engine of Fig. 4, Plate I., is that the working pair is BA in the first and BC in the second. So again, in Plate III., the only difference between the water wheel of Fig. 2 and the horse gear of Fig. 3 is in the nature of the driving link, which in the first case is gravity acting on the falling water, and in the second a living agent.

A striking example of the balance of forces in a machine occurs in the hydraulic rivetting machines. Here the working pair is a small hydraulic cylinder and its ram, between which the rivet is compressed. This cylinder is suspended from a crane by chains, and can be moved into any position, as it communicates with the accumulator (Part V.) by a flexible pipe. Any portable machine, however, is an example of the same kind: machines which require foundations have no complete

PLATE VII.

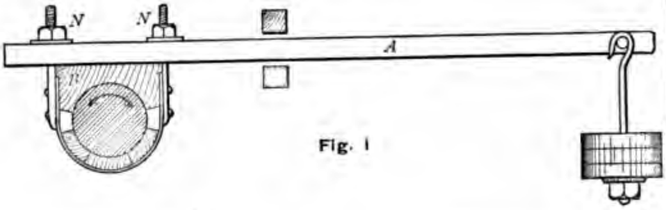


Fig. 1

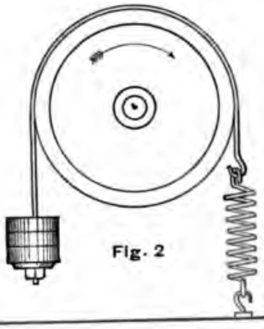


Fig. 2

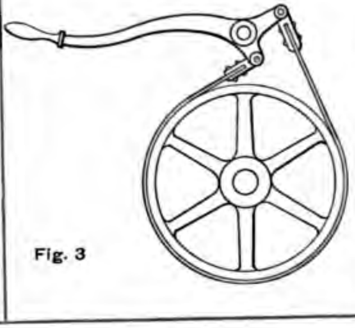
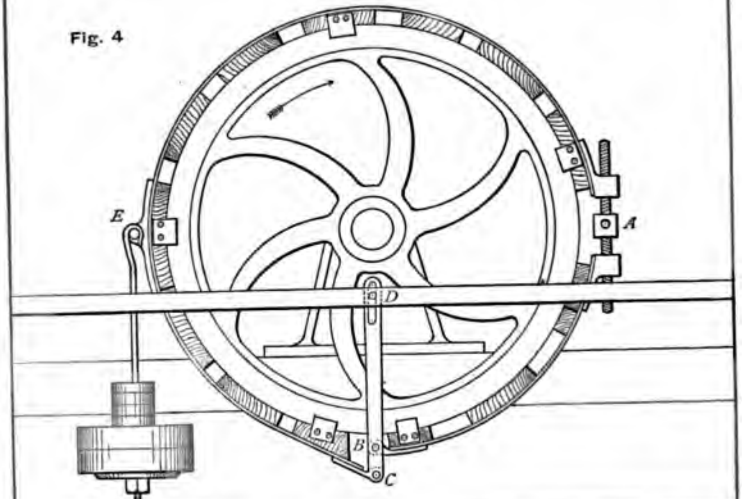


Fig. 3

Fig. 4



To face page 261.

frame apart from the solid ground which connects their parts together.

143. *Dynamometers.*—It is often a question of great practical importance to determine by direct experiment the power which is being expended in driving a machine. Instruments for effecting this are called Dynamometers; they are of very various construction, and only a few simple cases can be mentioned here. The most common are those in which the instrument measures the driving effort, while the speed is independently determined, and the power thence obtained, as in Art. 97, page 188.

(1) In Fig. 4, Plate III., a common "transmission dynamometer" is represented. A shaft transmitting the power is divided into parts, and bevel wheels *B*, *D* attached to each. A lever *A*, turning about an axis concentric with the shaft, has a weight suspended from it, which is found by trial just to balance the driving couple, transmitted through the bevel wheels *C* attached to *A* and gearing with *BD*. The whole forms the train described on page 137. Here the driving couple is measured in the act of transmission, and the revolutions of the shaft being known the power can be found.

(2) In Fig. 1, Plate VII., a "friction dynamometer" is represented in one of the various forms in which it is applied. *A* is a lever from which a weight is suspended; *B* is a block fixed to *A*, which rests on a revolving drum. A strap passes below the drum, and is tightened by the nuts *N*, *N*, till the friction just balances the weight, which by trial is made to balance the driving couple tending to turn the shaft. Here the driving couple and, consequently, the power are determined as in the preceding case, from which it only differs in the way in which the power is employed. Instead of being transmitted to the machine it is all absorbed by a friction brake which replaces it for the time being. A modification is shown in Fig. 2, in which the strap passes over a wheel and is tightened by a suspended weight, the difference between which and the tension of a spring balance measures the driving couple. (See Appendix.)

(3) In both the preceding cases the driving effort and the speed of the driving pair are constant, but in the indicator universally employed to measure the power of steam and other heat engines, we find an example in which both vary. The driving effort is now measured for each position of the piston and a curve drawn which represents it; the area of this curve will be the work done per stroke, and divided by the length of the stroke will give the mean driving effort. This will be further explained in Part V.

(4) Instead of measuring the effort and the speed independently, and performing a calculation to obtain the power, an instrument may be constructed which performs the operation automatically. Such instruments are called "integrating dynamometers," or sometimes "power meters." They are a special variety of integrating Apparatus, on the construction of which the reader is referred to papers by Mr. Boys, in the *Proceedings of the Physical Society*, vols. iv., v.

144. *Stability of Machines. Balancing.*—In a machine with reciprocating parts the balance of forces (Art. 142) is destroyed by their inertia when the machine is in motion, and, in consequence, the machine must be attached to the earth or some massive structure by fastenings of sufficient strength. The straining actions on these fastenings will now be briefly considered.

Taking the case of a direct-acting horizontal steam engine, let P be the total pressure of the steam on the cylinder cover, then the pressure (P') transmitted to the crank pin is not equal to P, but there is a difference (S), given by the formula (Art 109 p 212; see also Ex 13, p. 220).

$$S = P - P' = \frac{WV_0^2}{ga^2} . x.$$

This difference will be a force acting on the engine as a whole, and straining the fastenings. The direction of this force is reversed twice every revolution, and its maximum value is obtained by putting $x = a$ in the above formula. In slow-moving engines the value of S is small, but at high piston speeds it becomes very great, and must be carefully provided against, especially when, as in locomotives, the engine cannot be attached to the ground.

In most cases there are two cranks at right angles, and therefore two forces S, S' given by the equations

$$S = \frac{WV_0^2}{ga} . \cos\theta ; \; S' = \frac{WV_0^2}{ga} . \sin\theta,$$

where θ is the angle the first crank makes with the line of centres. These two forces are equivalent to a single force (Fig. 116),

$$Q = S + S' = \frac{WV_0^2}{ga}(\sin\theta + \cos\theta),$$

acting midway between them, and a couple

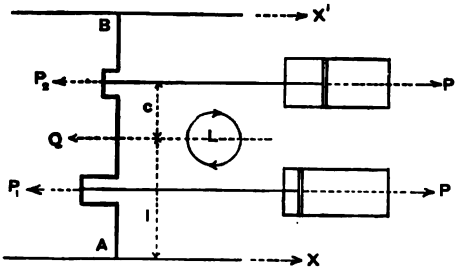
$$L = (S - S')c = \frac{WV_0^2}{ga} . c(\cos\theta - \sin\theta),$$

where c is the distance apart of the centre lines of the cylinders. The total effect therefore is the same as that of a single alternating force combined with an alternating couple, which tends to turn the engine as a whole about a vertical axis. The maximum values are

$$Q_0 = \frac{WV_0^2}{ga}\sqrt{2}\,;\; L_0 = \frac{WV_0^2}{ga}\sqrt{2c},$$

and they are each reversed twice in every revolution.

Fig. 116.



In locomotives this action produces dangerous oscillations at high speeds, and must therefore be counteracted by the introduction of suitably placed balance weights, so as to neutralize both the force and the couple.

Fig. 117.

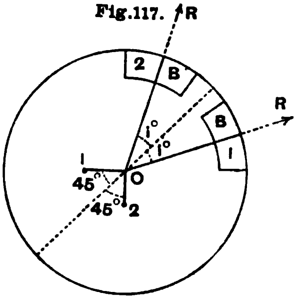


Fig. 117 shows a projection on a vertical plane of the two driving wheels and their cranks. On each wheel a balance weight is placed, occupying a segment between two or more spokes. The centre of gravity of each weight is in a radius nearly, but not exactly, opposite the nearer crank, the angle of inclination to the bisector being an angle i somewhat less than 45°. If B be the weight, r the radius of the circle in which its centre of gravity lies,

$$R = \frac{BV_0^2 r}{ga^2}$$

is its centrifugal force; and by rightly taking the values of B and i the horizontal components of these forces derived from the two balance weights, may be made to counteract both the force and the couple (Ex. 10, p. 266). In practice the weights are fixed approximately by a formula derived in this way, and the final adjustment is performed by trial. The engine is suspended by chains, and its oscillations, when perfectly adjusted, are very small even at very high speeds. (See Appendix.)

In high speed marine engines similar forces arise, of great magnitude, which must add considerably to the strain on the fastenings, but no attempt is commonly made to balance them.

When the speed of a machine is excessive, reversal of stress must be avoided (see Exs. 17, 18, p. 220), and the greatest care is necessary that the axis of rotation of each rotating piece passes through its centre of gravity, and coincides with one of the axes of inertia of the piece (Art. 132). The magnitude of the forces which arise, in case of any error, may be judged of from the results of Exs. 13, 16, page 267. The vibrations due to these forces will, however, in some cases be greatest at some particular speed—depending on the natural period of vibration of the frame of the machine—which could only be determined by trial. (Ch. XVI.)

In similar machines the forces due to inertia will be in a fixed proportion to the weight of the pieces, when the revolutions vary inversely as the square root of the linear dimensions of the machine.

145. *Straining Actions on the Parts of a Machine due to their Inertia.*—Another important effect of the inertia of a piece is to produce straining actions upon it. An important example is that of a ring rotating about its centre: the centrifugal force produces a tension on the ring which may be thus determined.

Suppose Fig. 121, p. 278, to represent the ring. Let the velocity of periphery be V, the weight W, and the radius r, then the centrifugal force on the small portion BB' of length z is

$$S = W . \frac{z}{2\pi r} . \frac{V^2}{g} .$$

Resolve this in a given direction and sum the resolved parts, as in the article to which this figure refers, then the total is

$$P = W . \frac{2r}{2\pi r} . \frac{V^2}{gr} = \frac{W}{\pi r} . \frac{V^2}{g} .$$

The stress to which this gives rise is evidently

$$q = \frac{W}{2\pi r A} . \frac{V^2}{g} = w . \frac{V^2}{g} ,$$

where A is the sectional area of the ring and w is the weight of unit volume. The result here obtained is of great importance; it shows that the "centrifugal tension" of a revolving ring is independent of the radius for a given speed of periphery. Hence the result also applies to every point of a flexible element, such as a belt, whatever be the form of the surfaces over which it is stretched. In high-speed belts the tension is considerably increased by this cause, and additional strength has to be provided (Ex. 12, p. 267).

Another example of the straining actions due to inertia occurs in the motion of a rod, the ends of which describe given curves.

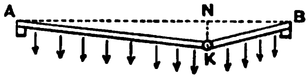
Shearing and bending are produced, and at high speeds the magnitude of the stress thus arising is very great. Two common examples are given on pages 266, 267, but the limits of this work do not permit us to pursue the subject.

In similar machines the intensity of the stress occasioned by the straining actions we are here considering will be the same if the revolutions vary inversely as the linear dimensions of the machine.

146. *Virtual Machines.*—It has already been pointed out (Art. 94) that a machine may be regarded as a mechanism with two additional links applied as straining links, or, what is the same thing, a frame with one straining link (Art. 43). Further, as also remarked in the article cited, the external forces on any structure may be regarded as a set of straining links. It follows then that if in any framework or other structure one of its parts suffer a change of form or size of any kind, the rest remaining rigid, we shall have a machine in which the driving links exert a known stress and the working link is the bar in question. The principle of work then enables us to determine the stress on the bar, for the stress ratio must be the reciprocal of the velocity ratio. A machine thus formed may be called a "virtual machine," its movements being only supposed for the purpose of the calculation, not actually existing. It is especially in applying this method that we find in treatises on statics the principle of work employed under the title "principle of virtual velocities."

We must content ourselves with a single example of this method. *AB* (Fig. 118) is a beam supported at the ends and loaded uniformly. Imagine the beam broken at *K*, and the pieces united by a stiff hinge, the friction of which is exactly equal to the bending moment *M*, then if the hinge be supposed gradually to yield under the weight, so that the joint *K* descends through the small space $KN(=y)$,

Fig.118.



$$\text{Energy exerted} = \tfrac{1}{2}yw(AK + BK).$$

$$\text{Work done} = M(i_1 + i_2) = M\left(\frac{y}{AK} + \frac{y}{BK}\right),$$

where i_1, i_2 are the angles *AK*, *BK* make with the horizontal.

Equating the two

$$M\left(\frac{1}{AK} + \frac{1}{BK}\right) = \tfrac{1}{2}w(AK + BK),$$

which gives the known value (p. 39),

$$M = \tfrac{1}{2}w.AK.BK.$$

The advantage of this method is that it leads directly to the required result, without the introduction of unknown quantities which require to be afterwards eliminated.

EXAMPLES.

1. In Ex. 1, page 197, suppose the gun to weigh 35 tons, what additional powder will be required to provide for recoil? *Ans.* 1 lb. nearly.

2. Two vessels of displacements 8,000 and 5,000 tons are moving at 6 knots and 4 knots respectively. One is going north and the other south-west; find the energy of collision. *Ans.* 11,700 foot-tons.

3. Find the height of a governor revolving at 75 revolutions per 1′. *Ans.* 6·24″.

4. Find the dimensions of a Farcot governor to revolve at 40 revolutions per 1′, with the arms inclined at 30° to the vertical, and to be parabolic for small displacements. *Ans.* Height of governor = 22″. Length of arms = 34″. Length of cross-piece to which arms are attached = 8½″. More generally, if θ be the inclination, l the length of the arms, the length of the cross-piece is $2l \, . \, \sin^3\theta$.

5. In a simple governor revolving at 40 revolutions per 1′ find the rise of the balls in consequence of an increase of speed to 41 revolutions. Also find the weight of ball necessary to overcome a frictional resistance of 1 lb., the linkwork being arranged so that the slider rises at the same rate as the balls. *Ans.* Rise of balls = 1·1″. Weight of each ball = 5 lbs.

6. The balls of a governor weigh 5 lbs. each and it is loaded with 50 lbs. The linkwork is such that the slider rises and falls twice as fast as the balls. Find the height for a speed of 200 revolutions per 1′, and, if the speed be altered 2 per cent., find the tendency to move the regulating apparatus. How much is this tendency increased by the loading? If the engine is required to work at three fourths its original speed, by how much should the load on the governor be diminished? *Ans.* Height = 9″·7. Tendency = 2·2 lbs. (increased 11 times).

7. A uniform rod is hinged to a vertical spindle and revolves at a given number of revolutions; find its position. Deduce the effect of the weight of the arms of a governor on its height. *Ans.* Height of rod = $\frac{3}{2} \, . \, g/A^2$. Height of governor is increased in the ratio $1 + \frac{1}{2}n : 1 + \frac{1}{3}n$ where n is the ratio of the weight of the arm to the weight of the ball.

8. In Ex. 6, p. 120, find the ratio in which the bending moment at each point is affected by the inertia of the rod.

Every point of the rod describes relatively to the engine a circle and the centrifugal force of any portion of the rod = 18·6 times the weight. In the lowest position the centrifugal force acts with gravity, and so in this position the bending action is the same as if the weight of the material of the rod were 19·6 times its true weight.

9. In a horizontal marine engine with two cranks at right angles distant 8 feet from one another, weight of reciprocating parts attached to each crank 10 tons, revolutions 75 per minute, stroke 4 feet. Find the alternating force and couple due to inertia. *Ans.* Alternating force = 54·2 tons. Alternating couple = 216·8 foot tons.

10. An inside cylinder locomotive is running at 50 miles per hour, find the alternating force and couple. Also find the magnitude and position of suitable balance weights, the diameter of driving wheels being 6 feet, the distance between centre lines of cylinders 2′ 6″, stroke 2′, weight of one piston and rods 300 lbs. Horizontal distance apart of balance weights 4′ 9″. Diameter of weight circle 4′ 6″. *Ans.* Alternating force = 7,871 lbs. Alternating couple = 9,839 foot-lbs. $B = 106·5$ lbs. $i = 27\frac{3}{4}°$.

11. A fly-wheel 20 feet diameter revolves at 30 revolutions per 1′. Assuming weight of iron 450 lbs. per cubic foot, find the intensity of the stress on the transverse section of the rim, assuming it unaffected by the arms. *Ans.* 96 lbs. per sq. inch.

12. A leather belt runs at 2,400 feet per 1′, find how much its tension is increased by centrifugal action, the weight of leather being taken as 60 lbs. per cubic foot. *Ans.* 20·5 lbs. per square inch.

13. If r be the radius of the circle described by the centre of gravity of a rotating body, h the height due to the revolutions (page 254), show that the centrifugal force is

$$R = W \cdot \frac{r}{h}.$$

Obtain the numerical result (1) for a wheel weighing 100 lbs. with centre of gravity one sixteenth of an inch out of centre, revolving at 1000 revolutions per minute, (2) for a piece weighing 10 lbs. revolving at 300 revolutions per minute in a circle 1 foot diameter. *Ans.* (1) 178 lbs. (2) 154 lbs.

NOTE.—The formulæ of Art. 144 can all be expressed most simply in terms of h.

14. In Question 8 suppose the connecting rod of uniform transverse section, find how much the bending moment upon it due to its weight is increased by the effect of inertia.

Here the bending moment is greatest (very approximately) when the crank is at right angles to the connecting rod, and the forces due to inertia then consist (also very approximately) of a set of forces perpendicular to the rod, and varying as the distance from the crosshead pin. At the crank pin we have simply the centrifugal force due to the revolutions and length of crank. Thus the curve of loads is a straight line (p. 61) whence, proceeding by the methods of Chap. III., we find for the maximum moment

$$M = \frac{Wl}{9\sqrt{3}} \cdot \frac{a}{h},$$

where l is the length of rod, a the length of crank, h the height due to the revolutions. In the numerical example the effect of inertia is about $9\frac{1}{2}$ times that of the weight W.

15. A body rotates about an axis OE, lying in a principal plane through its centre of gravity G, and inclined to a principal axis OG at an angle θ. Show that the moment of the centrifugal forces about O is

$$L = W\frac{k^2 - k'^2}{h} \cdot \sin\theta \cdot \cos\theta,$$

where h is the height due to the revolutions, and k', k are the radii of gyration about OG, and a line through O, perpendicular to OG in the plane GOE, respectively. Deduce the height of a compound revolving pendulum.

16. A disc rotates about an axis through its centre at 1000 revolutions per minute. The disc is intended to be perpendicular to the axis, but is out of truth by $\frac{1}{100}$th of the radius: find the centrifugal couple. *Ans.* If r be the radius in inches the couple in inch-lbs. is

$$L = \frac{Wr^2}{13{\cdot}1}.$$

17. In question 10 find the alternate increase and diminution of the pressure of the driving wheel on the rail due to the inertia of the balance weight. *Ans.* 5,900 lbs.

NOTE.—This force of more than $2\frac{1}{2}$ tons produces great straining actions on both the wheel and the rails.

18. The power of a portable engine is tested by passing a strap over the fly-wheel, which is 4 feet 6 inches diameter, fixing one end and suspending a weight from the other. The weight is 300 lbs., and the tension of the fixed end is found by a spring balance to be 195 lbs.: what is the power when running at 160 revolutions per minute. *Ans.* 7·8 H.P.

PART IV.

STIFFNESS AND STRENGTH OF MATERIALS.

PART IV.—STIFFNESS AND STRENGTH OF MATERIALS.

147. *Introductory Remarks.* The straining actions which tend to cause a body or a structure to separate into parts A and B in the manner explained in Part I. are counteracted by the mutual action between the parts at each point of the real or ideal surface which divides them. In other words (see Art. 1), a STRESS exists at each point of the surface, the elements of which are A's action on B and B's action on A. If we consider the total amount of the stress, these elements each form one element of the straining actions on A and B respectively; but for our present purpose it is needful to consider, not the total amount, but the intensity of the stress. This in general varies from point to point, and at each point is measured by the stress per unit of area on any small area enclosing the point.

Either element (say A) may be regarded either as A's action *on* B, or as the resistance which A offers to the action *of* B, in other words stress may be regarded in two aspects, either as the cause tending to produce separation into parts, or as the resistance to such separation. It is under the first aspect that we shall chiefly regard stress, generally employing the word resistance when we wish to express the second idea. Stress then may be described as the straining action on the ultimate particles of a body. Conversely a straining action as defined in Ch. II. may also be described as the "resultant stress" on the section we are considering.

If the stress exceeds a certain limit, separation into parts occurs, and this limiting intensity of stress varies for different material and measures the Strength of the material.

Accompanying the tendency to separation into parts we invariably find changes of dimension in the body and each of its parts, for no body in nature is absolutely rigid. Such changes are called STRAINS, and are of two kinds, changes of volume and changes of figure, or, in other words, changes of size and changes of shape. Changes of size in any dimension are measured by the ratio of the change to the original dimension considered; changes of shape consist in the alteration of relative angular position or distortion of the parts considered, and are measured by the absolute magnitude of the alterations in question. In most cases which concern us, both kinds of change take place together and are of exceeding smallness.

The strains produced in solid bodies by the action of forces depend on the nature of the material and on the kind of stress.

Bodies are either solid or fluid. A fluid may be defined as material which offers no resistance to change of shape, but only to change of volume, especially diminution of volume, so that any distorting stress, however small, will cause indefinite change of shape if sufficient time be allowed. On the other hand a solid body will resist a distorting stress for an indefinite time, provided that stress be not too great. In a fluid body at rest only one kind of stress can exist, namely, a pressure equal in all directions; hence often called "fluid" stress.

There are two extreme conditions in which a solid body may exist, the Elastic state and the Plastic state. Elasticity is the power a body possesses of returning to its original shape and dimensions after the forces which have been applied to it are removed. All bodies possess this property to a greater or less extent, and most (perhaps all) possess it to a great degree of perfection if the strains to which it has been exposed are not too great. Even so unlikely a material as soft clay is elastic if the force applied to it is very small. This may be shown by suspending a long filament, formed by forcing clay through a small orifice, by one end and twisting the other, to which an index is attached: on release the index returns to its original position.* In perfectly elastic material the recovery of size and shape on removal of the forces is complete, unless the temperature has meanwhile varied: and the materials of construction may be regarded as approximately satisfying this condition, provided a certain limit stress be not overpassed. This is called the Elastic Strength of the material. It is also described as the "limit of elasticity."

When, on the other hand, the forces applied to the body are comparatively great, the material in many cases approaches the other extreme condition, the plastic state. In this state any forces causing a distorting stress beyond a certain limit, and so applied that disruption does not occur, will produce indefinite distortion, so that the material behaves like a fluid. Thus soft clay, lead, copper, or even malleable iron may be moulded into different shapes or drawn out into wire. In intermediate cases a body may exhibit the properties of the elastic and the plastic states combined.

---

* See Robison's *Mechanical Philosophy*, vol. I., page 375. The original observation is said to have been made by Coulomb. Though frequently quoted it does not appear to have been verified.

We commence by studying matter in the perfectly elastic state. There are two different kinds of elasticity,—Elasticity of Volume and Elasticity of Figure. A fluid possesses the first kind only, since by definition it has no power of resisting change of shape: the second is characteristic of solids. In general a change of dimensions involves both a change of size and a change of shape, so that both kinds of elasticity are called into play together. In perfectly elastic material the strain produced by a given stress is always proportional to the stress, being found by dividing the stress by a co-efficient or " modulus " of elasticity, depending on the kind of stress and the nature of the material. This property having been discovered by Robert Hooke is known as Hooke's Law. Further, if the stress be relaxed in the slightest degree the strain diminishes, that is, in perfectly elastic material, the elastic forces are completely "reversible" (p. 182).

The magnitude of the stress produced by the action of given forces upon a body depends very much on whether they are applied all at once or are supposed to be at first very small and gradually to increase to their actual amounts. The next four chapters will be limited to the action of a gradually applied load on perfectly elastic material. The experimental part of the subject is placed in the last chapter (Ch. XVIII.), but should be referred to constantly as required.

CHAPTER XII.

SIMPLE TENSION, COMPRESSION, AND BENDING OF PERFECTLY ELASTIC MATERIAL.

SECTION I.—TENSION AND COMPRESSION.

148. *Simple Tension.*—The effect of forces acting on a bar has already been explained in Chapter II. to consist in the production of certain straining actions which we called Tension, Compression, Bending, Shearing, and Twisting, and we now go on to consider the changes of form and size which the bar undergoes and the stress produced at each point on the supposition that the material of the bar is perfectly elastic.

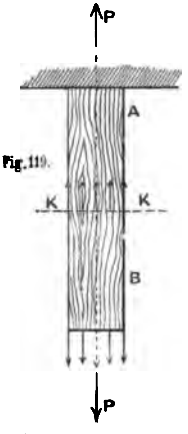


Fig. 119.

Let AB (Fig. 119) be a bar subjected to the action of equal and opposite forces applied at the ends in the same straight line. At any transverse section KK there will be a tendency to separate into two parts A, B, which is counteracted by a mutual action between the parts at each point of the section, which, in accordance with our previous definitions, is called the Tensile Stress at the point. The total amount of the stress will be P; but the intensity will depend on the area of the section (A), so that P/A is the mean intensity of stress or the stress per unit of area. The stress may be the same at all points of the section. We then say it is uniformly distributed, and the intensity at all points $= P/A$.

In order that the intensity of the stress may be the same at every point of every transverse section of the bar, it is *theoretically* necessary that the load P should be applied in a uniformly distributed manner all over the end B. Then if the material is perfectly homogeneous each elementary portion of KB will be strained alike, and the uniformly distributed load at B will be balanced by a uniformly distributed stress over any section KK. In such a case the line of action of the resultant of the applied load P passes through the centre of gravity or centre of position of the transverse section KK. Unless it does so the equilibrium of the portion KB is not possible by means of a uniformly distributed stress over the section. But from experience it appears that for uniformity of stress it is not absolutely necessary for the load to be applied in this distributed manner. It may be applied for instance by pressure on a projecting collar; and yet *if the line of application of the load traverses the centre of gravity of the sectional area*, the material, if homogeneous, will so yield as *practically* to produce at a section a little distant from the place of application of the load a stress of uniform intensity. This is a particular case of a principle which will be further referred to hereafter.

If the applied load is increased, the stress on the section is proportionately increased, until at last the material yields under it and the bar breaks. If W = breaking load, the corresponding stress measured by W/A is a quantity which depends on the nature of the material. If we call it f, then the breaking or ultimate load $= Af$.

Accompanying the application of the load producing a tensile stress, an increase of length and diminution of transverse dimension is observed. In metallic bodies the alterations are exceedingly small if the limit of elasticity is not exceeded (see Table II., Ch. XVIII.), and therefore in estimating the stress on the section it is not worth while to take account of the slight alteration in the area of the transverse section. Under the same load the change of length is proportional to the length. If x be the total change of length, and l the original length, then the extension per unit of length is

$$e = \frac{x}{l}.$$

On account of the smallness of e it is immaterial whether l is taken as the original or altered length of a metallic bar.

As already stated (Art. 147), it is usual to restrict the word *strain* to mean the alteration of the dimension and form which bodies undergo and to use the word *stress* when referring to the elastic forces which accompany the strain. Thus e is a measure of the tensile strain produced in the bar, whilst p is a measure of the accompanying tensile stress. Since by Hooke's law the extension of the bar is proportional to the force producing it, it follows that the strain is proportional to the accompanying stress. Thus p and e may be connected by some constant the value of which depends on the nature of the material. We may write

$$p = Ee,$$

in which E is called the modulus of elasticity of the material, which, when the stress p is expressed in pounds per square inch, has for wrought iron a value of about 28,000,000.

Putting for e its value x/l, we have the general relation,

$$\frac{p}{E} = \frac{x}{l}.$$

The transverse strain, that is, the contraction per unit of transverse dimension, is from one third to one fourth the longitudinal strain.

149. *Work done in Stretching a Rod.*—Having found the relation between the tensile stress and strain, we will now consider how much work must be done in order to stretch it.

Let a load of gradually increasing amount be applied to the bar, the bar will stretch equal amounts for equal increments of load: or the elongation of the bar will for all loads be proportional to the load. This may be represented graphically. Suppose the load P' produces the extension shown, greatly exaggerated, by B_0B' (Fig. 120), and we set off an ordinate $B'N'$ to represent P' on some scale, and do that for any number of loads, taking, for example, BN to represent P, which produces the extension $B_0B = x$; then all the points N will lie on the sloping line passing through B_0. Having done this, the area of the triangle B_0BN will represent the quantity of work done on the bar in stretching it the amount $B_0B = x$. Thus

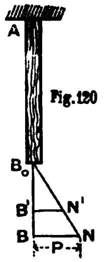


Fig. 120

$$\text{Work done} = \tfrac{1}{2}Px.$$

The energy thus exerted is stored up in the stretched bar, and may be recovered if the bar is allowed under a gradually diminished load to contract. In the perfectly elastic bar the contraction will be exactly the same as the extension, and there will be no loss of energy in stretching it. In other words the elastic forces are "reversible." But if the elasticity is imperfect, some of the energy expended in stretching the bar is employed in producing molecular changes, as for example, change of temperature. On contraction this amount of energy will not be restored.

We can express the work done in stretching the bar otherwise. For P put its value $= pA$, and for x its value $= pl/E$. The substitution of these values of P and x will give

$$\text{Work done} = \tfrac{1}{2}pA \, . \frac{pl}{E} = \frac{p^2}{E}\frac{Al}{2} = \frac{p^2}{E} \times \tfrac{1}{2} \text{ volume}.$$

Thus the work required to produce a given stress p is proportional to the volume, or, what is the same thing, to the weight, of the bar.

If the stress produced is increased up to the elastic limit, or, as it is often called, the *proof stress*, so that $p = f$, then $\frac{f^2}{E} \cdot \frac{\text{Volume}}{2}$ expresses the greatest amount of work which can be done on, and stored in the bar without injuring it or impairing its elasticity.

This is called the *resilience* of the bar. The quantity f^2/E, the value of which depends on the nature of the material, is called the *modulus of resilience*, and, as we shall see hereafter, furnishes a measure of the resistance of the material to impact in those cases in which the limits of elasticity are not exceeded (Chap. XVI.). A table of co-efficients of strength and elasticity for materials commonly used in construction will be found at the end of Chapter XVIII.

150. *Thin Pipes and Spheres under Internal Fluid Pressure.*—We now pass on to consider an important case of simple tension: that of a thin cylindrical shell subjected to internal fluid pressure. A cylinder with rigid ends and a sphere are cases of a vessel under internal fluid pressure which tends to preserve its form. The equilibrium in these two cases is stable, for if the vessel suffers deformation the internal pressure tends to make it recover its original true form. Vessels, the sides of which are flat, tend, by bulging, to assume these forms, and the tendency must be resisted by staying the surfaces in some way. If, as generally happens, there is acting also an external fluid pressure less than the internal, then, in what follows, the intensity of the internal pressure must be taken to be the excess of the internal over the external pressure.

Let p be the intensity of the fluid pressure in pounds per square inch, d the diameter, t the thickness of the shell, and l the length of the cylinder. Suppose in some way that the ends are maintained perfectly rigid, and for convenience let them be flat. There are two principal ways in which the strength of the shell can be estimated.

First, consider the tendency to tear asunder longitudinally, parallel to the axis of the cylinder. Imagine the cylinder divided into two parts by a plane passing through the axis of the cylinder. On each half cylinder there is a pressure P due to the resultant fluid pressure on that half which tends to produce a separation at the section imagined. The separation is prevented by the resistance to tearing which the metal of the shell offers, calling into action a uniform tensile stress at the two sections made by the imaginary plane through the axis of the cylinder.

Let $q =$ intensity of tensile stress produced; then the area over which the stress acts being $2tl$, the total resistance to tearing is $q \times 2tl$, which must also be the tendency to tear $= P$. In a transverse section take two points B, B' (Fig. 121) near together. The surface of the shell, $BB' \times l$, is acted upon by a normal pressure p per unit of area. The pressure $p . BB . l$ may be taken to act in a radius drawn to the middle point of BB', making an angle θ with the direction of the resultant force P. The resolved part of this pressure in the direction of P

$$=pl . BB' . \cos\theta = pl . NN',$$

NN' being the projection of BB' on the plane of section. Summing up the pressures on all the small arcs BB', composing the semicircle, we obtain the total separating force,

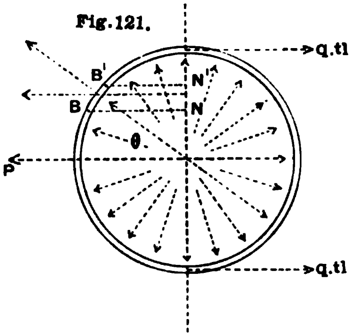
$$P = pl . \Sigma NN' = p . l . d,$$

$$\therefore\ 2qtl = pld,$$

$$\text{or } q = \frac{pd}{2t};$$

thus the tensile stress is directly proportional to the diameter, and inversely proportional to the thickness of the cylindrical shell. For greatest accuracy d should be taken as the mean of the internal and external diameters. The formula just obtained is true only when the thickness is small compared with the diameter. If t is large, the stress is not uniform over the section; the formula will then give the mean stress if d be understood to mean the internal diameter.

Fig. 121.



We next consider the tendency for the cylinder to tear across a transverse section. The total pressure on each end of the cylindrical shell is the separating force, and the resistance to separation is due to the tensile stress, q' suppose, called into action over the annular area $\pi d . t$ of the transverse section.

$$\therefore\ \pi dt . q' = \frac{\pi}{4} d^2 p\ ;\ \text{or } q' = \frac{pd}{4t}.$$

This is just half the stress on the longitudinal section. If the vessel is spherical in form, the stress produced on all sections of the sphere through the centre is the same as at the transverse section of the cylinder. It should be observed that we have here assumed that the transverse stress has no influence on the resistance to longitudinal tearing (Art. 222), and that the pressure on the ends is not provided against by longitudinal stays.

The formula just obtained is used to estimate the strength of a boiler which is more or less cylindrical; but since the boiler is made up of plates overlapping each other, connected together at the edges by rivets, and since also a line of rivets in a longitudinal section is generally found only for a portion of the length of the boiler, the question of strength is complicated. But a longitudinal section through the greatest number of rivet holes is the weakest section, and if for q we write f, where f is a co-efficient of strength to be determined from experience, the value of it depending, amongst other things, on the form of joint, then the formula

$$p = \frac{2ft}{d}, \text{ or } t = \frac{pd}{2f}$$

may be used as a semi-empirical formula to determine the greatest pressure which can be employed in a given boiler, or the thickness of metal required to sustain a given pressure. The value of the co-efficient for iron boilers with single rivetted joints is about 4,000 lbs. per square inch, or, when double rivetted, as is usual in large boilers, 5,500. With steel the value is about one-third greater. In large boilers at high pressure these values, however, have of late been considerably exceeded. (See Note.)

151. *Remarks on Tension.*--The results obtained in the present section are, strictly speaking, only applicable when the piece of material considered is of uniform transverse section, but they nevertheless may be used when the transverse section is variable, provided the rate of variation be not too great and the other conditions mentioned are strictly fulfilled. The intensity of the stress is then different at different parts of the bar, varying inversely as the transverse section, and in determining the elongation this must be taken into account.

In many cases of tension the effect of the weight of the tie and other circumstances introduces an additional stress, the amount of which is often imperfectly known. This is allowed for either by making a certain addition to the theoretical diameter or by the use of a factor of safety adapted to the particular case. On the other hand it also often happens, as in the case of ropes for example, that the strength of the material is greater in small sizes than large ones for reasons connected with the mode of manufacture.

152. *Simple Compression.*—When the forces applied to the ends of a bar act in a direction towards one another the bar is in a state of *compression*. If the bar is long compared with its transverse dimensions, then any slight disturbance from uniformity will cause it to bend sideways under the compressive force, and we have then, *not* simple compression, but compression compounded with bending, an important case to be considered hereafter. To obtain simple compression the ratio of length to smallest breadth should not exceed certain limits which depend on the nature of the material, viz., cast iron 5 to 1, wrought iron 10 to 1, steel 7 to 1. Further, it is necessary that the material be perfectly homogeneous and that the line of action of the load should be in the axis of the bar. Then the results we have obtained for simple tension apply to this case of simple compression

$$p = \frac{P}{A},$$

and the strength of the column is given by $P = Af$, where f is the coefficient of strength. The compression x which the column undergoes is connected with the stress by the equation

$$p = E\frac{x}{l}.$$

The modulus of elasticity E would, in a perfectly elastic body, be the same as for tension. In actual materials it sometimes appears to be less; but within the elastic limit only slightly less.

EXAMPLES.

1. A rod of iron 1 inch in diameter and 6 feet long is found to stretch one sixteenth inch under a load of 7½ tons. Find the intensity of stress on the transverse section and the modulus of elasticity in lbs. and tons per square inch.

Stress = 21,382 lbs. = 9·55 tons.

Modulus of elasticity = 24,631,855 lbs. = 10996·4 tons.

2. What should be the diameter of the stays of a boiler in which the pressure is 30 lbs. per square inch, allowing one stay to each 1½ square feet of surface and a stress of 3,500 lbs. per square inch of section of the iron? *Ans.* 1⅓ inches.

3. In Example 1 find the work stored up in the rod in foot-pounds. *Ans.* 43¾.

4. If in the last question the rod were originally 2″ diameter and half its length were turned down to a diameter of 1″. Compare the work stored in the rod with the result of the previous question.

Ratio = ⅝.

5. In Example 1 assume the given load of 7½ tons to be the proof load; find the modulus of resilience. *Ans.* 18·56 in inch-lb. units.

6. Find the thickness of plates of a cylindrical boiler 4′ 2″ diameter to sustain a pressure of 50 lbs. per square inch, taking the co-efficient of strength of plate at 4,000 lbs. *Ans.* $\frac{5}{16}$″.

7. A spherical shell 4′ diameter ¼″ thick is under internal fluid pressure of 1,000 lbs. per square inch. Find the intensity of stress on a section of the sphere taken through the centre. *Ans.* 48,000 lbs. per square inch.

8. Find the necessary thickness of a copper steam pipe 4″ diameter for a steam pressure of 100 lbs. above the atmosphere, the safe stress for copper being taken as 1,000 lbs. per square inch. *Ans.* ·2″.

9. A circular iron tank, diameter 16 feet, with vertical sides ½″ thick, is filled with water to a depth of 12 feet: find the stress on the sides at the bottom. How should the thickness vary for uniform strength throughout? *Ans.* 1,024 lbs. per square inch.

10. What length of iron suspension rod will just carry its own weight, the stress being limited to 4 tons per square inch, and what will be the extension under this load? *Ans.* 2,700 feet.

11. The end of a beam 10″ broad rests on a wall of masonry; if it be loaded with 10 tons what length of bearing surface is necessary, the safe crushing stress for stone being 150 lbs. per square inch. *Ans.* 15″.

12. Find the diameter of bearing surface at the base for a column carrying 20 tons, the stress allowed being as in the last question. *Ans.* 20″ nearly.

13. Compare the weight of the shell of a cylindrical boiler with the weight of water it contains when full. *Ans.* Ratio = 15·5p/f.

SECTION II.—SIMPLE BENDING.

153. *Proof that the Stress at each Point varies as its Distance from the Neutral Axis.*—The nature of the straining action producing bending has been sufficiently explained in the third section of Chapter II., and we shall now consider the kind of stress which results on the ultimate particles of a solid bar of uniform transverse section and of perfectly elastic material. The bar is supposed symmetrical about a plane through its geometrical axis, and the bending is supposed to take place in this plane which may be called the Plane of Bending. In the first instance the bending is supposed to be "simple," that is, it is not combined with shearing as is most often the case in practice, but is due to a uniform bending moment (see Art. 21). The curvature of the beam is then uniform, that is to say, it is bent into a circular arc. The investigation consists of three parts.

Fig. 122 shows a longitudinal section AB and a transverse section LL through the centre of the beam; by symmetry it follows that if the bending moment be applied to both ends in exactly the same way, that transverse section, if plane before bending, will be still plane after bending, for there is no reason for deviation in one direction rather than another. It will be seen presently that if the bending moment be applied to the ends of the beam in a particular way all transverse sections will be in the same condition, and we may therefore assume that not only the central section, but any other sections KK we please to take, will remain plane notwithstanding the bending of the beam. All such sections, if produced, will meet in a line the intersection of which by the plane of bending will be a point O which is the common centre of the circular arcs KL, PP, NN, &c., formed by the intersection of the same plane with originally plane longitudinal layers. These layers after bending have a double curvature, one in the plane of bending, the other in the transverse plane; the transverse bending however need not be considered at present, and the transverse section of the layers may be treated as straight lines. Before bending the layers were all of the same length, being cut off by parallel planes, but now they will vary in length since they lie between planes radiating from an axis O. We shall find presently that some layers must be lengthened and some shortened, an intermediate layer, NN in the figure, being unaltered in length. This layer is called the Neutral Surface and the transverse section of that layer SS is called the Neutral Axis, the last expression being always used in reference to a *transverse* section, *not* a longitudinal section. Let the radius of the neutral surface be *R*. The more the beam is bent, that is the less *R* is, the greater will be the stress produced by the bending action; and the first step in the investigation is to obtain the relation between the stress produced at any point of a transverse section and the radius of curvature *R*. If we bisect *SS* in *N* and draw *LNL* at right angles to *SNS*, it is necessary that the section of the beam should be symmetrical on each side of *LNL*; with this restriction the section may be any shape we please.

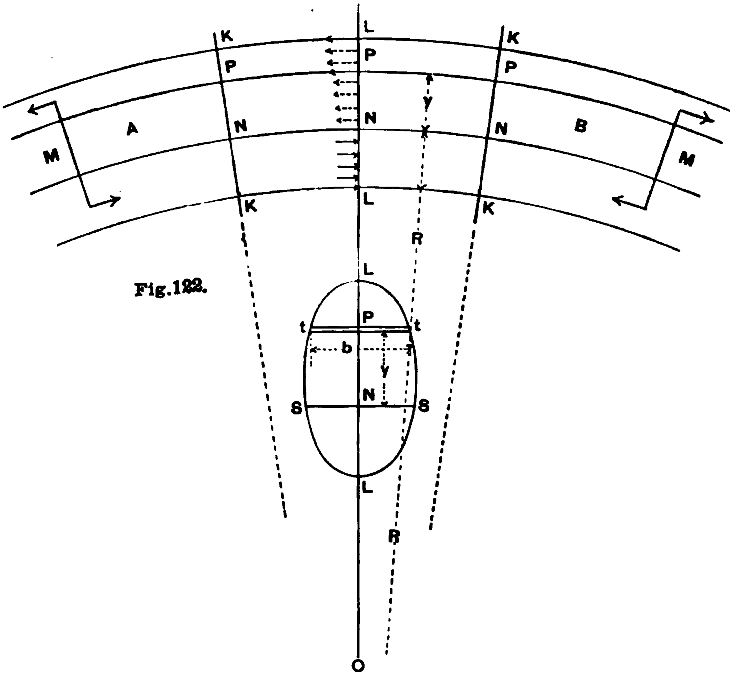


Fig. 122.

Now consider any layer *PP* of the beam between the planes *LL* and *KK* which is at the distance *y* from the neutral surface *NN* or neutral axis *SNS*. This layer will be curved to a circle whose radius is $R + y$, and it must undergo an alteration of length from *NN* which it had before bending, to *PP* which it now has. Thus the alteration of length per unit of length, that is, the strain $e = \frac{PP - NN}{NN}$, but since arcs are proportional to radii $\frac{PP}{NN} = \frac{R+y}{R}$,

$$\therefore \text{the strain } e = \frac{PP - NN}{NN} = \frac{y}{R}.$$

If the layer we are considering is taken below the neutral surface, the strain, which will then be compression, will be given by the same expression $e = y/R$, e and y both being negative.

Accompanying the longitudinal strain just estimated there must be a longitudinal stress proportional to the strain. Let p be the intensity of that stress, then

$$p = Ee,$$

where E is a modulus of elasticity. If we imagine the beam divided into elementary longitudinal bars, and if we imagine each of those bars independent of the others, it will follow that E is the same modulus of elasticity as we have previously employed in Section I. of this chapter. This, however, implies that the bar can freely contract and expand laterally when stretched and compressed, and we therefore could not be sure *a priori* that the union of the bars into a solid mass would not cause the value of E to be different from that for simple stretching, and to vary for different layers of the beam. It will be seen hereafter, however, that there are good reasons for the assumption.

Accordingly we write

$$p = E \cdot \frac{y}{R},$$

where E is the ordinary (also called Young's) modulus of elasticity. If y is taken below the neutral axis then p is negative, signifying that the stress is now compressive. In perfectly elastic material the value of E is the same for compression as for tension, and so, within the limits of elasticity, the same equation will apply for all parts of the transverse section.

Thus the stress at any point of the transverse section of the bar is proportional to its distance from the neutral axis.

154. *Determination of Position of Neutral Axis.*—The second step in the investigation is to find the position of the neutral axis, which may be done by dividing the beam into two portions, A and B, by a section LL, and considering the horizontal equilibrium of either portion, say B. The external forces, being vertical, have no horizontal component, and we have therefore only to take account of the internal molecular forces which act at the section LL. Above the neutral axis the action of LA is a tendency to pull B to the left; but below the neutral axis, the tendency is to thrust B to the right. In order that it may remain in equilibrium, and not move horizontally, it is necessary that the total pull should equal the total thrust; or the total horizontal force at the section must be zero. To estimate the horizontal force, consider the force acting on a thin strip of the transverse section, of breadth b, and thickness t, distant y from the neutral axis. The thrust or pull on this elementary strip $= p \,.\, b \,.\, t$.

Summing the forces on all the strips composing the sectional area, we must have

$$\Sigma p \,.\, bt = 0\,;$$

but $p = Ey/R$ where E and R are the same for all strips of the section.

$$\therefore \frac{E}{R} \,.\, \Sigma bt \,.\, y = 0.$$

That is to say, the sum of the products of each elementary area into its distance from the neutral axis must be zero.

This can be true only if the axis passes through the centre of gravity of the section; for it is the same thing as saying that the moment of the area about the neutral axis is to be zero.

155. *Determination of the Moment of Resistance.*—The third and last step in the investigation is to obtain the connection between the bending moment applied, and the stress which is produced by it. Again, considering either portion, AL or BL, of the beam, say AL, the external forces on A produce a bending moment or couple, M, which has to be resisted by the internal stresses called into action at the section K; so that the total moment of these stresses must be equal to M. The moment of the resisting stresses, being a couple, may be estimated about any axis with the same result. For convenience we will estimate it about the neutral axis of the section.

Let us again consider the elementary strip of area bt, distant y from neutral axis, on which the intensity of stress is p, the force, pull, or thrust, on this strip being pbt. The moment of the force $= p \,.\, bt \,.\, y$. Seeing that forces on all elementary strips, whether pull or thrust, all tend to turn the piece AL the same way, the total moment of the stresses will be found by summing all terms, $p \,.\, bty$, for the whole area of the section.

$$\therefore M = \Sigma p \,.\, bty.$$

Since $p = Ey/R$, substitute, and remember that E/R is the same for all strips, then

$$M = \frac{E}{R} \Sigma b \,.\, t \,.\, y^2.$$

In this formula the area of each strip has to be multiplied by the square of its distance from the neutral axis and the sum of the products taken. This, or an analogous sum, is of constant occurrence in mechanics, and has a name assigned to it. Σbty is the simple moment of an area about an axis. Σbty^2 may be called the moment of the second degree, but the common name is the *Moment of Inertia;* because a similar sum (differing only from this in involving the mass) occurs in dynamics under that name. To distinguish the two cases area-moment and mass-moment, the former is sometimes called the geometrical moment of inertia.

Let I denote the moment of inertia, so that $I=\Sigma bty^2$, the value of which for any form of section can be obtained by geometry, then

$$M=\frac{E}{R}I, \text{ or } \frac{M}{I}=\frac{E}{R},$$

thus connecting the curvature of the beam with the moment producing it. Having previously found $p/y=E/R$, we can now connect the moment with the stress by writing

$$\frac{p}{y}=\frac{M}{I}.$$

This equation may be employed to determine the strength of a beam to resist bending. The limit of strength is reached when either the greatest safe tensile stress on one side of the neutral axis, or the greatest safe compressive stress on the other side of the neutral axis is called into action. Thus in the equation $p/y=M/I$ we must put $p=f_1$, the co-efficient of strength under tension, or $p=f_2$, the co-efficient of strength under compression; and for y, either y_1, the distance of the most remote point on the stretched side, or y_2, the distance of the most remote point on the compressed side, so that

$$M=\frac{f_1}{y_1}I, \text{ or } \frac{f_2}{y_2}I.$$

The strength of the beam, or maximum moment of resistance to bending, is measured by the least of these quantities.

y_1 or y_2 is readily determined from geometry, the form of the section of the beam being given. It may be most conveniently expressed as a fraction of the depth of the beam. Thus y_1 or y_2 may be put $=qh$, where the co-efficient q has different values. In a rectangular section $q=\frac{1}{2}$, in a triangular section $q=\frac{1}{3}$ or $\frac{2}{3}$, and so on.

Next to express the value of I. It will be found that whatever be the form of the section, I may always be written $=nAh^2$, A being the area of the section of the beam, h the depth in the direction of bending, and n a numerical co-efficient, the value of which depends on the form of the section.

For a rectangular section,

$n=\frac{1}{12}$, so that $I=\frac{1}{12}Ah^2$,

„ elliptical or circular „

$n=\frac{1}{16}$ „ $I=\frac{1}{16}Ah^2$, For a triangular,

$$n = \tfrac{1}{18}, \text{ so that } I = \tfrac{1}{18}Ah^2,$$

and so on.

Therefore assuming q and n known, we can write

$$M = \frac{f}{qh}nAh^2 = f\frac{n}{q} \cdot Ah,$$

a formula which shows that for sections in which n/q is the same, the moment of resistance to bending is proportional to the product of the area and depth of the beam. Sections with the same n and q are said to be of the *same type*. They are often, but not correctly, said to be *similar*.

In estimating the numerical value of M, care must be taken with the units. It is generally advisable to use the inch unit throughout.

156. *Remarks on Theory of Bending.*—In the foregoing theory of simple bending it is supposed

(1) That the bar is homogeneous and of uniform transverse section and perfectly elastic;

(2) That sections plane before bending are plane after bending, for which it is theoretically necessary that the bending moment should be uniform, and applied at the ends of the bar in a particular way;

(3) That longitudinal layers of the beam expand and contract laterally in the same way, as if they were disconnected from each other (see Art. 208).

These assumptions are not obvious *a priori*, and require justification, which at the present stage of the subject we are not in position to give; for the present it may be stated that if the material be homogeneous and perfectly elastic, the equations hold good even though the transverse sections and the curvature vary and however the bending moment is applied. The *strength* of the material, however, is not generally the same, as if the layers were disconnected, and co-efficients of strength require therefore to be determined by special experiment on transverse strength (Art. 217).

157. *Calculation of Moments of Inertia.*—We have frequently to deal with beams of complex section, in which case to determine I it is convenient to divide the section up into simple areas, the I of each of which is known, and the total moment of inertia of the section will be the sum of these I's. In employing this process we require to know the relation between the moments of inertia of an area about two axes parallel to one another, one being the neutral axis. We make use of a general theorem which may be thus proved.

Let A be an area of which we know the moment of inertia about the neutral axis, SS (Fig. 123), and we require to know the moment of inertia about any parallel axis, XX, distant y_0 from SS. Dividing the area into strips of breadth b, and thickness t.

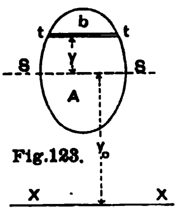


Fig. 123.

Moment of Inertia required $I = \Sigma b\,.\,t\,.\,(y + y_0)^2$.

$$= \Sigma bty^2 + 2y_0\Sigma bt\,.\,y + {y_0}^2\Sigma b\,.\,t.$$

Now $\Sigma bty^2 =$ moment of inertia about neutral axis, $\Sigma bt\,.\,y = 0$, because the neutral axis passes through the centre of gravity of the section, and $\Sigma bt =$ Area A.

$$\therefore I = I_0 + A{y_0}^2.$$

The moment of inertia of an area about any axis is, therefore, determined by adding to the moment of inertia of the area about a parallel axis through the centre of gravity the product of the area into the square of the distance between the two axes.

This theorem, together with previously quoted values of I_0, will enable us to determine the following results, which will be useful in application to beams—

Rectangle about its base, ... $I = \frac{1}{3}Ay^2$.
Triangle „ „ ... ... $I = \frac{1}{6}Ay^2$.
Triangle about a parallel to its base through vertex, $I = \frac{1}{2}Ay^2$.

Many other forms will divide up into rectangles or triangles, or both; for example, the moment of inertia of a trapezoid about the neutral axis may be readily determined by taking, for the area above the neutral axis, the I for a rectangle about one end, and triangles about the base. For the area below, a rectangle about one end and triangles about the vertex, and add the results.

158. *Beams of I Section with equal Flanges.*—The case of a beam of I section is very important.

First, suppose the flanges of equal breadth and thickness, and the web of uniform thickness b', the depth being h', b being the breadth of the flange, and h the whole depth of the beam. The moment of inertia of the section may be taken as the difference of the moments of inertia of two rectangles (see Fig. 124).

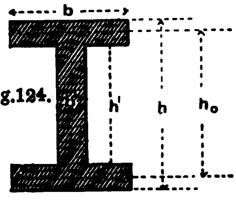
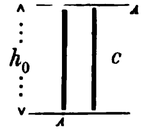


Fig. 124.

$$I = b\tfrac{1}{12}h^3 - \tfrac{1}{12}(b - b')h'^3.$$

This is the accurate value of I, and when the flanges are thick this expression for I must be used; but if the flanges are thin compared with the depth, a very close approximation can be obtained with less trouble by supposing each flange to be concentrated in its centre line, and taking for the depth of the beam the distance h_0 to the centre of flanges.

If A = area of each flange and C = area of web,



then $I = A\frac{h_0^2}{4} + A\frac{h_0^2}{4} + \frac{1}{12}Ch_0^2 = \frac{h_0^2}{2}\left(A + \frac{C}{6}\right)$

Putting $p = f$ and $y = \frac{1}{2}h_0$, in the formula $\frac{p}{y} = \frac{M}{I}$,

$$M = \frac{f}{\frac{1}{2}h_0}\,\frac{h_0^2}{2}\left(A + \frac{C}{6}\right) = fh\left(A + \frac{C}{6}\right).$$

Since the total area of the flanges is $2A$ it appears that, area for area, the web has only one-third the resistance to bending of the flanges.

We previously deduced an approximate expression for the strength of an I beam, viz.,

$$M = Hh = fhA \text{ (see Art. 27)},$$

in which the effect of the web in resisting bending was neglected, the whole of the bending action being supposed to be taken by the flanges. The present formula shows the amount of the error involved in that assumption. In using this approximation when h the effective depth is reckoned from centre to centre of the flanges, two errors are made, one in supposing the resistance to bending of the web neglected, and the other, often much greater, in supposing the mean stress on the flange equal to the maximum, hence it is better to take for the effective depth

$$h = \frac{h_0^2}{h'},$$

where h' is the outside depth and h_0 the depth from centre to centre of flanges.

159. *Ratio of Depth to Span in I Beams.*—The formula just obtained for the moment of resistance of a beam of I section shows that the greater the depth of the beam and the thinner the web the stronger will the beam be for the same weight of material, or in other words that the best distribution of material is as far away from the neutral axis as possible. The practical limitation to this is that a certain thickness of web is necessary to hold the flanges together and give sufficient power of resistance to lateral forces and to the direct action of any part of the load which may rest on the upper flange. Hence the weight of web rapidly increases as the depth increases, and a certain ratio of depth to span is best as regards economy of material (see Ex. 17, page 295). This is especially important in large girders in which economy of material is the primary consideration. In smaller beams the proper ratio of depth to span is generally in great measure a question of stiffness, a part of the subject to be considered in Chapter XIII. The moment of resistance of I sections of practical proportions is generally about double that of a rectangular section of equal area. The straining actions on the web will be considered in Ch. XV.

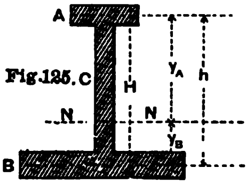
160. *Proportions of I Beams for Equal Strength.*—Materials in general are not equally strong under tension and compression, so that a beam whose section is symmetrical above and below the neutral axis will yield on one side before the material on the other side of the neutral axis has reached its limiting stress. Accordingly we might obtain a more economical distribution of material if we were to take some from the stronger side and put it on the weaker, so that the limiting tensile on one side and the limiting compressive stress on the other may be produced simultaneously. The section of the beam will be different above and below the neutral axis, which will not now be at the centre of depth of the beam, but in such a position that the distances to the top and bottom of the beam are in the proportion of the greatest allowed stresses to one another. The neutral axis in all cases must pass through the centre of gravity of the section.

Let f_A, f_B be the co-efficients of strength under compression and tension respectively, y_A, y_B distances of the most strained layer from neutral axis, then the beam will be strongest when

$$\frac{y_A}{f_A}=\frac{y_B}{f_B}=\frac{y_A+y_B}{f_A+f_B}=\frac{h}{f_A+f_B}.$$

For simplicity of calculation we will consider a beam (Fig. 125) in which the web is of uniform thickness throughout the depth, and so of rectangular section, and each flange also of rectangular section, and determine the relation which should hold between the areas of flanges and web for maximum strength of beam, and the moment of resistance to bending where this condition is satisfied. We will further suppose each flange to be concentrated in its centre line.

Fig.125.



Let A = area of compressed flange, B = area of stretched flange, C = area of web. Since the neutral axis is at the centre of gravity of the section, we obtain, by taking moments about that axis,

$$A \, . \, y_A + C\frac{y_A-y_B}{2}=By_B;$$

or, substituting the previously given values of y_A and y_B,

$$Af_A + C\frac{f_A-f_B}{2}=Bf_B.$$

Supposing f_A and f_B known, A, B, and C must be such as to satisfy this relation. We have some liberty of choice between these quantities, and frequently find one of the flanges omitted, so producing a beam of **T** or ⊥ section.

In a cast-iron beam, where the resistance to compression is greater than for tension, the compressed flange A may be omitted. Putting $A=0$ we get $C=\frac{2f_B}{f_A-f_B}B$, and supposing $\frac{f_A}{f_B}=4$; $C=\frac{2}{3}B$, or $B=1\frac{1}{2}C$. In a wrought-iron beam on the other hand f_A/f_B is about $\frac{3}{5}$, and the stretched flange B is the area to be omitted. Putting $B=0$, we find $A=\frac{f_B-f_A}{2f_A}C=\frac{1}{3}C$.

Otherwise we may assume the depth and thickness of the web to be given (Art. 159), then the equation

$$Af_A+C\,.\frac{f_A-f_B}{2}=Bf_B,$$

furnishes a relation between the areas of the flanges. For example, in cast iron, if we assume $f_A=4f_B$, we find

$$B=4A+\tfrac{3}{2}C.$$

Having decided on the proportions between the parts of the section we can now calculate the moments of inertia and resistance. Still considering the flanges concentrated in their centre lines,

$$I=Ay_A^2+By_B^2+\tfrac{1}{3}C\,.\frac{y_A}{h}\,.\,y_A^2+\tfrac{1}{3}C\,.\frac{y_B}{h}\,.\,y_B^2.$$

$$=Ay_A^2+By_B^2+\tfrac{1}{3}C\,.\frac{y_A^3+y_B^3}{h},$$

a result which admits of ready calculation. Further

$$\frac{M}{I}=\frac{f_A}{y_A}=\frac{f_B}{y_B}=\frac{f_A+f_B}{h},$$

whence we obtain

$$M=(f_A+f_B)\frac{I}{h}.$$

The calculation just now made is one which has been frequently given in dealing with beams of I section,* but in applying it to actual examples it should be remembered that the results are obtained on the supposition that the flanges are concentrated in their centre lines, and are consequently only approximate when the co-efficients f_A, f_B mean the intensities of the stress at those centre lines, *not* at the surface of the beam where the stress is greatest. If, for example, F_A be the maximum stress on the flange A

$$F_A=f_A\,.\frac{y_A+\frac{1}{2}t_A}{y_A},$$

---

* See Rankine's *Civil Engineering*, page 257.

where t_A is the thickness of the flange. The difference is especially great in the case of the larger flange of cast-iron beams, and the true ratio of maximum compressive and tensile stress is much less than it appears in the preceding article. On the other hand, in extreme cases, such as we are now considering, the stress may not be uniformly distributed along a line parallel to the neutral axis.

Extensive experiments were made on cast-iron beams by Hodgkinson, with the object of determining the best proportions between the flanges, with the result that rupture always took place by tearing asunder of the lower flange, unless it was at least six times the size of the compressed flange. This proportion is rarely adopted in practice, from the difficulties of obtaining a sound casting, and the necessity of having sufficient lateral strength. Nor is it certain that the proportions which are best for resisting the ultimate load are also best in the case of the working load; it is, in fact, probable that a smaller proportion is better even on the score of strength. If we take $f_A = 2\frac{1}{2}f_B$, instead of $4f_B$, we find

$$B = 2\tfrac{1}{2}A + \tfrac{3}{4}C,$$

which agrees more closely with practice. The ratio of maximum compressive and tensile strength is in this case about 2, which, according to some authorities, is the ratio of *elastic* strengths in the two cases.

In wrought-iron beams the areas of the flanges are usually equal, and this is correct if the elastic strength, and not the ultimate strength, is regarded as fixing the proper proportions, and if there be sufficient provision against the yielding of the top flange by lateral flexure. Small-sized beams of this kind are rolled in one piece, while large girders are constructed of iron or steel plates and angle irons, rivetted together. Some of the forms they assume are shown in Plate VIII., Ch. XVIII.

In making calculations respecting girders, approximate methods may be used for preliminary tentative calculations, but should be checked by a subsequent accurate determination of the neutral axis and moment of inertia. A previous reduction of the section to an equivalent solid section is required when, as is often the case, all parts of the section do not offer the same elastic resistance to the stress applied to them, either because they are not sufficiently rigidly connected or from the material being different. This is especially the case in determining the resistance to the longitudinal bending of a vessel occasioned by the unequal distribution of weight and buoyancy already considered in Chapter III. On this important question the reader is referred to a treatise on Naval Architecture by Mr. W. H. White. In many cases of built-up girders the shearing action which generally exists has considerable influence, a matter for subsequent consideration (Ch. XV.). The effect of the weight of the girder itself has been considered in Ch. IV. (See also Ex. 13, p. 294, and Ex. 11, p. 338.)

161. *Beams of Uniform Strength.*—A beam of uniform strength is one in which the maximum stress is the same on all sections. For beams of the same transverse section throughout this can only be the case when the bending moment is uniform, but, by properly varying the section, it is possible to satisfy the condition however the bending moment vary. For this purpose we have only to consider the equation

$$M = f.\frac{n}{q}.Ah,$$

which must now be satisfied at all sections. Suppose

$$A = kbh,$$

where k is a numerical factor depending on the type of section, then

$$M = f.\frac{nk}{q}.bh^2.$$

All sections of the beam being supposed of the same type we have only to make Ah or bh^2 vary as M, that is as the ordinates of the curve of bending moments. The principal cases are—

(1) Depth uniform. Here the breadth must vary as the bending moment, whence it is clear that the curve of moments may be taken as representing the half plan of the beam.

(2) Sectional Area uniform. Here the depth must vary as the bending moment, that is, the curve of moments may be taken to represent the elevation or half elevation of the beam.

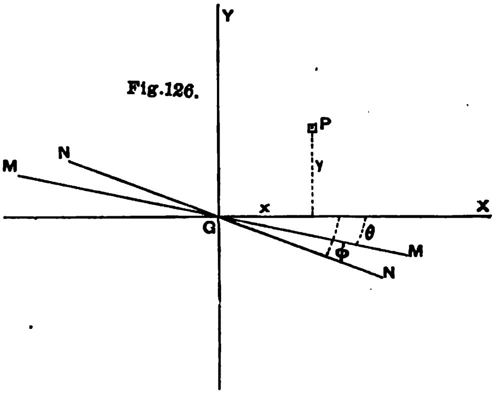
(3) Breadth uniform. Here the elevation or half elevation of the beam must be a curve, the co-ordinates of which are the square roots of the co-ordinates of the curve of moments.

(4) Ratio of breadth to depth constant. Here the half plan and half elevation are each curves, the ordinates of which are the cube roots of the ordinates of the curve of moments.

The first, third, and fourth of these cases are common in practice with some modifications occasioned by the necessity of providing strength at sections of the beam where the bending moment vanishes, as it usually does at one or both ends.

162. *Unsymmetrical Bending.*—It occasionally happens that the plane of the bending moment is not a principal plane of the beam, as for example when a vessel heels over, the plane of longitudinal bending will not coincide with the plane of symmetry of the vessel which is obviously the plane of the masts. The neutral axis does not now coincide with the axis of the bending couple, though in other respects the theory of bending still holds good.

Fig. 126.



In Fig. 126 let *MM* be the axis of the bending moment *M*, inclined at an angle θ to the principal axis of inertia *GX*, *GY* of the plane section. Then the couple *M* may be resolved into two components $M \cos \theta$ and $M \sin \theta$, each of which will produce stress at any point *P* as if the other did not exist. Let p be the stress, x, y the co-ordinates of *P* referred to the axes *GX*, *GY*, the moments of inertia about which are I_1, I_2, then

$$p = \frac{M \,.\, \cos \theta \,.\, y}{I_1} + \frac{M \,.\, \sin \theta \,.\, x}{I_2}.$$

The position of the neutral axis *NN* is found by putting $p = 0$, then the angle ϕ which it makes with *GX* is given by

$$\tan \phi = -\frac{y}{x} = \frac{I_1}{I_2} \,.\, \tan \theta.$$

This equation shows that the neutral axis is parallel to a line joining the centres of the circles into which the beam would be bent by the component couples supposed each to act alone.

The neutral axis being thus determined and laid down on the diagram the points can be found which lie at the greatest distance from that axis. At these points the stress will be greatest, and if *X*, *Y* be their co-ordinates, still referred to the axes *GX*, *GY*, the moment of resistance will be determined by the equation

$$\left\{ f = M \frac{Y \,.\, \cos \theta}{I_1} + \frac{X \,.\, \sin \theta}{I_2} \right\}.$$

For a different method of expressing the moment of resistance see Rankine's Applied Mechanics, p. 314.

EXAMPLES.

1. A bar of iron 2″ diameter is bent into the arc of a circle 372′ diameter. Find in tons per square inch, 1st, the greatest stress at any point of the transverse section; 2nd, the stress on a line parallel to the neutral axis half an inch from the centre, E being taken = 29,000,000. *Ans.* Maximum stress = 5·8. Stress at ½″ from centre = 2·9.

2. Find the diameter of the smallest circle into which the bar of the last question can be bent; the stress being limited to 4 tons per square inch. *Ans.* Diameter = 540 feet.

3. Find the position of the neutral axis of a trapezoidal section; the top side being 3″, bottom 6″, and depth 8″. Also find the ratio of maximum tensile and compressive stresses. *Ans.* Neutral axis 3·56 inches from bottom. Ratio of stresses 5 to 4.

4. A cast-iron beam is of I section with top flange 3″ broad and 1″ thick and bottom flange 8″ broad and 2″ thick; the web is trapezoidal in section ½″ thick at top and 1″ at bottom; total outside depth of beam 16″. Find the position of the neutral axis and the ratio of maximum tensile and compressive stresses. *Ans.* Neutral axis 4·81 inches from bottom. Ratio of stresses 3 to 7.

5. A wrought iron beam of rectangular section is 9″ deep, 3″ broad, and 10 feet long. Find how much it will carry loaded in the centre, allowing a co-efficient of 3 tons per square inch. Also deduce the load the same beam will bear when set flatways. *Ans.* When upright load = 4·05 tons. When set flatways load = 1·35 tons.

6. A piece of oak of uniform circular section is 16″ diameter and 12 feet long. It is supported at the two ends and loaded at a point 5 feet from one end. How great may the load be, allowing a stress of ½ ton per square inch? *Ans.* Load may be 5·74 tons.

7. In Example 5 suppose half the weight of metal formed into a beam of I section, of the same depth, each flange being equal to the web; what load will the beam carry? *Ans.* Load may then be 4¾ tons.

8. Find the moment of resistance to bending of the section given in Example 4, the co-efficient for tension being 1 ton per square inch. *Ans.* I = 798 inch units. Moment of resistance to bending = 166·4 inch-tons.

9. Suppose the skin and plate deck of an iron vessel to have the following dimensions at the midship section, measured at the middle of the thickness of the plates. Find the position of the neutral axis and moment of resistance to bending. Breadth 48′ and total depth 24′, the bilges being quadrants of 12′ radius. Thickness of plate ⅝″ all round, and co-efficient of strength 4 tons in compression.

Ans. Neutral axis 13″ above centre of depth. Moment of resistance to hogging = 32,500 ft.-tons. To sagging 39,000.

10. What should be the sectional area of a T beam of wrought iron to carry 4 tons uniformly distributed? Span 20′, depth of beam 10″. Co-efficient for compression 3 tons, and for tension 5 tons? *Ans.* Area = 13·7 square inches.

11. If, in the last question, the flange is made equal to the web instead of being proportioned for equal strength, show that to carry the same load the beam must be about one quarter heavier.

12. In Example 8 find the moments of inertia and resistance on the supposition that the flanges are concentrated at the centre lines, and thus by comparison with previous results show the amount of the error involved in the assumption. *Ans.* Moment of inertia = 861·5 inch units. Moment of resistance = 227 inch-tons.

13. Show that the limiting span (Art. 41) of a beam of uniform transverse section is

$$L = \lambda \, . \, \frac{8n}{Nq},$$

where N is the ratio of span to depth, and the rest of the notation is the same as on pages 82 and 286. Obtain the numerical result for a wrought iron beam of rectangular section, taking λ from Table I., Ch. XVIII., and supposing $N = 12$.

Ans. $L = 336$ ft.; in an ordinary I section the result would be doubled. For the case of large girders see page 339.

14. If l be the length of an iron rod in feet, d its diameter in inches, just to carry its own weight when supported at the ends, show that when the stress allowed is 4 tons per square inch $l = \sqrt{224d}$.

15. If I_1, I_2 be the moments of inertia of two plane areas A_1, A_2, about their neutral axes which are supposed parallel at distance apart z, show that the moment of inertia of their sum or difference about their common neutral axis is $I = I_1 \pm I_2 + z^2 \cdot \dfrac{A_1 A_2}{A_1 \pm A_2}$. Apply this formula to the trapezoidal section of Question 3. *Ans.* $I = 185$ inch units nearly.

16. Find the moment of resistance to bending of a beam of I section, each flange consisting of a pair of angle irons $3\tfrac{1}{2}'' \times \tfrac{1}{2}''$ rivetted to a web ·37″ thick and 16″ deep between them. Assuming it 24 feet span, find the load it would carry in the middle, using a coefficient of 3 tons per square inch. *Ans.* $M = 288$ inch-tons. $W = 4$ tons.

17. If it be assumed that for constructive reasons the thickness of web of an I beam with equal flanges must be a given fraction of the depth, show that for greatest economy of material the sectional area of the web should be equal to the joint sectional area of the flanges. Prove that in this case $M = \tfrac{1}{3} f \cdot Sh$.

18. In a cast-iron beam of I section of equal strength for which $f_A = 2\tfrac{1}{2} f_B$; if it be assumed that for constructive reasons the thickness of the web should be a given fraction of the depth, show that for greatest economy of material the large flange, the web, and the small flange should be in the proportion 25, 20, 4. Prove also that the moment of resistance is given by the same formula as in Question 17 supposing $2/f = 1/f_A + 1/f_B$.

19. A beam of rectangular section of breadth one half the depth is bent by a couple the plane of which is inclined at 45° to the axes of the section. Find the neutral axis, and compare the moment of resistance to bending with that about either axis. *Ans.* Ratio $= 2\sqrt{2}/3$ and $\sqrt{2}/3$.

20. If a beam be originally curved in the form of a circular arc of radius R_0, instead of being straight, show that the neutral axis does not pass through the centre of gravity of the section. In a rectangular section of depth h show that the deviation is, approximately,

$$z = \frac{h^2}{12R_0}.$$

21. In the preceding question if R_0 is large show that the equations of bending are

$$\frac{p}{y} = E\left(\frac{1}{R_0} - \frac{1}{R}\right) = \frac{M}{I}.$$

REFERENCE.

For the graphical determination of moments of inertia the reader is referred to the treatises cited on page 73.

CHAPTER XIII.

DEFLECTION AND SLOPE OF BEAMS.

163. *Deflection due to the Maximum Bending Moment.*—It is not only necessary that a beam should be strong enough to support the load to which it is subjected, it is also necessary that its changes of form should not be too great, or in other words, that it should be sufficiently stiff, and we next proceed to determine under what conditions this will be the case.

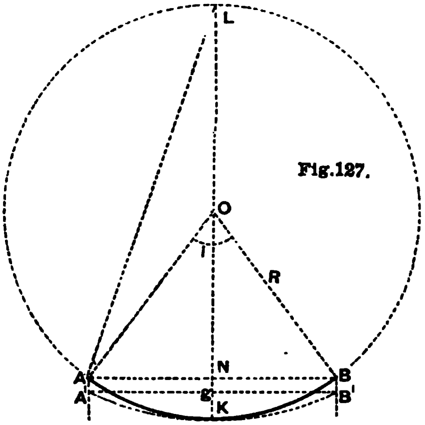


Fig.127.

The question is simplest when the beam is bent into an arc of a circle, we have then

$$\frac{p}{y}=\frac{M}{I}=\frac{E}{R}=\text{constant.}$$

Two cases may be especially mentioned—

(1) Depth uniform. We then have p constant, that the beam is of uniform strength. (See Case 1 of Art. 161.)

(2) Sectional area uniform. We then have, since

$$M=\frac{E}{R}I=n\ .\ \frac{E}{R}\ .\ Ah^2,$$

the depth of the beam varying as the square root of the bending moment, as in Case 3 of the same article. Let l be the length of the beam, i the angle its two ends make with one another, then since i is also the angle subtended by the beam at the centre

$$i=\frac{l}{R}=\frac{Ml}{EI}.$$

If the beam be supported at the ends i is twice the angle which the ends make with the horizontal, an angle called the Slope at the ends. Let AB be the beam (Fig. 127), O the centre of the circle into which it is bent, KL the diameter of the circle through K the middle point of the beam. Then KN is the deflection which is given by a known proposition of Euclid

$$KN \,.\, NL = AN^2.$$

Hence remembering that the diameter of the circle is very large* we have, if δ be the deflection,

$$\delta = \frac{l^2}{8R} = \frac{Ml^2}{8EI}.$$

This formula gives the deflection in any case where the curvature is uniform.

When the transverse section is uniform the curvature varies. Unless the bending moment be likewise uniform, the deflection curve is not then a circle AKB, but for the same maximum bending moment a flatter curve $A'KB'$. Thus the deflection is less than that calculated by the above formula, which may be described as the "deflection due to the maximum moment." The actual deflection may conveniently be expressed as a fraction of that due to the maximum moment. It is possible to construct the deflection curve graphically by observing that the curvature at every point is proportional to the bending moment. We have then only to strike a succession of arcs with radii inversely proportional to the ordinates of the curve of bending moment. It is however more convenient to proceed by an analytical method.† The fraction is least when the beam is least curved, which is evidently the case when it is loaded in the middle, and we shall show presently that it is then two-thirds, while, when uniformly loaded, it is five-sixths.

164. *General Equation of Deflection Curve.*—It was shown above that

$$i = \frac{M}{EI} \,.\, l.$$

If the bending moment vary, then we must replace l by an element of the length ds and i by the corresponding element of the angle; we shall then have an equation

$$\frac{di}{ds} = \frac{M}{EI},$$

which by integration will furnish i. It will generally be convenient to reckon i from a horizontal tangent and it then means the slope of the beam at the point considered. To perform the integration it is in most cases necessary to suppose the slope of the beam small, as it actually is in most important cases in practice, and we may then replace ds the element of arc by dx, the corresponding element of a horizontal tangent AN (Fig. 128) taken as axis of x, whence

---

* For clearness it is made small in the figure.

† Readers who have no knowledge of the Calculus may pass over the next four articles.

$$\frac{di}{dx}=\frac{M}{EI}\text{ approximately,}$$

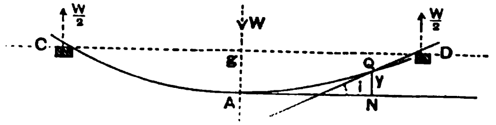
an equation which can generally be integrated because M is usually a function of x.

The deviation y of any point Q of the beam from the straight line AN can now be found since $dy/dx=i$, from which we further obtain the fundamental equation

$$\frac{d^2y}{dx^2}=\frac{M}{EI},$$

which applies to all cases where the bending of the beam is occasioned by a transverse load. We shall first give some elementary examples of the determination of the deflection and slope of a beam and then consider the question more generally.

Fig. 128.



165. *Elementary Cases of Deflection and Slope.—Case I.* Suppose a beam supported at the ends and loaded in the middle.

In Fig. 128 CD is the beam resting on supports at C, D, and loaded in the middle with a weight W. Take the centre A as origin and the horizontal tangent at A as axis of x, then if l be the whole length

$$\frac{d^2y}{dx^2}=\frac{M}{EI}=\frac{\frac{W}{2}\left(\frac{l}{2}-x\right)}{EI}$$

$$\therefore i=\frac{dy}{dx}=\frac{\frac{W}{2}\left(\frac{l}{2}x-\frac{1}{2}x^2\right)}{EI}$$

is the slope of the beam at Q, no constant being required since i is zero when $x=0$.

If $x=l/2$ we get the slope at the ends of the beam

$$i_1=\frac{Wl^2}{16EI}.$$

Integrating a second time

$$y=\frac{\frac{W}{2}\left(\frac{1}{4}lx^2-\frac{1}{6}x^3\right)}{EI}.$$

As before no constant is required because $y=0$ when $x=0$.

If now we put $x = l/2$ we get the elevation of D above AN or, what is the same thing, the depression Ag of A below the level of the supports. This is called the Deflection of the beam; if we denote it by δ,

$$\delta = \frac{\frac{W}{2}\left(\frac{1}{16}l^3 - \frac{1}{48}l^3\right)}{EI} = \frac{Wl^3}{48EI},$$

a result which we may also write

$$\delta = \frac{2}{3} \cdot \frac{M_0 l^2}{8EI} = \frac{2}{3} \cdot \delta_0,$$

where M_0 is the maximum moment and δ_0 the deflection due to it.

Case II. Let the beam be supported at the ends and loaded uniformly with w pounds per foot run. It will be sufficient to give the results, which are obtained in precisely the same way, remembering that the bending moment is now $\frac{1}{2}w(a^2 - x^2)$ where a is the half span. We have

$$i_1 = \frac{wa^3}{3EI} = \frac{Wl^2}{24EI}; \quad \delta = \frac{5}{24} \cdot \frac{wa^4}{EI} = \frac{5}{384} \cdot \frac{Wl^3}{EI}.$$

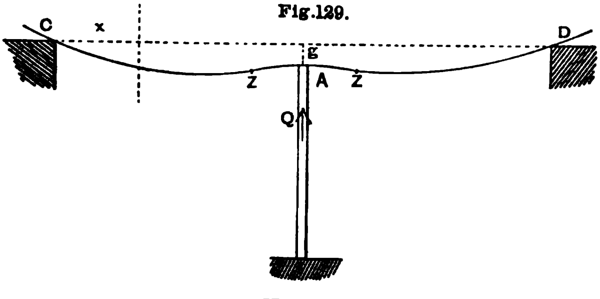
The value of δ may be expressed as in the previous case in terms of the deflection due to the maximum moment. We have $\delta = \frac{5}{6} \cdot \delta_0$.

166. *Beam propped in the Middle.*—When a beam is acted on by several loads the deflection and slope due to the whole is the sum of those due to each load taken separately. An important example is

Case III. Beam supported at the ends and propped in the middle, uniformly loaded. (Fig. 129.)

Here the deflection of the beam is the difference between the downward deflection due to the uniform load and the upward deflection due to the thrust Q of the prop. Hence we write down at once for the deflection at the centre,

Fig. 129.



$$\delta = \frac{5}{384} \cdot \frac{Wl}{EI} - \frac{Ql^3}{48EI},$$

an equation which may be used to determine the load carried by the prop when its length is given, and conversely.

First suppose the centre of the beam propped at the same level as the supports, then $\delta = 0$, and

$$Q = \frac{5 \times 48W}{384} = \tfrac{5}{8}W,$$

so that the prop in this case carries five-eighths of the weight of the beam, the supports C, D only carrying three-eighths. Each supporting force is $\frac{3}{16}wl$, l being as before the whole length of the beam; hence the bending moment at a point distant x from C is given by the formula

$$M = \tfrac{3}{16}wlx - \tfrac{1}{2}wx^2 = \tfrac{1}{2}wx(\tfrac{3}{8}l - x),$$

from which it appears that the beam is bent downwards until a point Z is reached, such that

$$CZ = \tfrac{3}{8}l = \tfrac{3}{4}AC.$$

Here the bending moment is zero, that is, Z is a "point of contrary flexure" or "virtual joint." (Compare Art. 38.)

Beyond Z the beam is bent upwards, and at the centre A we get, by putting $x = \frac{1}{2}l$,

$$-M_0 = \tfrac{1}{32}wl^2.$$

The case here discussed is also that of a beam one end of which is fixed horizontally and the other supported at exactly the same level.

Let us next inquire what will be the effect of supposing the centre of the beam propped somewhat out of the horizontal line through the supports at the ends. Let us suppose δ to be $1/n^{\text{th}}$ the deflection of the beam when the prop is removed, then

$$\frac{1}{n}\cdot\frac{5}{384}\cdot\frac{Wl^3}{EI} = \frac{5}{384}\cdot\frac{Wl^3}{EI} - \frac{Ql^3}{48EI},$$

that is

$$Q = \tfrac{5}{8}W\left(1 - \frac{1}{n}\right),$$

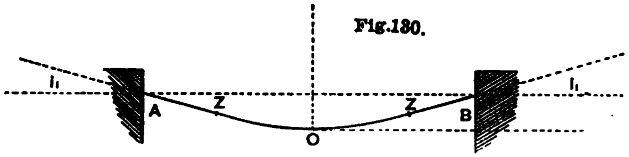
a formula which gives the load on the prop. If, for example, $n = 5$, $Q = \frac{1}{2}W$, or if $n = -5$, $Q = \frac{3}{4}W$; thus if the centre of the beam be out of level, by as much as one fifth the deflection when the prop is wholly removed, the load on the prop will vary between $\frac{1}{2}W$ and $\frac{3}{4}W$, a result which shows the care necessary in adjustment to obtain a definite result.

167. *Beam fixed at the Ends.—Case IV.* Uniformly loaded beam, with ends fixed at a given slope.

In Fig. 130 AB is a uniformly loaded beam, with the ends A, B fixed not horizontally but for greater generality at a slope i. Here the central part of the beam will be bent downwards and the end parts upwards; at Z, Z there will be virtual joints; let $OZ = r$, then taking O as origin the bending moment at any point between O and Z is

$$M=\tfrac{1}{2}w(r^2-x^2),$$

Fig. 130.



a formula which will also hold for points beyond Z, as can be seen from Art. 38, or proved independently. We have then

$$\frac{d^2y}{dx}=\frac{\frac{1}{2}w(r^2-x^2)}{EI};$$

$$i=\frac{\frac{1}{2}w(r^2x-\frac{1}{3}x^3)}{EI}.$$

No constant is required, because i is zero at O. Let a be the half span OA, or OB, then putting $x=a$, we get for the slope at the ends

$$i_1=\frac{\frac{1}{2}w(r^2a-\frac{1}{3}a^3)}{EI},$$

a formula from which r can be determined if i_1 be given. If $r=a$, we get the case where the ends are free; let the slope then be i_0 we have

$$i_0=\frac{wa^3}{3EI} \text{ as before (p. 298).}$$

Now, assume the actual slope to be $1/n^{\text{th}}$ of this, we get

$$\frac{1}{n}\cdot\frac{wa^3}{3EI}=\frac{\frac{1}{2}w(r^2a-\frac{1}{3}a^3)}{EI};$$

that is,

$$r^2=\tfrac{1}{3}a^2\left(1+\frac{2}{n}\right).$$

If the ends are fixed exactly horizontal, then

$$r^2=\tfrac{1}{3}a^2,$$

and by substitution we find for the bending moment at the centre and the ends

$$M_0=\tfrac{1}{6}wa^2;\ M_A=M_B=\tfrac{1}{3}wa^2.$$

If the ends were free, the bending moment at the centre would have been $\frac{1}{2}wa^2$, so that the beam will be strengthened in the proportion 3 : 2. The formula obtained above, however, shows that a small error in adjustment of the ends will make a great difference in the results.

It is theoretically possible so to adjust the ends that the bending moments at the centre and the ends shall be equal, in which case the beam will be strongest. For this we have only to put

$$\tfrac{1}{2}wr^2=\tfrac{1}{2}w(a^2-r^2),$$

that is

$$r^2=\tfrac{1}{2}a^2,$$

whence by substitution we get

$$n = 4\,;$$

that is, the ends should be fixed at one fourth the slope which they have when free, and the strength of the beam will then be doubled.

By proceeding to a second integration the deflection of the beam can be found. In particular when the ends of the beam are horizontal it can be shown that the deflection is only one fifth of its value when the ends are free.

The graphical representation of the bending moments in Cases III., IV., is easily effected, as in Fig. 42, p. 76.

168. *Stiffness of a Beam.*—The stiffness of a beam is measured by the ratio of the deflection to the span. In practice, the deflection is limited to 1 or 2 inches per 100 feet of span when under the working load; that is, the ratio in question is $\frac{1}{600}$th to $\frac{1}{1200}$th. It appears from what has been said that if M_0 be the maximum moment the deflection is given by

$$\delta = k\,.\,\frac{M_0 l^2}{8EI},$$

where k is a fraction, which, in beams of uniform section, varies from two-thirds to unity, depending on the way in which the beam is loaded.* Hence the greatest moment which the beam will bear consistently with its being sufficiently stiff is

$$M_0 = \frac{8E\delta}{kl}\cdot\frac{I}{l}.$$

If we express I as usual in terms of the sectional area and depth, we get

$$M_0 = s\frac{n}{k}A\frac{h^2}{l},$$

where s is a co-efficient depending on the material and on the admissible deflection which may be called the "Co-efficient of Stiffness."

We thus obtain a value for the moment of resistance of a beam which depends on its stiffness, not on its strength, and if that value be less than that previously obtained for strength (p. 286), we must evidently employ the new formula in calculating dimensions. On comparing the two, we find that they will give the same result if

$$\frac{sh}{kl} = \frac{f}{q}\,;\ \text{or}\ \frac{h}{l} = \frac{fk}{qs}\,;$$

that is to say, for a certain definite ratio of depth to span, and if there is no other reason for fixing on this ratio, it will be best to choose the value thus determined. The two formulæ then give the same result. In large girders a greater depth is generally desirable, then the strength formula must be used; while in small beams it may often be convenient or necessary to have a smaller depth, and then the stiffness formula must be employed.

---

* When the transverse section is not uniform the co-efficient k may be greater than unity.

169. *General Graphical Method.*—The foregoing simple examples of the determination of the deflection and slope of a beam are perhaps those of most practical use, but, by the aid of graphical processes, there is no difficulty in generalizing the results which are of considerable theoretical interest. We can, however, afford space only for a hasty sketch.

The general equations given in Art. 164 show that the angle (i) between two tangents to the deflection curve of a beam is proportional to the area of the curve of bending moments intercepted between two ordinates at the points considered. Starting from the lowest point of the deflection curve, let us now imagine a curve drawn, the ordinate of which represents that area reckoned from the starting point, then that curve will represent the slope of the beam at every point, and may therefore properly be called the "Curve of Slope." But referring again to the general equations we see that the ordinate of the deflection curve reckoned upwards from the horizontal tangent at the lowest point, is connected with the slope in the same way as the slope with the bending moment, and is consequently proportional to the area of the curve of slope. Thus it appears, on reference to Chapter III., that the curves of Deflection, Slope, and Bending Moment are related to each other in the same way as the curves of Bending Moment, Shearing Force, and Load. The five curves, in fact, form a continuous series each derived from the next succeeding by a process of graphical integration.

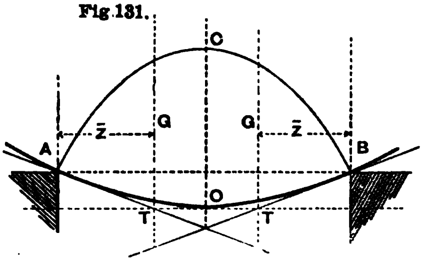
We now see that any property connecting together the second three quantities must also be true for the first three. For example, we know, from the properties of the funicular polygon, that two tangents in the curve of moments intersect in a point vertically below the centre of gravity of the area of the corresponding curve of loads (see Arts. 31, 35). It must therefore be true that two tangents to the deflection curve intersect vertically below the centre of gravity of the corresponding area of the curve of moments, a useful property, which can be proved directly without much difficulty.

The deflection curve of a beam may therefore be constructed in the same way that the funicular polygon is constructed in Art. 35, the perpendicular distance (H) of the pole from the load line in the diagram of forces being made equal to EI. To do this we have only to divide the moment curve into convenient vertical strips and regard each as representing a weight. Set down these ideal weights as a vertical line and choose a pole at a distance from the line equal to EI, measured (on account of the largeness of E) on a scale less in a given ratio. Now, construct the polygon and draw its closing line, the intercept multiplied by the scale ratio is the deflection of the beam. A parallel to the closing line in the diagram of forces gives the slopes at the extremities of the beam which correspond to the supporting forces of the loaded beam in the original case.

We have hitherto supposed the beam to be of uniform stiffness throughout; if not, let the quantity EI, which is now variable, be E_0I_0 at some datum section. Reduce the ordinates of the curve of moments in the proportion E_0I_0 to EI, then the reduced curve is to be employed in the way just described for the original curve.

170. *Examples of Graphical Method. Theorem of Three Moments.*—Let us now take some examples.

Fig. 131.



Case I.—Symmetrically loaded beam, of flexibility also symmetrical about the centre. Let ACB (Fig. 131) be the curve of moments, reduced if necessary, AOB the deflection curve; both curves, of course, will be symmetrical about the centre vertical, then from what has been said, tangents at A, B to the deflection curve intersect the tangent at O in points T vertically below the centres of gravity of the two equal areas ACO, BCO. Hence if S be the area of the whole curve of moments, $\bar{z}$ the horizontal distance of either point T from the nearer end,

$$i_0 = \frac{S}{2EI}; \delta = \bar{z} . i_0 = \frac{S . \bar{z}}{2EI}$$

must be the slope of the ends of the beam and its deflection.

Case II. Beam continuous over several, spans loaded in any way. (Fig. 132.) Let ACO', BDO' be the moment curves due to the load on two spans AO', BO' of a beam AOB, continuous over three supports A, O, B, of which the centre O is somewhat below the level of A, B. Being continuous, there will be bending moments at A, O, B, which are represented in the diagram by AE, $O'L$, BF. Joining EL, FL, the actual bending moment at each point of the beam will be represented by the intercept between the line ELF and the curves of moments due to the load and corresponding supporting forces. (See Art. 38.) The curve *AOB* is the deflection curve, *AT*, *BT* are the

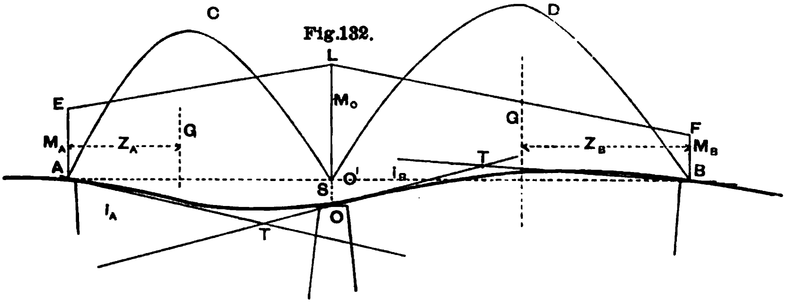


Fig.132.

tangents at *A*, *B* and *TOT* is the tangent at *O*, intersecting *AT*, *BT* in the points *T*.

Now, let i_A be the angle between the tangents at *O* and *A*, then, as before,

$$i_A = \frac{S}{EI},$$

where *S* is the area of a curve representing the actual bending moment at each point. In the present case *S* is the difference of two areas, one the noment curve for the load, the other the trapezoid *EO'* for the moments M_A, M_0.

$$\therefore S = A - \frac{M_A + M_0}{2} \cdot l_A,$$

where *A* is the area of the moment curve *ACO'* and l_A is the span *AO'*. Let the horizontal distance from *A* of the common centre of gravity of the two curves be *x*; then, as before, *x* is also the horizontal distance of *T* from *A*, and

$$y_A = \frac{Sx}{EI}, \text{ as before.}$$

To find *x*, let z_A be the horizontal distance of the centre of gravity of *ACS* from *A*, then

$$Sx = Az_A - M_A l_A \cdot \frac{l_A}{2} - \frac{M_0 - M_A}{2} \cdot l_A \cdot \tfrac{2}{3} l_A;$$
$$= Az_A - \tfrac{1}{6} M_A \cdot l_A^2 - \tfrac{1}{3} M_0 \cdot l_A^2.$$

We have thus found y_A the distance of *A* from the tangent through *O*; and y_B the corresponding distance of *B*, is written down by change of letters.

Assuming now the depression of *O*, the centre of the beam, below the level of the two other supports to be δ, it appears from the geometry of the diagram that

$$\frac{y_A - \delta}{l_A} = \frac{-y_B + \delta}{l_B};$$

or

$$\frac{y_A}{l_A} + \frac{y_B}{l_B} = +\delta\left(\frac{1}{l_A} + \frac{1}{l_B}\right);$$

hence dividing the values of y_A, y_B by l_A, l_B respectively, and adding

$$A\frac{z_A}{l_A} + B\,.\,\frac{z_B}{l_B} - \tfrac{1}{3}M_0(l_A + l_B) - \tfrac{1}{6}M_A l_A - \tfrac{1}{6}M_B l_B = \delta\left(\frac{1}{l_A} + \frac{1}{l_B}\right)EI.$$

This equation connects the bending moments at three points of support of a continuous beam, the centre support being below the end supports by the small quantity δ. It can readily be extended to the case where the flexibility of the beam is variable by reducing the moment curves as previously explained, then the moments M, which are the results of the calculation, will, in the first instance, be reduced, and can afterwards be increased to their true values.

The above equation is the most general form of the famous Theorem of Three Moments, originally discovered by Clapeyron, which is much employed in questions relating to continuous beams—a somewhat large subject, on which we have not space to enter.

171. *Resilience of a Bent Beam.*—The work done in bending a beam by a uniform bending moment M is evidently $\tfrac{1}{2}Mi$, where i is the angle which the two ends of the beam make with each other, as in Art. 163; hence by substitution for i we find for the work U,

$$U = \frac{M^2}{2EI}\,.\,l;$$

and if the bending moment vary,

$$U = \int \frac{M^2}{2EI}\,.\,dx.$$

An important case is when the beam is of uniform strength, then we have

$$p = \frac{My}{I} = \text{constant} = \frac{M_0 y_0}{I_0},$$

where the suffix 0 refers to a datum section. Then

$$U = \frac{M_0^2}{2EI_0}\int \frac{I}{I_0}\,.\,\frac{y_0^2}{y^2}\,.\,dx.$$

Assuming now the section (A), though varying, to remain of the same type,

$$\frac{I}{I_0} = \frac{Ay^2}{A_0 y_0^2}.$$

If, therefore, we call V the volume of the beam,

$$U = \frac{M_0{}^2}{2EI_0} \cdot \frac{V}{A_0} = \frac{p^2}{2E} \cdot \frac{I_0}{A_{00}y^2} V.$$

With the notation of Art. 155 this gives

$$U = \frac{p^2}{E} \cdot \frac{n}{2q^2} \cdot V$$

For the resilience we have only to change p into f, the proof strength. It thus appears that in beams of uniform strength with transverse sections of the same type the resilience is proportional to the volume, and less than that of a stretched or compressed bar, as might have been foreseen from general considerations. The ratio of reduction is $q^2 : n$, being 3 : 1 in rectangular sections, 4 : 1 in elliptic sections. When the beam is not of uniform strength the ratio of reduction must be greater for the same type of section. The reduction is of course least in I sections of uniform strength.

The function U is of great importance in the theory of continuous beams and other similar structures, the relative yielding of the several parts of the structure being always such that this function is less than it would be for any other distribution of stress and strain. It may be called the Elastic Potential, and when known all the equations necessary to determine the distribution of stress may be found by simple differentiation. (See Appendix.)

EXAMPLES.

1. If l be the length of an iron rod in feet, d its diameter in inches, just to carry its own weight with a deflection of 1 inch per 100 feet of span, show that

$$l = \sqrt{233d^2}.$$

Compare this result with that of Ex. 14, p. 295, and state what formula is to be used when both stiffness and strength are required.

2. Find the ratio of depth to span in a beam of rectangular section loaded in the middle, assuming stress $= 8{,}000$, $E = 28{,}000{,}000$, deflection $= \frac{\text{span}}{1200}$. *Ans.* $\frac{1}{17 \cdot 5}$.

3. A beam is supported at the ends and loaded at a point distant a, b from the supports with a weight W, show that the depression of the weight below the points of support is $\frac{Wa^2b^2}{3EI(a+b)}$.

4. In the last question deduce the work done in bending the beam, and verify the result by direct calculation. (See Art. 20.)

5. A dam is supported by a row of uprights which take the whole horizontal pressure of the water. The uprights may be regarded as fixed at their base at the bottom of the water, while their upper ends at the water level are retained in the vertical by suitable struts sloping at 45°, the intermediate part remaining unsupported. Find the bending moment at any point of the upright, and show that the thrust on the struts is about two sevenths the horizontal pressure of the water.

6. A timber balk 20 feet long of square section supports 160 square feet of a floor, find the dimensions that the deflection of the floor, when loaded with 60 lbs. per square foot, may not exceed $\frac{1}{2}$ inch. *Answer*, $12\frac{3}{4}''$.

7. A shaft carries a load equal to m times its weight (1) distributed uniformly, (2) concentrated in the middle. Considering it as a beam fixed at the ends, find the distance apart of bearings for a stiffness of $\frac{1}{1200^{\text{th}}}$. *Ans.* If l be the distance apart in feet, d diameter in inches, then for a wrought iron or steel shaft

$$(1)\ l = 10{\cdot}5\sqrt[3]{\frac{d^2}{m+1}}; \qquad (2)\ l = 8{\cdot}3\sqrt[3]{\frac{d^2}{m+\frac{1}{2}}}.$$

8. A beam originally curved, as in Ex. 21, p. 295, is fixed at one end and loaded in any way. If i be the change of slope at any point and X, Y the displacements parallel to axes of x, y of the point consequent on any load, prove that

$$\frac{di}{ds} = \frac{M}{EI}; \quad \frac{dX}{dy} = -i; \quad \frac{dY}{dx} = i.$$

Apply these formulæ to find the straining actions at any point of one of the rings of a chain of circular links.

CHAPTER XIV.

TENSION OR COMPRESSION COMPOUNDED WITH BENDING CRUSHING BY BENDING.

172. *General Formula for the Stress due to a Thrust or Pull in combination with a Bending Moment.*—The bars of a frame and the parts of other structures are often exposed, not only to a pull or thrust alone, or to a bending action alone, but to the two together; and the total stress at any point of a transverse section is then the sum of that due to each taken separately. That is to say, if H be the thrust, reckoned negative if a pull, M the bending moment, the stress at any point distant y from the neutral axis of the bending (see Art. 155), reckoned positive on the compressed side, must be given by

$$p = \frac{H}{A} + \frac{My}{I} = \frac{H}{A}\left\{1 + \frac{q}{n} \cdot \frac{M}{Hh}\right\},$$

the notation being as in the article cited.

This formula shows how the effect of a thrust or pull is increased by a bending action: it has many important applications, some of which we shall now briefly indicate.

173. *Strut or Tie under the Action of a Force parallel to its Axis in cases where Lateral Flexure may be neglected.*—*Case I.* Bar under the action of a force in a principal plane parallel to its axis.

Let z be the distance from the axis of the line of action of the force, then

$$M = Hz\,;\ p = \frac{H}{A}\left(1 + \frac{q}{n} \cdot \frac{z}{h}\right).$$

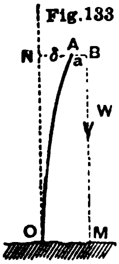
For example, let the section be circular, then $n = \frac{1}{16}$, $q = \frac{1}{2}$, and we find

$$p = \frac{H}{A}\left(1 + \frac{8z}{h}\right),$$

from whence it appears that a deviation from the axis of $\frac{1}{16}$th the diameter of a rod increases the effect of a thrust or pull 50 per cent. Similarly it can be shown that if the line of action of the force lie outside the middle fourth of the diameter of a circular section, or the middle third of a rectangular section, the maximum stress will be more than double the mean, and at certain points the stress will be reversed. In designing a structure, then, the greatest care must be exercised that the line of action of a thrust or pull lies in the axis of the piece which is subjected to it; to effect which, the joints, through which such straining actions are exerted, must be so designed that the resultant stress at the joint is applied at the centre of gravity of the section of the piece. This is a condition which cannot always be satisfied, and allowance in any case must be made for errors in workmanship. In practical construction it is the joints which require most attention, being most often the cause of failure. In frames which are incompletely braced the friction of pin joints causes the line of action of the stress to deviate from the axis. (See Ch. XVIII.)

The effect is increased in the case of a thrust and diminished in the case of a pull by the curvature of the piece, which increases or diminishes z. Fig. 133 shows the axis of a column, under the action of a weight W, suspended from a short cross piece of length a. The column bends laterally, as shown in an exaggerated way in the figure. The inclination of AB to the horizontal is so small that the difference between the actual and the projected length of AB may be disregarded; the bending moment at O is therefore $W(a+\delta)$, where δ is the lateral deviation AN of the top of the pillar. This deviation we will in the first instance suppose small compared with a, and then determine the condition that this may actually be the case. Neglecting it, the axis of the pillar is bent by the uniform bending moment Wa into a circular arc of radius R, and as in Art. 163

Fig. 133



$$\delta \, . \, 2R = l^2,$$

substituting for R its value (Art. 155) we get

$$\delta = \frac{Ml^2}{2EI} = \frac{Wal^2}{2EI};$$

whence we find

$$\frac{\delta}{a} = \frac{Wl^2}{2EI}.$$

The condition, then, that the lateral deviation should be small is that W should be much less than $2EI/l^2$, and if this condition be satisfied the stress will not be much increased beyond that indicated by the formula given above. The very important cases in which W is large will be treated presently.

In the case of a pull this restriction on the use of the formula need not be attended to, the effect of the deviation being to diminish the stress.

174. *Effect of a Thrust on a Loaded Beam.—Case II.* Uniformly loaded beam supported at the ends and subject to compression.

Let the load be W and the thrust H, then

$$p = \frac{H}{A}\left\{1 + \frac{q}{n} \cdot \frac{\frac{1}{8}Wl}{Hh}\right\}.$$

For example, let the section be rectangular, then $q = \frac{1}{2}$, $n = \frac{1}{12}$, and we find

$$p = \frac{H}{A}\left\{1 + \frac{3l}{4h} \cdot \frac{W}{H}\right\}.$$

Let us further suppose the ratio of depth to span one sixteenth then

$$p = \frac{H}{A} \cdot \left(1 + 12\frac{W}{H}\right) = \frac{W}{A}\left(12 + \frac{H}{W}\right),$$

which shows how greatly the effect of a thrust is increased by a moderate bending moment.

If the deflection be supposed 1 inch in 100 feet then H will in consequence produce an additional bending action at the centre equal to $Hl/1200$, which will be equivalent to an addition to W of $H/150$. For safety H ought not to exceed $3W$, and the stress due to the bending action of the uniform load on the beam will then be increased about 25 per cent. by the effect of the thrust. This calculation shows why it is often necessary to support a beam at points not too far apart by suitable trussing even when support is not required to give sufficient stiffness. Theoretically a proper "camber" given to the beam will counteract the bending action, and, conversely, a small accidental deflection will increase it.

175. *Remarks on the Application of the General Formula.*—The formula given above in Art. 172 is much used in questions relating to the stability of chimneys, piers, and other structures in masonry and brickwork. The stress on horizontal sections of such structures varies uniformly or nearly so, and the formula then shows where the stress is greatest and also where it becomes zero, tension usually not being permissible. It must be borne in mind however that the bending is frequently unsymmetrical, so that the axis of the bending moment will not coincide with the neutral axis of the bending stress on the section (Art. 162). The stability of blockwork and earthwork structures is a large subject which will not be considered in this treatise.

176. *Straining Actions due to Forces Normal to the Section.*—The reasoning of this section shows that when a structure is acted on by forces some or all of which have components normal to a given section, the straining actions due to the normal components will in general depend on the relative yielding of the several parts of the section (Art. 42). These normal components however can always be reduced to a single force, acting through any proposed point in the section, and a couple, and if the point be properly chosen according to the nature of the structure at the section that single force will be a simple thrust or pull; thus in the cases we have mentioned the point is the centre of gravity of the section. Having done this the couple will be so much addition to the bending action. An important example of this is the case of a vessel floating in the water in which the horizontal longitudinal component of the fluid pressure generally produces bending, the arm of the bending couple being the distance of the intersection of the line of action of the resultant with the section considered, from the neutral axis of the "equivalent girder."

177. *Maximum Crushing Load of a Pillar.*—When the compressing force is sufficiently great it produces a strong tendency to bend the pillar even though there be no lateral force. We have already seen that the condition that this shall not be the case is that W shall be small compared with the quantity $2EI/l^2$, and we now proceed to inquire the effect produced when W has a larger value. All these cases come under the head of what is called Crushing by Bending, and are very common and important in practice.

As in the case of the deflection of a beam the question is much more simple when the pillar bends into an arc of a circle, which it will do in various cases explained in Art. 163. The case which we select is that in which the sectional area remains constant and the thickness varies. Such a pillar is of uniform strength when very slightly bent, and when more bent the weakest point is at the base. As the breadth becomes great at the summit this form could not be practically applied without modification, but the conclusions derived by considering it may be applied with slight modifications to the cases which occur in practice.*

When the load is applied exactly at the centre the elevation of such a pillar is a semi-ellipse with vertex at the summit; when not exactly at the centre the ellipse is truncated. For the present purpose it is not necessary to consider this point further, as the form is not intended for practical application.

---

* The case where the thickness is uniform has been considered by Dr. Young in his *Natural Philosophy* (see Young's works, Peacock's edition, p. 139), who shows that the outline is a circular arc, as follows at once from Art. 161. The compressive stress however near the summit of the pillar is then very great.

Assuming then the form of the bent pillar to be a circular arc we have as before

$$\delta=\frac{Ml^2}{2EI},$$

but we have now, since we cannot neglect δ,

$$M=W(a+\delta).$$

Hence by substitution we find

$$\delta=\frac{W(a+\delta)l}{2EI},$$

where I is the moment of inertia at the base, from which we find

$$\delta=\frac{a}{\frac{2EI}{Wl^2}-1}.$$

This result shows that the pillar bends laterally more and more as W increases, and breaks with some value of W which we will find presently by substitution in the formula of Art. 172.

First, however, observe that if $a=0$, that is, if the line of action of the load pass through the centre of the pillar at its summit, then $\delta=0$ unless the denominator of the fraction be also zero, that is, unless

$$W=2\,.\,\frac{EI}{l^2}.$$

The interpretation of this is, that if W be less than the value just given the pillar will not bend at all, but if disturbed laterally will return to the upright position when the disturbing force is removed. If W have exactly that value then, when put over into any inclined position the pillar will remain there in a state of neutral equilibrium, while the smallest increase of W above this limit will cause the pillar to bend over indefinitely and so break. Thus the foregoing equation may be regarded as giving the crushing load of the pillar under certain conditions to be defined more exactly presently.

If the pillar had not bent into the arc of a circle as has been just supposed, we should have arrived at exactly the same formula except that the co-efficient 2 is replaced by a not very different number depending on the circumstances of the particular case. If the transverse section be uniform then the pillar bends into a curve of sines and we must replace 2 by $\pi^2/4$ or 2·47, thus obtaining

$$W=\frac{\pi^2}{4}\,.\,\frac{EI}{l^2},$$

a formula which having been first obtained by Euler is known as Euler's Formula. It applies directly to a column fixed firmly in the ground and entirely free at the upper end; it can however easily be modified to suit the cases more common in practice where the ends of the column are constrained to lie in the same vertical line. There will be three such cases shown in Figs. 134, 134*a*, 134*b*.

Fig.134.

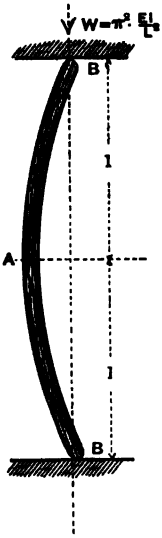


Fig.134a.

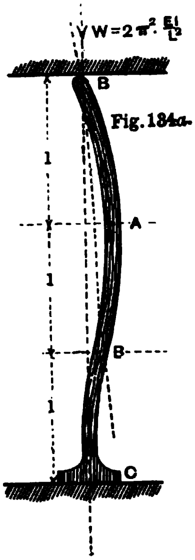
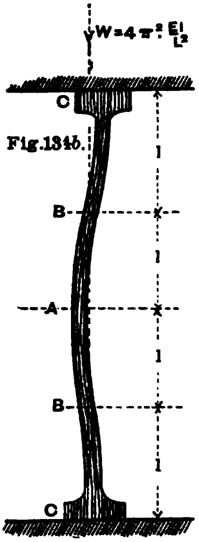


Fig.134b.



In the first the ends of the pillar are rounded and it bends laterally in the curve *BAB*; each half *AB* is then in the position of the pillar originally considered, except that the base instead of the summit is free to move laterally, hence to get the crushing load we have only to replace l by $L/2$, where L is the whole height of the pillar. In the third both ends of the column are flat, which has the effect of retaining the axis in the vertical at top and bottom, so that lateral bending takes place in the curve *CBABC*, being a curve with two points of contrary flexure or "virtual joints." Here the four pieces, *CB*, *BA*, *AB*, *BC*, are all in the same condition and must be of the same length; each is in the condition of the pillar originally considered; to get the crushing load then we have only to replace l by $L/4$, where l is the whole length. In the second the pillar bends into a curve *BABC* which has one point of contrary flexure *B*, the other being at the summit; if this point were in the same vertical as the summit then the pillar would be divided into three equal parts and we should get the crushing load by writing $L/3$ for l in the original formula. As the figure shows however, *B* must be a little out of the vertical, and this slightly diminishes the crushing load which we get approximately by writing $L/2\sqrt{2}$ for l.

We thus obtain the three formulæ,

$$W = \pi^2 . \frac{EI}{L^2}; \quad W = 2\pi^2 . \frac{EI}{L^2}; \quad W = 4\pi^2 . \frac{EI}{L^2},$$

for the three cases in question with a uniform section. If the pillar be bent into a circle as described above, then π^2 is to be replaced by 8.

178. *Manner in which a Pillar crushes. Formula for Lateral Deviation.*—The value of W here found is the maximum load which a pillar, free to deflect laterally, can sustain under any circumstances; but, in order that it may actually be sustained, the pillar must be perfectly straight, the material must be perfectly homogeneous, and the line of action of the load must be exactly in the axis. These conditions cannot be accurately satisfied, and consequently a lateral deflection is produced, which increases indefinitely as the load approaches the theoretical maximum. This may be expressed by supposing that a is not zero, but some known quantity depending on the degree of accuracy with which the conditions are satisfied, and which may be called the "effective" deviation; since, when the pillar is straight and homogeneous, it will be the actual deviation of the line of action of the load from the axis. Let W_0 be the theoretical maximum load as calculated from the preceding formulæ and W the actual load, then

$$\delta = \frac{a}{\frac{W_0}{W} - 1} = a . \frac{W}{W_0 - W} \quad \text{(p. 313)};$$

thus we see that a load of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ the theoretical maximum produces a lateral deflection of $1a$, $2a$, $3a$, increasing the deviation of the load from the axis of the column to $2a$, $3a$, $4a$. These numbers are only exact when the pillar is so formed as to bend into the arc of a circle, when this is not the case they follow a more complicated law of the same general character depending on the type of pillar and the nature of the deviation. For our purpose the simple case is sufficient. It is convenient to express the load in pounds per square inch of the area (A) of the pillar at its base, then we may write with the notation of Art. 155

$$p_0 = \frac{W_0}{A} = \pi^2 . nE . \frac{h^2}{L^2};$$

for the case where the pillar is rounded at both ends, the number π^2 being replaced by $2\pi^2$ or $4\pi^2$ in the two other cases of the last article. Similarly writing $p = W/A$ for the actual load on the pillar, we get by substitution

$$\delta = a . \frac{p}{p_0 - p}, \quad \text{or } a + \delta = a . \frac{p_0}{p_0 - p}.$$

The deviation is accompanied by an increase in the maximum stress (f) on the transverse section, which is given by the formula

$$f = \frac{H}{A}\left(1 + \frac{q}{n} \cdot \frac{M}{Hh}\right) \qquad \text{p. 309),}$$

from which we get, replacing H by W and M by $W(a + \delta)$,

$$f = p\left(1 + \frac{qa}{nh} \cdot \frac{p_0}{p_0 - p}\right),$$

a result which shows that f increases indefinitely as p approaches p_0, so that the pillar must break before the theoretical maximum is reached, however small the original deviation is. The greatest value of f must be the elastic strength, for as soon as this is past an additional lateral deviation at the most compressed part will occur, sooner or later accompanied by rupture.

The formula may be written in the more convenient form,

$$\left(\frac{f}{p} - 1\right)\left(1 - \frac{p}{p_0}\right) = \frac{qa}{nh},$$

in which it is worth while to observe that the right-hand side is unity for the deviation necessary to produce double stress when the pillar is so short that no sensible augmentation of the deviation is produced by lateral bending. In materials like cast iron which have a low tenacity, very long pillars give way by tension on the convex side: the formula then becomes

$$\left(\frac{f'}{p} - 1\right)\left(1 - \frac{p}{p_0}\right) = \frac{qa}{nh},$$

where f' is the tensile stress at the elastic limit. The two formulæ give the same result if

$$p = \frac{f - f'}{2}.$$

For loads greater than this the first formula applies, and for small loads the second. In pillars flat, but not fixed at the ends, without capitals f' may be zero.

179. *Actual Crushing Load.*—We thus see that if a pillar were absolutely straight and homogeneous it would crush, by direct compression if p_0 were greater than f, and by lateral bending if p_0 were less than f, the crushing load being the least of these two quantities; but that the smallest deviation will be augmented by lateral bending, so that the actual crushing load will be less than the least of these quantities. Experience confirms this conclusion. When a long pillar is loaded we do not find that it remains straight till a certain definite load p_0 is reached, and then suddenly bends laterally. We find, on the contrary, that a perceptible lateral deflection is produced by a small load, which gradu-

ally increases as the load is increased, till rupture takes place, showing, as we might anticipate, that some small deviation existed originally. And as that deviation evidently depends upon accidental circumstances it is impossible, from imperfection of data, to find the actual crushing load of a pillar for those proportions of height to thickness, for which its effect is greatly augmented by a small deviation. The augmentation is on the whole greatest when

$$f = p_0 = \pi^2 . n . E . \frac{h^2}{L^2};$$

that is, when

$$\frac{L}{h} = \sqrt{\frac{\pi^2 n E}{f}}.$$

This gives, by taking the values of E and f from Table II., Ch. XVIII.,

Wrought Iron, $L = 38 \sqrt{\pi^2 n} . h = 30h$ (Circular Section).
Soft Steel, $L = 29 \sqrt{\pi^2 n} . h = 23h$ „
Hard Steel, $L = 23 \sqrt{\pi^2 n} . h = 18h$ „
Cast Iron, $L = 20 \sqrt{\pi^2 n} . h = 16h$ „

In the case of cast iron there is a difficulty in determining the value of f, but if we suppose that the elasticity of the material is not greatly impaired at half the ultimate crushing load, we get the value given. The case of timber is exceptional, and will be referred to further on. For pillars fixed or half-fixed at the ends the number π^2 is to be replaced by $4\pi^2$ or $2\pi^2$ as before.

Let us assume this condition satisfied, and let us imagine the pillar loaded with three fourths the theoretical maximum crushing load, then by substitution we find, $qa/nh = \frac{1}{3} . \frac{1}{4}$, or since $n/q = \frac{1}{8}$ for a circular section,

$$\frac{a}{h} = \frac{1}{96},$$

from which it will be seen how small a deviation will cause the pillar to crush under three fourths the theoretical maximum load, when the proportion of height to thickness is that just given. With a pillar of double this height deviation has little influence, and with a pillar of one third this height lateral flexure has little influence on the resistance to crushing.

On the whole, then, it would seem that the most rational way of designing pillars would be to calculate the theoretical maximum load, and then adopt a factor of safety depending on the value of the deviation found from the above formula; it is obvious that in some cases a much larger deviation may be considered likely than in others. For the case of thin tubes see Ch. XVIII.

180. *Gordon's Formula.*—A considerable part of our experimental knowledge respecting the strength of pillars is due to Hodgkinson.* His results show that in cast-iron pillars with flat ends, the length of which exceeds 100 diameters, the theoretical maximum is closely approached, while with shorter lengths the strength falls off considerably, as might be expected. In other respects the theoretical laws are approximately fulfilled, the principal difference being that columns with one or both ends rounded are somewhat stronger relatively to columns with flat ends than theory would indicate, an effect which may be partly due to imperfect fixing of the ends. Various empirical formulæ have been given to express the results of experiment on the crushing of pillars. That which has been most used is commonly known as Gordon's. It is so constructed as to agree in form with the theoretical formulæ in the extreme cases in which those formulæ give correct results. As modified by Rankine, only replacing r^2, the square of the radius of gyration, by nh^2, in the notation of this work the formula is

$$\frac{W}{A} = \frac{f}{1 + \frac{l^2}{cnh^2}},$$

which becomes, when l/h is small,

$$W = Af,$$

and when l/h is large,

$$W = \frac{cnfAh^2}{l^2} = cf \cdot \frac{I}{l^2};$$

while for intermediate values it gives intermediate results.

If we compare this last with Euler's formula for a column with flat ends, we get

$$c = 4\pi^2 \frac{E}{f},$$

and this may be called the "theoretical" value of the constant c. The values actually used for c are somewhat different, being deduced from such experiments as have been made, and the results for different forms of section are not always consistent. Rankine gives

VALUE OF CONSTANTS.

	Value of f.	Value of c.
Wrought Iron, ...	36,000	36,000
Cast Iron, ...	80,000	6,400
Dry Timber, ...	7,200	3,000

---

* *Phil. Trans.*, 1840, Part II. An abridgment is given in Hodgkinson's work on Cast Iron, cited at the end of Chapter XVIII.

These values refer to struts fixed at the ends and to the crushing load. If one end be rounded, the value of c must be divided by 2, and if both ends are rounded, by 4. A large factor of safety must be employed, for reasons already sufficiently indicated.

Rankine's formula has been very extensively tested for the case of wrought columns of large size of various transverse sections, constructed of rivetted plates, and has been found to give good results.*

In the case of timber Hodgkinson found, from a limited number of experiments on struts of oak and red pine of small dimensions, a formula which agrees with the formula for the theoretical maximum crushing load when the value of E in that formula is taken as about 900,000 lbs. per square inch. It is possible that the low lateral tenacity of this material increases its flexibility under a heavy crushing load. The formula gives a crushing stress greater than the direct resistance to crushing of the material when L is less than $20h$, which seems hardly probable, and the lower values given by Gordon's formula appear preferable. In the case of steel the value of f may be expected to be increased and the value of c diminished in the ratio of the direct resistance to crushing of steel and wrought iron respectively.

Calculations made by Gordon's formula may be tested by calculating the deviation a by the formula on p. 316; the magnitude of this will be to some extent a measure of the safety of the proposed load. In all cases of struts of large size subject to a heavy load, special care is necessary in considering all the circumstances—if a deflection be occasioned by the unsupported weight of the strut itself, or if, as is often the case, it be constructed of rivetted plates, a large margin of safety is desirable. So also in pieces forming part of a machine in which a bending action may be produced by inertia and friction, or which are subject to shocks, the simple thrust alone is often a very imperfect measure of the stress to which they are subject.

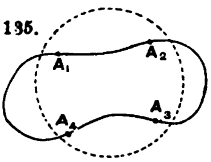
Returning to the case of a long slender column we observe that the resistance to crushing depends solely on the stiffness and not on the strength being proportional to the modulus of elasticity. Hence a long column is stronger when made of wrought iron than when made of cast iron, although with short columns the reverse is true. It appears from Gordon's formula that for a ratio of length to diameter of about 26½ the two materials are equally strong. In very long columns steel is not stronger than iron, for its modulus of elasticity is not very different; in shorter lengths however the greater resistance to direct crushing of steel gives it an advantage.

---

* "Minutes of Proceedings of the Institution of Civil Engineers," vol. liv.

181. *Collapse of Flues.* —There are other cases of crushing by bending. An important one is that of the yielding of a thin tube under *external* fluid pressure. The strength of a tube under external fluid pressure is as different from that of a tube under internal pressure as the strength of a bar under compression is different to its strength under tension.

Fig. 135.



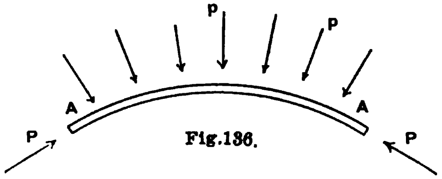


Fig. 136.

A tube perfectly uniform in thickness made of perfectly homogeneous hard material and subject to perfectly uniform normal pressure externally, would theoretically maintain its form until it yielded by the direct crushing of the material. But when the pressure exceeds a certain limit the tube is in a state of unstable equilibrium, and any deviation from perfect accuracy in the above conditions will cause the tube to yield by collapsing, the collapsing being accompanied by bulging. If the tube is very long it will collapse in the manner shown in Fig. 135, the circumference dividing itself up into four arcs two of which are concave outwards and the other two convex. A want of exactness in the construction will in practice generally prevent the collapsing from being symmetrical. Each portion of tube between the points A is under the action of forces applied at the ends towards one another, which crush it by lateral bending just as a long column is crushed. Just before collapsing, each segment AA (Fig. 136), of length s say, will be under the action of a thrust P suppose, applied at the ends tangentially. Equilibrium is maintained by fluid pressure of intensity p on the convex side. When the pressure exceeds a certain limit the equilibrium is unstable, some accidental circumstance determining the position of the point A of contrary flexure, and the consequent length s of any arc.

As shown on page 278 the thrust per inch length of the tube may be taken as approximately proportional to pd. Thus if t = thickness of tube, we may expect that the collapsing pressure would be given by a formula like that which expresses the crushing load of a long slender rod of rectangular section, namely, $pd = k't^3/s^2$ where k' is an elastic co-efficient. All other things being equal, the diameter alone varying, the length s of an arc AA would be proportional to the diameter of the tube d, and, under those circumstances, the collapsing pressure of a thin tube (see Appendix), would probably vary with t^3/d^3. But the length of the tube, as well as the diameter, influences the value of s. In all practical cases, as in all those on which experiments were made, the ends of the tube are rigidly constructed, and very much support the tube in the neighbourhood from collapsing; thus the proximity of the ends has an important effect in determining the length of the arcs into which the circumference divides itself. If the length of the tube is decreased a limit will be reached below which the tube on collapsing divides itself up into six arcs, three concave and three convex, as shown in Fig. 137. Then the length of each arc will bear a smaller proportion to the diameter than in the long tube. A still shorter tube will, when it collapses, divide it into eight arcs, and so on. Thus the length s is in some way dependent on the length of the tube. The correctness of this reasoning is borne out by experiments made by Fairbairn and others. In Fairbairn's experiments the tubes were made of rivetted wrought-iron plates. The ends were made rigid by a strong stay placed within the tube, keeping the ends apart. The tube thus constructed was placed in a larger cylinder of wrought iron and external pressure was applied by forcing water in. The pressure being gradually increased the tube will at last suddenly collapse, making a noise which indicates the instant of the occurrence. The results of the experiments showed that the collapsing pressure may be approximately expressed by the formula

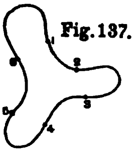


Fig. 137.

$$p = k\frac{t^2}{ld},$$

the dimensions being all in inches, the co-efficient $k = 9{,}672{,}000$. This formula must not be used for extreme cases nor for tubes of thickness less than $\frac{3}{8}$ inch.

Since a short tube is so much stronger than a long one, we have an explanation of the advantage of rivetting a T iron ring around a boiler furnace tube, which amounts to a virtual shortening of the length of the tube. Other formulæ have been proposed, some of which represent the results of experiment more closely, but the materials at present available do not admit of the construction of a satisfactory formula.*

EXAMPLES.

1. Find the thickness of metal of a cast-iron column fixed at the ends, 1 foot mean diameter, 20 feet high, to carry 100 tons. Factor of safety, 8. *Ans.*—Thickness $1''$.

2. Find the crushing load of a wrought-iron pillar $3''$ diameter, 10 feet high, free at the ends. *Ans.*—Crushing load = 66,218 lbs. = 30 tons nearly.

---

* See a paper by Professor W. C. Unwin, *Minutes of the Proceedings of the Institution of Civil Engineers*, from which the preceding remarks are partly taken. Some other cases of crushing by bending will be given in the Appendix.

3. If in last question the pillar were of rectangular section of breadth double the thickness, what sectional area would be required for equal strength? *Ans.*—Sectional area = 9·4 square inches instead of 7 square inches as before.

4. Find the collapsing pressure, according to Fairbairn's formula, of a cylindrical boiler flue $\frac{7}{16}''$ thick, 48″ diameter, and 30 feet long. *Ans.*—Collapsing pressure = 107 lbs.

5. In Ex. 1 calculate the deviation of the line of action of the load from the axis to produce a maximum stress of 10,000 lbs. per square inch. *Ans.* 1·8″.

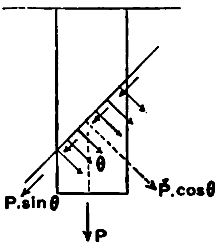
6. In Ex. 2 calculate the deviation to produce a maximum stress of 9,000 lbs. per square inch with a load of 11,000 lbs. or of 22,000 lbs. *Ans.*—1·5 or ·5.

CHAPTER XV.

SHEARING AND TORSION OF ELASTIC MATERIAL.

182. *Distinction between Tangential and Normal Stress.—Equality of Tangential Stress on Planes at Right Angles.*—In the cases we have hitherto considered of simple tension, compression, and bending, the stress on the section under consideration has been at all points normal to the section. But we may take our section inclined at any angle to the stress, and the mutual action is then not normal to the section. The particles on each side of the section partly act on one another in the direction of the section itself, and so constitute a stress analogous to friction, resisting the sliding of one portion relatively to the other. Such a stress is called *tangential* or *shearing stress*, being the stress called into action by shearing.

Fig.138.



Let us return to the case of the stretched bar carrying a load P (Fig. 138). On a transverse section of the bar only a normal stress is produced. Now suppose we take an oblique section, whose normal makes an angle θ with the axis of the bar, and let us resolve the force P into two components, one perpendicular and the other parallel to the section. The normal component $P\cos\theta$ tends to produce a direct separation at the section, producing a tensile stress similar in character to that on a transverse section, but of less intensity.

If A = area of transverse section of bar, then $A\sec\theta$ = area of oblique section; the intensity of the normal stress

$$p_n=\frac{P\cos\theta}{A\sec\theta}=\frac{P}{A}\cos^2\theta=p\cos^2\theta,\text{ where }p=\frac{P}{A}.$$

The other component $P\sin\theta$ produces a tangential or shearing stress of intensity

$$p_t=\frac{P\sin\theta}{A\sec\theta}=p\sin\theta\cos\theta.$$

Similarly if the bar is subjected to a compressive instead of a tensile load.

Many materials which offer great resistance to direct compression yield by sliding across an oblique plane. Now p_t is a maximum when $\theta = 45$, this is therefore approximately the angle of separation. The same maximum stress, the value of which is $p/2$, occurs on another plane sloping the other way at an angle of 45°. We sometimes find fracture to occur across two oblique planes; sometimes across one only.

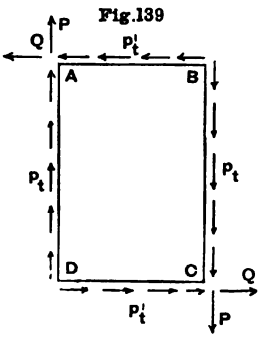
If in $p_t = p \sin\theta \cos\theta$ we change θ into $90 + \theta$, p_t has the same value; so that the intensity of the tangential stresses on two planes at right angles to one another is the same. This is true generally in all cases of stress, as will be seen presently.

183. *Tangential Stress equivalent to a Pair of Equal and Opposite Normal Stresses. Distorting Stress.*—In the example we have just considered we have both shearing and normal stress; but there are cases in which there is only a shearing stress. Let $ABCD$ (Fig. 139) be a rectangular plate of thickness t. Over the surfaces BC and AD suppose a tangential stress to be applied of intensity p_t. Calling b and a the length of the sides of the plate, the total amount of the tangential stress on each side is

$$P = p_t \,.\, bt.$$

To prevent the turning of the plate, suppose the forces P balanced by the application of an uniform stress over the surfaces BA and DC of intensity p'_t. The amount of the force on each of these sides,

Fig.139



$$Q = p'_t \,.\, a \,.\, t.$$

Since equilibrium is produced, the moment of the couple P must be equal to the moment of the couple Q.

$$\therefore p_t \,.\, bt \,.\, a = p'_t \,.\, at \,.\, b;$$

$$\text{or } p_t = p'_t:$$

that is, the intensity of the stress is the same on BA as on AD.

Shearing therefore cannot exist along one plane only: It must be accompanied by a shearing stress of equal intensity along a plane at right angles. Such a pair of stresses unaccompanied by normal stress constitute a Simple Distorting Stress, so called because it distorts the elements of the body.

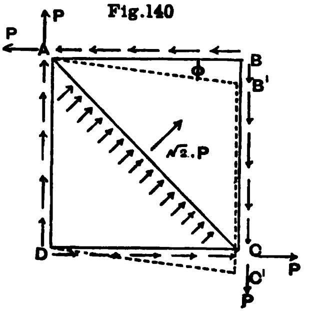


Fig.140

Let us now assume, for simplicity, the plate to be square (Fig. 140). The effect of the forces is to produce a change of form, which, in perfectly elastic bodies, is exactly proportional to the shearing force which produces it. The square *ABCD* becomes a rhombus *AB′C′D*, the angle of distortion ϕ being proportional to the stress p_t. We may write

$$p_t = C\phi,$$

where the co-officient *C* is a kind of Modulus of Elasticity, but of a different nature from that previously employed. The volume of the elastic body *A* is in general practically unaltered. Under the action of the forces it has simply undergone a change of form or figure, and the co-efficient *C* which connects the change of form with the stress producing it, is a co-efficient of elasticity of figure. It is sometimes called the *modulus of transverse elasticity*, but preferably the *co-efficient of rigidity.*

The ordinary (Young's) modulus of elasticity *E* connects the stress and strain in a bar when it undergoes changes both of volume and figure. The co-efficient of rigidity *C* for metallic bodies is generally less than $\frac{3}{8}E$, and for wrought-iron bars may be taken as 10 to $10\frac{1}{2}$ millions.

Let us now take a section of the square plate (Fig. 140) along one of the diagonals and consider the forces which act on the two sides of the triangular upper portion. Resolve these forces parallel and perpendicular to the diagonal. The components of the two *P*'s along the diagonal balance one another, and there will be no tendency for this triangular portion to slide relatively to the other; that is to say, there is no shearing stress on the diagonal section. But the other components, perpendicular to the diagonal, cause the upper triangular portion to press on the lower with a force

$$2\frac{P}{\sqrt{2}} = \sqrt{2}\,.\,P.$$

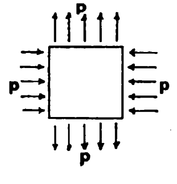
If we divide this force by the area of the diagonal section over which it is distributed, we obtain the intensity of this normal stress,

$$p_n = \frac{\sqrt{2}\,.\,P}{\sqrt{2}\,.\,at} = p_t.$$

On the diagonal section *AC* which we have been considering, this stress is compressive, but if we take the section along *BD*, the other diagonal, we find by the same reasoning a stress of the same magnitude, but tensile.

Thus it appears that a shearing stress on any plane necessarily involves tensile and compressive stresses of equal intensity on planes at 45° to this plane, so that a simple distorting stress, which was defined above as a pair of shearing stresses on planes at right angles, may also be defined as a pair of normal stresses of equal intensity and of opposite sign, as shown in Fig. 141.

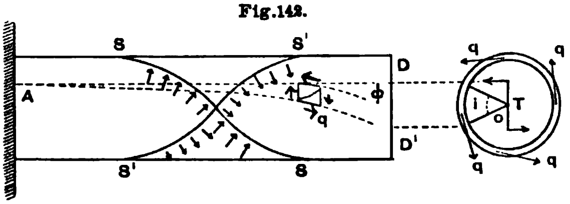
Fig. 141.



We now proceed with various examples of this kind of stress, commencing with the case of torsion. Torsion was mentioned as one of the five simple straining actions to which a bar as a whole may be exposed. It is produced by a pair of equal couples applied at the ends of the bar, the axis of the couples being the axis of the bar.

When we consider the nature of the elastic forces called into action amongst the particles of the bar, Torsion reduces to a case of Shearing. To understand this, we will begin with a simple case. Imagine a thin tube (Fig. 142) with one end fixed, and the other acted on by a uniform tangential stress of intensity q. Let t be the thickness and d the mean diameter of the tube, then

Fig. 142.



Sectional area of tube $= \pi dt$ approximately;
Total shearing force $= q\pi dt$;

and since the force on each unit of area of the section acts approximately at the same distance from the centre of the tube, the total twisting moment $= q\pi dt \times \frac{1}{2}d = \frac{1}{2}q\pi d^2t$. This twisting moment is balanced by the resistance to turning offered at the fixed end. At any transverse section KK of the tube there will be produced a uniform stress of intensity q.

Let us now consider a small square traced on the surface of the tube, with two sides on two transverse sections. If we take the square small enough we may treat it as a plane square. To balance the shearing stress q, which acts on the sides of the square lying in the transverse planes, a shearing stress of equal intensity is, as explained above, called into action on the other two sides of the square, in the direction of the length of the tube, so that, if the tube were cut by longitudinal slits, the power of resistance to torsion would be as effectually destroyed as if it were cut by transverse slits. But if we make spiral slits at an angle of 45°, as shown at *SS* in Fig. 142; supposing the slits indefinitely fine, and no material removed, the strength of the tube to resist torsion in the direction shown would not be impaired. The material of the tube would then be divided into spirally-bent ribands, which would be in tension along their length, and in compression laterally, the ribands being caused to press against one another. Along a second set of spirals such as *S′S′*, longitudinal compression and lateral tension exist; the lateral forces are indicated in both cases by arrows in the figure.

So much for the state of stress induced in the tube by the torsion. Next as to the change of form which accompanies the stress. The square will be distorted into a rhombus. A straight line *AD*, drawn on the surface parallel to the axis of the tube passing through the centre of the square, will be twisted into a spiral *AD′*, the angle of the spiral being the angle of distortion of the square. Let θ be that angle, then

$$q = C\theta, \text{ where } C \text{ is the co-efficient of rigidity.}$$

The effect of this is that, relatively to the end *A*, the end *D* is twisted round through an angle $DOD' = i$ suppose, called the angle of torsion.

In circular measure $i = \dfrac{\text{arc } DD'}{r}$ (r = radius of tube). Also arc $DD' = l\theta$, θ being a small angle. Therefore $i = l\theta/r$. Since also $\theta = q/C$, we have the angle of torsion $i = ql/Cr$, in terms of the stress. From this we may express the angle of torsion in terms of the twisting moment producing the torsion.

184. *Torsion of a Shaft.*—We now pass on to the consideration of the torsion of a solid cylindrical shaft. First, let us imagine the shaft to be made up of a number of concentric tubes exactly fitting one another, and let us further imagine that at the end of each tube a suitable twisting moment is applied, so that each tube is twisted round through exactly the same angle. This effect will be produced by applying over the section at the end of each elementary tube a tangential stress, which is proportional to the radius of the tube. If we make $q/r = q_1/r_1$, where q_1 and r_1 refer to the outside tube, then the angle of torsion will be the same for all the tubes, and they will not tend to turn relatively to one another, but all together. We may then suppose them united together again in a solid mass. If the stress applied be proportional to the distance from the centre, the shaft will twist just as if it were a set of tubes, each being subjected to the same stress and strain as if it were an independent tube.

Now in the actual case of the twisting of a solid shaft, all portions from the outside inwards to the centre must turn through the same angle, and hence the shearing stress at any point of the section of the shaft must be proportional to its distance from the centre. This is true except very near the point of application of the twisting moment. Suppose, for example, the twisting moment is applied by means of a wheel keyed on the shaft, then in the immediate neighbourhood of the key-way, the stress will not be as stated, but at a short distance along the shaft the stress distributes itself in the manner described. This is another instance of the general principle already employed in the case of stretching and bending.

The total resistance to torsion of the solid shaft is the sum of the twisting moments of all the concentric tubes into which it may be imagined to be divided. Thus

$$T = \Sigma 2\pi r^2 tq\text{ ; in which } q = r\,.\,\frac{q_1}{r_1}.$$

$$\therefore T = \Sigma_0^{r_1} 2\pi r^2 tr \frac{q_1}{r_1} = \frac{q_1}{r_1}\Sigma_0^{r_1} 2\pi rt\,.\,r^2,$$

that is, the product of the sectional area of each tube multiplied by the distance squared of the area from the axis of the shaft must be taken and summed. The result is called the Polar Moment of Inertia, which may be written I. Its value is $\frac{1}{2}\pi r_1^4$. Thus

$$T = \frac{q_1}{r_1} I = \frac{q_1}{r}\frac{\pi}{2} r_1^4 = \frac{\pi}{2} q_1 r_1^3.$$

It is not to be supposed that the strength of a shaft of any section to resist torsion is proportional to the polar moment of inertia of that section. In non-circular sections the stress is generally greatest not at the points farthest away from the centre, but more often at those which are nearest the centre. The cases of a rectangle, an ellipse and various other forms have been investigated by M. St. Venant who has obtained the annexed results.*

RELATIVE STRENGTHS OF SHAFTS OF THE SAME SECTIONAL AREA.

FORM OF SECTION.	STRENGTH.
Circular, - - - - - - - -	1
Square, - - - - - - - -	·738
Rectangle with sides in the ratio $n : 1$, -	$\sqrt{\frac{2}{n + 1/n}} \times$ ·738
Ellipse with axes in the ratio $n : 1$, - -	$\sqrt{n}$ $(n < 1)$

---

* Diagrams and particulars with respect to M. St. Venant's results will be found in Sir W. Thomson's *Treatise on Natural Philosophy*, 1st ed., vol. 1, p. 545.

Dropping the suffixes, taking r to be the outside radius, we can write the moment of resistance to torsion of the shaft,

$$T = \tfrac{1}{2}\pi f r^3, \text{ or } \tfrac{1}{16}\pi f d^3;$$

where f is the co-efficient of strength of the material to resist shearing. Thus the strength under torsion is proportional to the cube of the diameter. The formula shows that, assuming f to be the same in each case, the strength of a shaft to resist a twisting moment is double its strength to resist a bending moment. Since $i = ql/Cr$ we can eliminate q, and thus obtain

$$i = \frac{2}{\pi C} \cdot \frac{l}{r^4} \cdot T.$$

185. *Diameter of Shaft to transmit a Given Power.*—Having determined the diameter of shaft required to take a given twisting moment we are now able to obtain a solution of the practical question, What diameter of shaft is required to transmit a given horse-power at a given number of revolutions per minute?

Let T_0 = *mean* twisting moment transmitted in inch-tons, then $T_0 \times 2\pi N$ = work transmitted per minute in inch-tons, where N= revolutions per minute of shaft.

Let HP denote the horse-power to be transmitted, then

$$T_0 \times 2\pi N = \frac{33000 \times 12}{2240} H.P.$$

$$\therefore T_0 = \frac{33000 \times 12}{2240 \times 2\pi} \frac{H.P.}{N}.$$

Now the shaft must be strong enough to take not only the mean but the maximum twisting moment.

We may express the maximum in terms of the mean by writing $T = KT_0$, where K is a co-efficient whose value is different in different cases and T = maximum twisting moment, but

$$T = \frac{\pi}{16} f d^3 \text{ or } d^3 = \frac{16T}{\pi f}.$$

$$\therefore d^3 = \frac{16 \times 33000 \times 12}{2\pi^2 \times 2240} \frac{K}{f} \frac{H.P.}{N},$$

and

$$d = 5{\cdot}233 \sqrt[3]{\frac{K}{f} \frac{H.P.}{N}}.$$

The value of f depends in some measure on the fluctuation to which the twisting moment is subject, but under ordinary circumstances should not exceed 3½ tons per square inch (Art. 221) for wrought iron, or, probably, about 4 tons for steel, and 2½ tons for cast iron. The value of K, the ratio of maximum to mean twisting moment, depends on the circumstances discussed in Chapter X. We may assume it equal to 1½ under ordinary circumstances, allowing a small addition for the bending due to the weight of the shaft. On substitution we obtain for wrought iron

$$d = 4\sqrt[3]{\frac{H.P.}{N}}.$$

This formula agrees closely with the best practice in screw-propeller shafting.

When the amount of bending to which the shaft is subject is considerable, as in the case of crank shafts, the diameter determined by this formula is too small. It will be seen hereafter that when all the forces acting on the shaft are known, a value of K can be calculated which gives the effect of bending. If we assume $K = 2$, the co-efficient 4 in the above formula will be replaced by 4·5, and this agrees closely with practice in the crank shafts of marine screw engines. In other cases a still larger value may be necessary.

In the formula for the angle of torsion

$$i = \frac{ql}{Cr};$$

if we replace q by its working value for wrought iron (7,200 lbs.), C by 10,500,000 lbs., and i by the circular measure of 1°, we find

$$l = 12.7d,$$

showing that under the working stress the shaft twists through 1° for each 12¾ diameters in its length. For many purposes this is much too small, and the dimensions of a shaft then depend on stiffness, not on strength, as in the case of beams (Art. 168). The greatest angle of torsion permissible depends in great measure on the irregularity of the resistance, and no general rule can therefore be laid down for it. If the angle of torsion be given and the length, the diameter will depend on the fourth root of the twisting moment, as shown by the formula of Art. 184. In this, as in other cases where dimensions depend on stiffness, not on strength, steel has no advantage over iron, because the co-efficients of elasticity of the two materials are the same, or nearly so. A hollow shaft is both stronger and stiffer than a solid shaft of the same length and weight.

186. *Distance apart of Bearings.*—The distance apart of the bearings of a shaft depends on the stiffness necessary to resist the bending due to the weight of the shaft itself, and of any pulleys or wheels upon it, together with the tension of belts and other similar forces. If the total load be equivalent to m times the weight of the shaft itself uniformly distributed, the length between bearings for a wrought iron or steel

shaft d inches diameter will be given approximately for a stiffness of $\frac{1}{1200}$th by Ex. 7, p. 308.

When, as in screw propeller shafting, the bearings are liable to get out of line, too great stiffness in a shaft will produce great straining actions upon it.

187. *Web of a Beam of I Section.*—Torsion is one of the few cases in practice where a simple distorting stress occurs alone and not in combination with other kinds of stress. It generally happens that a normal stress is combined with it; such, for example, is the case in the web of a beam of I section, to which we next proceed to direct our attention. Taking a transverse section, the normal stress at a point distant y from the neutral axis is given by the formula

$$\frac{p}{y}=\frac{M}{I},$$

and is therefore the same for the same values of M and I, whether the web be thin or thick, while it will be shown presently that the tangential stress is greater the thinner the web, and becomes the most important element when the web is thin.

Let us suppose, for simplicity, the flanges equal, and also that the beam is supported at the ends and loaded in the centre with a weight W.

As we have previously seen, the flanges will sustain the greater portion of the bending moment, the web carrying only a small portion of it, $\frac{1}{6}$, if the area of the web equals the area of each flange. For simplicity, let us imagine the flanges to take the whole of the bending. Let K_1 and K_2 (Fig. 143) be two transverse sections of the beam at distances x_1 and x_2 from the centre of the beam, $2a$ being the span of the beam, the bending moment at the first section,

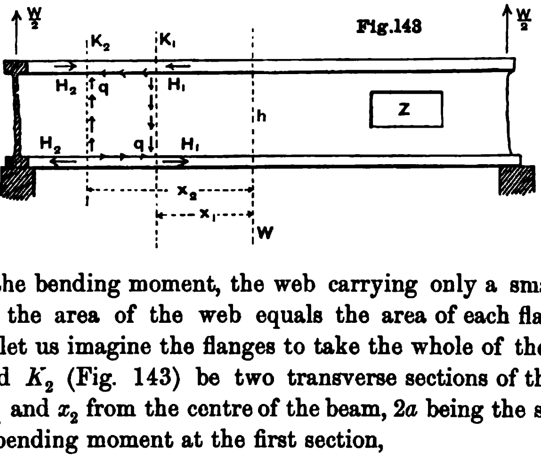


Fig.143

$$M=\tfrac{1}{2}W(a-x_1) \text{ and at the second } M_2=\tfrac{1}{2}W(a-x_2).$$

Now, supposing the flanges to take the whole of the bending the stress H produced on the flanges is given by the formula

$$Hh=M. \quad \text{Thus at } K_1 \text{ we have } H_1=\frac{W(a-x_1)}{2h},$$

$$\text{and at } K_2 \text{ we have } H_2 = \frac{W(a-x_2)}{2h},$$

and similar forces on the bottom flange only reversed in direction. There will thus be a resultant force $H_1 - H_2$ tending to push the portion K_1K_2 of the flange to the left,

$$H_1 - H_2 = \frac{W(x_2 - x_1)}{2h}.$$

This force is balanced by the resistance of the web to shearing along the line of junction with the flange.

Since $H_1 - H_2$ is proportional to the length of K_1K_2, the shearing force per unit of length of web $= W/2h$. If we suppose t to be the thickness of the web, the intensity of the shearing stress will be

$$q = \frac{W}{2ht}.$$

Thus, considering the portion of the web between the sections K_1 and K_2 apart by itself, we see that on the upper and lower horizontal edges of it, where it joins the flanges, it is subject to a shearing stress of intensity q. Now, to balance this stress there must act on the vertical sides KK a shearing stress of equal intensity q. Now, the shearing force for the vertical sections KK is $\frac{1}{2}W$. Supposing the web to be of rectangular section and of height h, then, assuming the whole of the shearing force to be borne by the web, the intensity of the shearing stress on the vertical sections is

$$q = \frac{W}{2ht}.$$

Therefore the assumption that the flanges take the whole of the bending moment is equivalent to supposing the web to take all the shearing. Assuming this, we see that the shearing stress, being uniformly distributed over the vertical section, will be accompanied by an equal shearing stress on any horizontal section. When considered alone, the effect of these shearing stresses on planes at right angles to one another is to produce tensile and compressive stresses on the web in directions making an angle of 45° with the horizontal and vertical planes; and thus the web may be superseded by an indefinite number of diagonal bars inclined at an angle of 45°, thus forming a lattice girder.

If the web is designed so as to be strong enough only to withstand the shearing stress, replacing q by f the co-efficient of strength against shearing f, we find

$$t = \frac{W}{2hf}.$$

The influence of the normal stress due to bending will be considered in a subsequent chapter. Its effect is greatly to increase the strain on the web (see Art. 202), which besides will in most cases exhibit weakness on account of the compressive stress in one of the diagonal directions. If the distance between the flanges is great, the web will be liable to yield by buckling or lateral flexure (see page 288). To prevent this, the web must be stiffened by angle irons rivetted on it. But the girder would then be made heavy, and it is therefore more economical to make large girders with openwork diagonal bracing.

We have in this investigation supposed the beam loaded in the middle, so that the shearing force is uniform throughout the length of each half, and the problem was thus simplified. But the same principles apply if the load be distributed in any manner. The shearing force will then vary from point to point along the beam.

188. *Distribution of Shearing Stress on the Section of a Beam.*—In beams of other types it is still true that the central parts of the beam are subject to shearing, but the total amount of the shearing stress being the same, its intensity is much less, because it is distributed over a greater area. The intensity at the centre of the beam is found as follows for a beam of uniform transverse section.

Suppose the beams supported at the ends and loaded in the middle as before, and take section $K_1K'_1$, $K_2K'_2$. Let NN be the neutral surface, SS the neutral axis (as in Fig. 122, Art. 153). Above the neutral surface the beam is compressed and below it it is stretched by equal forces. Let these forces be H_1 for the section K_1K_1, and H_2 for the section K_2K_2; then, reasoning as before, the shearing force in the neutral surface must be $H_1 - H_2$, and the intensity of that shearing stress, if b_0 be the breadth at the centre,

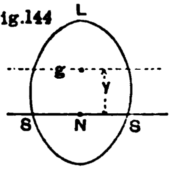
$$q = \frac{H_1 - H}{NN \,.\, b_0}.$$

Now to find H we have, in the notation of Art. 154,

$$H = \Sigma bt \,.\, p = \frac{M}{I} \,.\, kA\bar{y},$$

where kA is the area of that part of the section A which lies above the neutral axis (SLS in Fig. 144), and $\bar{y}$ is the distance of its centre of gravity (g) from that axis. The same result will be obtained if we take that part of the area which lies below the axis. We now have, as before, by substitution,

Fig.144



$$H_2 - H_1 = \frac{W(x_2 - x_1)}{2} \,.\, \frac{k \,.\, Ay}{I},$$

whence, as usual, replacing I by nAh^2, we find

$$q = \frac{W}{2n} \,.\, \frac{k \,.\, \bar{y}}{b_0 h^2}.$$

The total shearing stress on the section is $\frac{1}{2}W$, and therefore the mean intensity is

$$q_0 = \frac{W}{2A}.$$

Thus we obtain the ratio of the shearing stress on the neutral surface to the mean shearing stress on the whole transverse section.

$$\frac{q}{q_0} = \frac{A}{b_0 h} \cdot \frac{k \cdot \bar{y}}{nh}.$$

In the present case where the beam is loaded in the middle the shearing stress is the same at all points of the neutral surface, but in other methods of loading this will not be the case. The formula however in all cases gives the ratio in question correctly, which in most practical cases will be found to be greater than unity. In fact it is not difficult to see that the shearing stress must generally be greatest at the neutral surface, and must diminish to zero as we approach the external surface of the beam. The formula then gives the maximum shearing stress on the section. Even in practical cases, however, we sometimes find by a more general investigation similar to that just given that the shearing stress is not greatest at the neutral surface, but elsewhere. (See Ex. 13, page 339.)

Let us for example take a rectangular section, then

$$A = b_0 h : k = \tfrac{1}{2} : n = \tfrac{1}{12} : \bar{y} = \tfrac{1}{4}h.$$

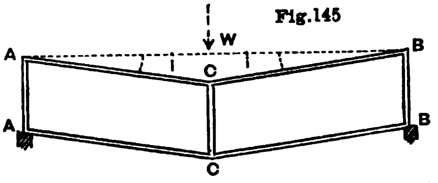
$$\therefore \frac{q}{q_0} = \frac{12}{8} = \frac{3}{2},$$

so that the greatest shearing stress is $1\frac{1}{2}$ times the mean. In like manner in a circular section it may be shown to be as 4 : 3. Other cases are given in the examples at the end of this chapter.

In all cases where a bar is subject to shearing and the sides of the bar are free from tangential stress, the stress on the transverse section will be increased in this way. In pin joints where the pin is an easy fit the only tangential stress on the sides of the pin will be due to friction and cannot be relied on.

189. *Deflection due to Shearing.*—A certain part of the deflection of a beam is due to the distortion of its central parts. Returning to the beam of I section, loaded in the middle, suppose the flanges hinged at the centres, and let vertical stiffening pieces AA, BB, CC, be rigidly connected to the web, but hinged to the flanges, then distortion of the web takes place, as shown

Fig.145



in a very exaggerated way in the figure (Fig. 145), causing a deflection δ of the beam such that

$$\frac{\delta}{\frac{1}{2}l} = i = \frac{q}{C} = \frac{W}{2htC},$$

where C as before is the co-efficient of rigidity, and q the shearing stress is expressed as before.

$$\therefore \delta = \frac{Wl}{4htC} = \frac{ql}{2C}.$$

For wrought iron take $q = 9{,}000$ for the working load, and $C = 9{,}000{,}000$, then

$$\delta = \frac{l}{2000},$$

which is above half the working deflection due to bending in ordinary cases.

This calculation, however, greatly exaggerates the deflection due to shearing even in a beam of I section, for the web cannot in general be so thin as to give a stress of 9,000 lbs. per square inch, and the effect is much less for a uniformly distributed load. Nevertheless in beams of this class the deflection due to shearing is a sensible part of the whole, the more so as in rivetted girders the union of the parts seldom renders them completely rigid. This is the principal reason why large girders show a considerably smaller modulus of elasticity when the deflection is calculated in the usual way than solid bars. In bars this part of the deflection is insensible, the distorting stress being small.

190. *Weakening of Beams by Insufficient Resistance to Longitudinal Shearing of the Web.*—If the central part of a beam be cut away as shown at *Z* in Fig. 143, the strength of the beam will be diminished and its deflection increased. This will be true even if there be only a narrow longitudinal slot at the neutral surface, but the weakening is the greater the more material is cut away, the condition of the beam in an extreme case becoming that of an *N* girder (Art. 25) without diagonal bracing. Imperfect union of the parts of the web along either a longitudinal or vertical section will have the same effect in a less degree. Wooden ships not unfrequently exhibit weakness due to this cause, and to counteract it diagonal riders of iron are introduced to take part of the shearing force. The ordinary formula for resistance to bending cannot be applied in such cases.

191. *Joints and Fastenings.*—Among the most important cases of shearing are those which occur in joints and fastenings of all kinds.

Such questions are generally very complex, considered as purely theoretical problems, and the direct results of experience are always required at every step to interpret and confirm theoretical conclusions.

When two pieces butt against each other the pressure is transmitted by contact only, and fastenings are therefore required not for transmission of stress but merely to retain the pieces in their relative positions. With tension it is otherwise; it is still necessary to have surfaces which press against one another, and these can only be obtained by the introduction of fastenings which transmit stress laterally, and are therefore subject to shearing and bending. The parts of a joint should be so proportioned as to be of equal strength. One of the simplest examples is that of a pin joint connecting two bars in tension as in a suspension chain with bar links. Fig. 1 (Plate VIII.) shows a pair of bars of rectangular section connected together by links *C* and *D* united as shown by pins passing through eyes at their extremities. In suspension chains there are generally four or five bars placed side by side, but the principle is the same in any case. The pull on the chain is balanced by the resistance to shearing of the pins, which have besides to resist bending. Let d be the diameter of the pins, b the breadth, t the thickness of one of the bars, t' the thickness, b' the breadth of the links which for equality of strength, that is to say, of sectional area, will be connected by the equation

$$2b't' = bt.$$

Let f be the co-efficient of strength for tension, then $\frac{4}{5}f$ (Art. 224) will be the co-efficient for shearing, whence remembering that the maximum shearing stress exceeds the mean in the ratio 4 : 3 as shown above,

$$P = btf = 2\pi\frac{d^2}{4} \cdot \tfrac{3}{5}f = \frac{3\pi}{10}fd^2.$$

According to this estimate the area for shearing should be five-thirds the area for tension, but the true ratio is probably not so great: the calculation supposes that the sides of the pin are subject to normal stress alone, whereas the tangential stress due to friction must be considerable. Besides the strength of iron such as is used for pins is greater than that of plates. As the calculation applies only to stress within the elastic limit, it is impossible to test it by experiment. In practice the areas are made nearly equal when nothing else is considered except resistance to shearing. When, however, such a joint is actually pulled asunder it frequently gives way in quite a different manner before shearing commences. Imagine a cylinder pressed down into a semicircular hollow which it very exactly fits, and let the material be elastic and soft compared with the cylinder, then, reasoning as in Art. 115, p. 225, it appears that the stress between the surfaces will be given by the equation

$$p = p_0 \,.\, \cos\,\theta,$$

and if P be the pressing force, l the length,

$$p_0 \,.\, \tfrac{1}{4}\pi dl = P \text{ or } p_0 = \frac{4P}{\pi dl}.$$

If the pin fits the eye exactly the pressure will follow this law so long as the tension is small. As the tension increases, however, the pressure becomes more uniformly distributed over the semi-cylinder, because the eye-hole tends to contract laterally as the links of a chain of rings would do under tension. The other extreme supposition would be to suppose it uniformly distributed, then

$$p_0 \,.\, dl = P \text{ or } p_0 = \frac{P}{dl}.$$

The actual pressure will be intermediate between these two values. If p_0 be too great the metal crushes under the pressure. The theoretical limit to p_0 will be considered hereafter (Art. 222); for the present it will be sufficient to say that the experiments of Sir C. Fox* have shown that the curved area should be at least equal to the sectional area under tension, that is to say we ought to have

$$\tfrac{1}{2}\pi dl = bt = \tfrac{1}{10}3\pi d^2.$$

To satisfy these conditions we must have for the ordinary case where the thickness of the eye is the same as that of the rest of the bar

$$d = \tfrac{5}{8}b : t = \tfrac{2}{5}b \text{ approximately.}$$

The first of these gives the diameter of pin recommended by Sir C. Fox and other authorities; the second gives the greatest thickness of link for which this diameter gives sufficient resistance to shearing, but the thickness in actual examples of suspension links is generally considerably less. The pin has also to resist bending, but of small amount in the present example. The sides and end of the eye are subject to tension, but it is not uniformly distributed, the question being similar to that of a thick hollow cylinder under internal fluid pressure. The mode in which the eye crushes and then fractures transversely by tension, is shown in Plate VIII., and further described in Chapter XVIII.

In rivetted joints the question is further complicated by the friction between the plates united by the rivets. On the subject of joints and fastenings the reader is referred to Prof. W. C. Unwin's work cited on page 121.

---

* Proceedings of the Royal Society, vol. xiv., p. 139.

EXAMPLES.

1. Find the diameter of a shaft for a twisting moment of 1,000 inch-tons; stress allowed being $3\frac{1}{2}$ tons per square inch. *Ans.* Diameter $= 11{\cdot}34''$.

2. From the result of the previous question deduce the diameter of a shaft to transmit 5,000 H.P. at 70 revolutions per minute. Maximum twisting moment $= \frac{3}{2}$ the mean. *Ans.* $16{\cdot}37''$.

3. The angle of torsion of a shaft is not to exceed 1° for each 10 feet of length. What must be the diameter for a twisting moment of 100 inch-tons—modulus of transverse elasticity, 10,500,000?

Compare the result with the diameter determined from consideration of strength, taking a co-efficient of $3\frac{1}{4}$ tons. *Ans.* Diameter determined from consideration of stiffness $= 6{\cdot}2''$. Diameter from consideration of strength $= 5{\cdot}2''$.

4. Show that the resilience of a twisted shaft is proportional to its weight.

$$\textit{Ans.}\ \text{Resilience} = \tfrac{1}{2}Ti = \frac{f^2}{C} \times \frac{\text{Volume}}{4}.$$

5. Compare the strengths of a solid wrought iron shaft and hollow steel shaft of the same external diameter, assuming the internal diameter of the hollow shaft half the external, and the co-efficient for steel $1\frac{1}{2}$ times that for iron.

6. The external diameter of a hollow shaft is double the internal. Compare its resistance to twisting with that of a solid shaft of the same weight and material.

$$\textit{Ans.}\ \text{Strength is greater in the ratio } \frac{5\sqrt{3}}{6} = 1{\cdot}443.$$

7. A pillar, whose sectional area is $1\frac{1}{2}$ square feet, is loaded with two tons. Find in lbs. per square inch the intensity of the tangential stress on a plane inclined at 15° to the axis of the pillar. *Ans.* Tangential stress $= 5{\cdot}18$ lbs.

8. In a single rivetted lap joint, the pitch of the rivets being three diameters or six times the thickness of the plates, find, 1st, the mean stress on the reduced area; 2nd, the shearing stress on the rivets; and, 3rd, the mean direct stress between rivet and plate: the tension of the joint being 4 tons per square inch of the original area, and the friction between the two surfaces of the plate in contact neglected.

Ans. Mean tension on reduced area - $= 6$ tons.
Shearing stress on rivet - - $= 7{\cdot}6$ tons.
Mean direct stress $\dfrac{4 \times \text{pitch} \times \text{thickness}}{\text{diameter} \times \text{thickness}} = 12$ tons per sq. in.

9. In a beam of I section with flanges and web which may be considered as rectangles, the thickness of each flange is one sixth the outside depth of the beam, and the breadth twice the thickness. The thickness of the web is half that of the flanges: find the ratio of maximum to mean shearing stress on the section. *Ans.* $\frac{18}{7}$.

10. In the last question find the fraction of the whole shearing force which is taken by the web. *Ans.* 80 per cent.

11. If the sectional area of the web of a flanged girder be proportional to the shearing force and the r^{th} power of the depth; find the most economical ratio of span to depth and the limiting span.

If the web be C and each flange A, as on page 288, the whole sectional area is $C + 2A = S$ and the moment of resistance to bending is

$$M = fh(\tfrac{1}{2}S - \tfrac{1}{3}C).$$

Assuming now $C = c \,.\, h^r$, where c is constant,

$$\frac{M}{fh} + \tfrac{1}{3}ch^r = \tfrac{1}{2}S,$$

and therefore, for a given value of M, S is least when

$$M = \tfrac{1}{2} \cdot \frac{r}{r+1} \cdot fSh : \quad C = \frac{3S}{2(r+1)}.$$

In a girder with lattice web the same formula for M holds good, but $S = C(r+1)$.

If now $F = f'C$, where F is the shearing force and f' is a co-efficient much less than the resistance to shearing on account of the necessary stiffening (Art. 187),

$$M = \tfrac{1}{3} \cdot r \frac{f}{f'} \cdot Fh,$$

a formula which will give the required ratio (N) for any given load. If the load be uniformly distributed

$$N = \frac{4r}{3} \cdot \frac{f}{f'}.$$

It is probable that in most cases $r = 2$ nearly, but that the value of f/f' will vary, according to the type of girder, from 2 to 4, being greatest for a continuous web.

The limiting span of a girder of uniform section is readily shown to be

$$L = \frac{4r}{r+1} \cdot \frac{\lambda}{N}. \qquad \text{(Comp. Ex. 13, p. 294.)}$$

The weight of a smaller girder of the same type is found as in Ch. IV.

On the influence of size on the strength of vessels, see papers by Mr. John and the late Mr. Froude in the *Transactions of the Institutions of Naval Architects* for 1874.

12. Show that the weight in lbs. of a shaft to transmit a given horse power at a given number of revolutions is

$$W = 21{,}000 \cdot \frac{K \cdot HP}{N\lambda} \cdot \frac{l}{d},$$

the value of λ being given as in Ch. XVIII., the proper co-efficient of resistance to shearing being used. The rest of the notation is explained on page 329.

The distance to which power can be transmitted by shafting with a given loss by friction is given by Ex. 18, p. 245, when the angle of torsion is immaterial, but in practice is generally limited by the necessity of having sufficient stiffness. The bending and twisting of shafts is considered in Chapters XVII., XVIII.

13. If a bar of square section be sheared diagonally show that the shearing stress at the neutral surface is equal to the mean shearing stress on the section. Also find where the shearing stress is a maximum and the ratio of maximum to mean. *Ans.* At a distance from the neutral surface equal to one-eighth the depth. Ratio = 1·125.

CHAPTER XVI.

IMPACT.

192. *Preliminary Remarks. General Equation of Impact.*—Hitherto the forces applied to the body or structure under consideration have been imagined to have been originally very small, and to have increased gradually to their actual amount. This is seldom exactly the case in practice, while it frequently happens that the load is applied all at once, or that it has a certain velocity at the instant it first comes in contact with the body. Such cases may all be included under the head of IMPACT, and will form the subject of the present chapter.

When a body in motion comes into contact with a second body against which it strikes, a mutual action takes place between them, which consists of a pair of equal and opposite forces, one acting on the striking body, the motion of which it changes, the other on the body struck, which it in general moves against some given resistance. Certain changes of figure and dimension, or, in other words, strains are likewise produced in both bodies, in consequence of the stress applied to them.

The simplest case is where the impact is direct and the resistance to motion has some definite value, as, for example, where a pile is driven by the action of a falling weight. Here let R be the resistance which the pile offers to be driven; that is to say, the load which, resting steadily on the pile, would just cause it to commence to sink; let W be the falling weight, h the height from which it falls, x the space through which the pile sinks in consequence of the blow; then the mutual action between the pile and the weight at the instant of impact consists of a pair of equal and opposite forces R. The whole height through which the weight falls is $h+x$, and the space through which the resistance is overcome is x; hence, equating energy exerted and work done, we have

$$W(h+x)=Rx.$$

This equation shows that any resistance, however great, can be overcome by any weight, however small; and also, that the force of the blow, as measured by the space the pile is driven, is proportional to its energy. We have however assumed that the whole energy of the blow is employed in driving the pile, whereas some of it will be expended in producing vibrations and in damaging the head of the pile and the bottom of the weight. As the pile is driven deeper, the resistance to being driven increases, and at length becomes equal to the crushing stress of the material: the pile then sinks no farther, the whole of the energy of the blow being wasted in crushing.

This last is also the case of impact of a flying shot against a soft plastic substance, which exerts during deformation a definite force uniform or variable which brings the weight to rest in a certain space. Suppose V the velocity of the shot, x the space, and R the mean resistance which the substance offers, then the kinetic energy of the shot is $WV^2/2g$, while the work done is Rx, equating which

$$W.\frac{V^2}{2g}=Rx.$$

Here the whole energy of the blow is spent in producing changes of figure in the body struck; but if the striking body had been soft, and the body which is struck hard and immoveable, the energy of the blow would have been employed in producing change in the shape of the striking body. Thus we may write down as the general equation of impact—

Energy of blow = Work done in overcoming the resistance to movement of the body struck.
+ Work done in internal changes in the striking body.
+ Work done in internal changes in the body struck.

Which of these three terms is the most important will depend on the relative magnitude of the resistance to movement, and the crushing stress of the materials of the two bodies. If either body have a sensible motion after impact, the corresponding kinetic energy must be taken account of in writing down the equation, as will be seen farther on.

193. *Augmentation of Stress by Impact in Perfectly Elastic Material.*—We now proceed to apply the equation to the case which most immediately concerns us, namely, that of impact on perfectly elastic material, including in this the effect of a load which is applied all at once.

We will suppose a structure or piece of material of any kind resting on immoveable supports, and struck by a body harder than itself, so that we may neglect all changes produced in the striking body.

Generally in both bodies there will also be produced vibrations, of the nature of those constituting sound, which absorb a certain amount of energy, but this we shall neglect. The whole energy of the blow then is supposed expended in straining the structure, or piece of material, struck by the blow.

Now the effect of impact is to produce a mutual action S, which represents a force applied to the structure at some definite point. In consequence of this the structure suffers deformation, and the point of application moves through a space x. The resistance to deformation is proportional to x, because the limit of elasticity is not exceeded; it therefore commences by being zero, and increases gradually till the velocity of the striking body is wholly destroyed. The mean value of the resistance is therefore one half its maximum value. During the first part of the period occupied by the impact the mutual action S is greater than the resistance, and during the second part less, as will be explained fully presently; but, when the maximum strain has been produced, the mean value during the whole period must be exactly equal to the mean resistance, the weight and the structure being at rest. The state of rest is only momentary, for the strained structure will immediately, in virtue of its elasticity, commence to return to its original form; but, for the moment, a strain has been produced, which is a measure of the effect of the blow, and which must not exceed the powers of endurance of the material.

Let now R be the maximum resistance, and let the blow consist in the falling of a weight W, through a height h above the point where it first comes in contact with the structure; then $h+x$ is the whole height fallen through, and it follows from what has been said that

$$W(h+x)=\tfrac{1}{2}Rx.$$

The resistance R may also be described as the "equivalent steady load," being the load which, if gradually applied at the point of impact would produce the same stress and strain which the structure actually experiences. We most conveniently compare it with W by supposing that we know the deflection δ which the structure would experience if the striking weight W were applied as a steady load at the point of impact; we then have, since the limits of elasticity are not exceeded,

$$\frac{x}{\delta}=\frac{R}{W}.$$

Substituting the value of x we get

$$\frac{R^2}{W^2}=\frac{2R}{W}+\frac{2h}{\delta}.$$

Let the height h be n times the deflection δ, then solving the quadratic, the positive root of which alone concerns us,

$$R = W(1 + \sqrt{2n+1}),$$

an equation which shows how the effect of a load is multiplied by impact.

194. *Sudden Application of a Load.*—A particular case is when $h = 0$, then $R = 2W$. So that if a load W is suddenly applied to a perfectly elastic body, from rest, not as a blow, it will produce a pressure just twice the weight. This case is so important that we will consider a special example in detail.

Let a long elastic string be secured at A. If a gradually increasing weight be applied the string will stretch, and the weight descend. Let the load required to produce any given extension be represented by the ordinates of the sloping line B_0NN_2 (Fig. 146). Next, instead of applying a gradually increasing load, let a weight W represented by B_0M_0 be applied all at once to the unstretched string. The string will of course stretch, and the weight descend. When it has reached B (Fig. 146) the tension of the string pulling upwards, being represented by BN, will be less than W acting downwards. Moreover, in the descent B_0B an amount of energy has been exerted by the weight represented by the area of the rectangle B_0M_0MB. At the same time the work which has been done in stretching the string is represented by the area of the triangle B_0NB. The excess of energy exerted over work done has been employed in giving velocity to the descending weight, and is stored as kinetic energy in the weight.

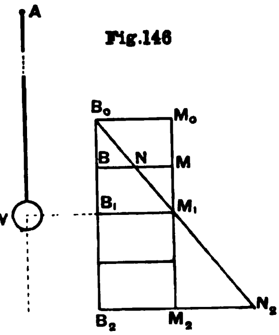


Fig.146

On reaching B_1, the tension of the string is just equal to the weight, but the stretching does not cease here. The weight has now its greatest velocity, which corresponds to an amount of kinetic energy represented by the triangle $B_0M_0M_1$. Although any further extension of the string causes the upward pull of the string to be greater than the weight W, yet the weight will go on descending until the energy that it has exerted is equal to the work done in stretching the string; then the kinetic energy will be exhausted and the weight will be brought to rest. This will occur when the area of the triangle $B_0N_2B_2$ equals the area of the rectangle $B_0M_0M_2B_2$, that is when $B_2N_2 = 2B_2M_2$, or $B_0B_2 = 2B_0B_1$.

We thus see that the tension of the string produced by the sudden application of the load is twice that due to the same load steadily applied.

The string will not remain extended so much as B_0B_2, for now the upward pull of the string, exceeding the weight, will cause it to rise. On reaching B_1 it will have the same velocity upwards that it had on first reaching B_1 downwards. This will carry it to B_0, from which it will again fall, and so on. Practically the internal friction due to imperfect elasticity, and the resistance of the air, will soon absorb the energy and bring the weight to rest at B_1.

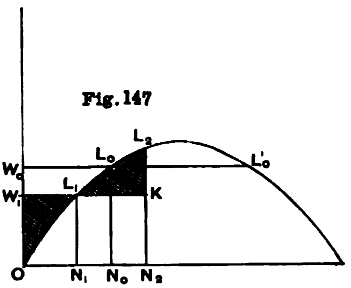


Fig. 147

195. *Action of a Gust of Wind on a Vessel.*—Another interesting example of the way in which the sudden application of a load augments its effect is furnished by the case of a vessel floating upright in the water and acted on by a sudden gust of wind, a question which, though not strictly belonging to this part of the subject involves exactly the same principle.

First, suppose no wind pressure, but that a gradually increasing couple is applied to heel the vessel.

If along a horizontal line (Fig. 147) angles of heel be marked off, such as ON, and for those points ordinates such as NL are set up to represent on some convenient scale the magnitude of the couple required to produce that angle of heel, a curve OL will be obtained, which we have already (p. 180) called the curve of *Statical Stability* of the ship.

Now suppose a steady wind pressure to be gradually applied. It will produce on the mast and sails a definite moment, on account of which the ship will incline to a certain angle, such that the ordinate of the curve of stability corresponding to that angle will represent the moment of the wind pressure. So long as the wind is constant, she will remain inclined at that angle. Next suppose the same wind pressure to be suddenly applied all at once, as by a gust to the ship floating upright at rest. The ship will heel over, and until she is inclined to some extent the wind moment will be greater than the righting moment, and the excess will cause the ship to acquire an angular velocity. Accordingly, when she arrives at the angle of heel corresponding to the moment of wind pressure on the stability curve, she does not come to rest, but inclines farther, until the energy exerted by the wind pressure is all taken up in overcoming the righting moment through the angle of inclination. The work thus done is represented by the area of the curve of stability standing above the angle of heel reached.

Let OW_1 represent the magnitude of the wind moment. The ship will incline until the area OL_2N_2 = area OW_1KN_2, or area OW_1L_1 = area L_1L_2K; that is, if the moment of wind pressure remains undiminished as the ship heels, which will hardly be true in practice. Suppose the moment of wind pressure OW_0 to be such that the area OW_0L_0 = the area $L_0L_2L'_0$. In this case the sudden gust of wind will carry the ship to such an angle ON'_0 that she will not again return; and the smallest additional pressure of wind will capsize the ship, although that same wind pressure applied gradually would incline the ship to the angle ON_0 only.

196. *Impact at High Velocities. Effect of Inertia.*—Returning to the general case of impact against a perfectly elastic structure (Art. 193), let us now take the other extreme case in which the height through which the weight falls is great compared with the deflection δ due to the same weight gradually applied; then, since n is great, our equation becomes

$$R = W \, . \, \sqrt{2n} = W\sqrt{\frac{2h}{\delta}},$$

which may be written in either of the forms

$$R = \sqrt{\frac{2W}{\delta}} \, . \, \sqrt{Wh} \ (1); \text{ or } x = \sqrt{2h\delta} \qquad (2.)$$

The first form shows that the stress produced by the impact is proportional to the square root of the energy of the blow, and the second, that the deflection occasioned by the fall of a given weight is proportional to the square root of the fall, or, what is the same thing, to the velocity of impact. These results are exact when the impact is horizontal, and the last has been verified by experiment. It is to be remembered that the limits of elasticity are supposed not to be exceeded: when a rail or carriage axle is tested by a falling weight, as is very commonly done, the energy of the blow is generally much in excess, and the piece of material suffers a great permanent set, the resistance is then approximately constant instead of increasing in proportion to the deflection. The effect of the blow is then more nearly directly proportional to its energy. It will be seen presently how small a blow matter is capable of sustaining without injury to its elasticity.

The effect of a blow, on a structure or piece of material as a whole, is diminished, on account of its inertia, by an amount which is greater the greater the velocity of impact, but which varies according to the relative mass and stiffness of its parts. In the act of yielding the parts of the body are set in motion, and the force required to do this is frequently greater than the crushing strength of the materials, so that a part of the energy of the blow is spent in local damage near the point of impact.

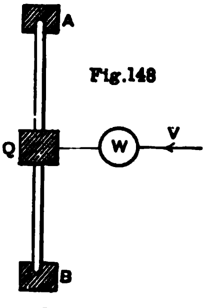


Fig.148

Figure 148 shows a narrow deep bar AB, the ends of which rest in recesses in the supports, which prevent them from moving horizontally, but do not otherwise fix them. The bar carries a weight Q in the centre, against which a second weight W moving horizontally strikes with velocity V. The bar being very flexible horizontally, the weight Q at the first instant of impact moves as it would do if free; that is, the two weights move onwards together with a common velocity v fixed by the consideration that the sum of the momenta of the two weights is the same before and after impact, so that

$$WV=(W+Q)v.$$

The energy of the two weights after impact is

$$(W+Q)\frac{v^2}{2g}=\frac{W^2}{W+Q}\cdot\frac{V^2}{2g},$$

showing that the energy of the blow has been diminished in the proportion $W : W+Q$. The loss is due to the expenditure of energy in damage to the weights.

If now, instead of a weight Q attached to the centre of a flexible bar, we suppose the bar less flexible and of weight Q, the effect of the blow is diminished by the same general cause, but not to the same extent: the diminution cannot be calculated exactly, but may be estimated by replacing Q in the preceding formula by kQ, where k is a fraction to be determined by experiment. In a series of elaborate experiments made by Hodgkinson on bars struck horizontally by a pendulum weight, it was found that k was $\frac{1}{2}$.

We are thus led to separate the energy of a blow into two parts:

$$E_1=\frac{W^2}{W+kQ}\cdot\frac{V^2}{2g}:\quad E_2=\frac{k\,.\,WQ}{W+kQ}\cdot\frac{V^2}{2g}.$$

The first of these strains the structure or piece of material as a whole, and the second does local damage at the point of impact. Hence the great difference which exists between the effect of two blows of the same energy, one of which is delivered at a low, and the other at a high velocity. At high velocities most of the energy is expended in local damage; at low velocities most is expended in straining the structure as a whole.

If the body which is struck be in motion, instead of resting on immoveable supports, as in Fig. 148, the energy of the blow will be diminished. This case has been considered in Ch. XI., p. 252, where it is shown that the energy of the collision is

$$E = \frac{WQ}{W+Q} \cdot \frac{V^2}{2g},$$

where V is the relative velocity of the bodies. Of this a part—represented, as before, by replacing Q by kQ—is spent in local damage and the rest in straining the structure as a whole.

The exceptional case where, as in the collision of billiard balls, the limit of elasticity is not exceeded at the point of impact, need not be here considered. The energy of local damage is, then, not wholly dissipated in internal changes: a part is recovered during the restitution of form which occurs in the second part of the process of impact, and increases the action on the structure as a whole. In ideal cases the whole may be thus recovered, but, in practice, a portion is always employed in producing local vibrations, and finally dissipated by internal friction.

197. *Vibrating Loads. Synchronism.*—The load on a structure may vary from time to time, continuously, or otherwise, and its effect will then, in general, be greatly augmented. Some simple examples will now be considered.

Returning to the case of the weight suspended from an elastic string (Fig. 146, p. 343); suppose in the first instance the weight at rest, then the corresponding extension (δ) is B_0B_1 in the figure and the position of the weight is B_1. Next imagine the weight raised vertically and suddenly released, it will oscillate about B_1 as a mean position. In any position B the tension of the string is represented by BN and the weight by BM, so that NM represents an unbalanced force which draws the weight downwards when it is above B_1, and upwards when below. Now NM is proportional to BB_1, and the weight therefore moves under a force always proportional to its distance from B_1.

This kind of motion is known as a "simple harmonic motion"; we have already had an example in the case of a piston moving in its cylinder; for in Ch. IX., p. 212, it was shown that the force necessary to move the piston varies as the distance from the centre of the stroke. In fact Fig. 99, p. 212, may be taken to represent the motion, the velocity of the weight in any position being represented by QN. From the formulæ given in the article cited it is easy to show that the time of a double vibration of the weight is given by

$$t = 2\pi\sqrt{\frac{\delta}{g}},$$

being the same as that of the small oscillations of a pendulum of length B_0B_1. It is dependent only on the elasticity of the string and the magnitude of the weight, not on the extent of the vibration.

The vibrations of any structure may be distinguished into general and local, that is into the vibrations of the structure as a whole, and the vibrations of its parts. All such vibratory motions are of the same general character, as in the simple case just described they take place in certain definite times depending on the inertia and elasticity of the structure and its parts, but not depending on the extent of the vibration.

Next suppose the weight (Fig. 146) oscillating about B_1, and let B be the extreme upward position. At the instant when the weight is at B imagine a small downward force P applied; the effect of this will be that the weight descends to a position B_2 before coming to rest, such that $B_1B_2 > BB_1$, instead of being equal to BB_1, as would otherwise be the case. Then suppose P removed, the weight will rise to a point as much above B as B_1B_2 is greater than BB_1. Again suppose P applied, then the weight will descend below B_2, and this process may be continued indefinitely. Thus it appears that a load P, however small, if applied and removed at intervals, *corresponding to the natural period of vibration of the weight* W, will produce a vibration of continually increasing extent, thus augmenting indefinitely the tension of the string, which will soon break, however small the original load W and its fluctuation P. If the weight P be applied as before at B, but removed and replaced at a different interval, the vibration will still augment, in the first instance, but the augmentation will be limited, and will be succeeded by a diminution, and so on indefinitely.

In the foregoing simple example numerical results could readily be obtained if necessary; in actual structures and machines the circumstances are much more complex, and calculations are therefore generally difficult, but the same general principles hold good. Whenever the load on a structure fluctuates the stress due to it is greater than that which corresponds to the maximum load: and the augmentation is greater the more nearly the period of fluctuation approaches the period of vibration of the whole structure, or of that part of it immediately affected by the load. Vibrations of the same period are often described as "synchronous."

As examples of a fluctuating load may be mentioned—

(1) When a company of soldiers march in regular time over a suspension bridge vibrations of the flexible structure are set up which are constantly augmented by synchronism. On a girder bridge the augmentation would be comparatively small, the period of vibration of the bridge being generally very different.

(2) In certain torpedo boats the vibration due to the action of the screw is excessive at one particular speed. This is an effect of synchronism between the revolutions of the screw and the period of bending vibrations of the boat in a horizontal plane.

(3) When a ship rolls broadside on to a series of equal waves the rolling is increased by the action of the waves, and is greatest when the period of the waves is equal to the period of rolling of the ship in still water.

One case of a fluctuating load can be completely worked out without much difficulty, and the result has been applied to various purposes. This is where the load fluctuates according to the harmonic law already considered for an elastic string. The calculation cannot be given here, but it may be mentioned that it is in this way that the late Mr. Froude arrived at his well known conclusions respecting the rolling of ships amongst waves.*

198. *Impact when the Limits of Elasticity are not Exceeded. Resilience.*—The effect of impact on perfectly elastic material may also be dealt with by considering the amount of energy stored up in the body in consequence of the deformation which each of its elementary parts have suffered. We have already seen that when a piece of material is subjected to a simple uniform longitudinal stress of intensity p, the amount of work U done by the stress is

$$U = \frac{p^2}{2E} \times \text{Volume.}$$

Let w be the weight of a unit of volume of the material, and W the weight of the body considered, then we may write our question

$$U = W.H$$

where H is a certain height given by

$$H = \frac{p^2}{2Ew},$$

and the whole elastic energy of the body may be measured by this height, which is the distance through which the body must fall to do an equivalent amount of work.

If for p we write f the elastic strength of the material, then we obtain what we have already called the Resilience of the body, and H becomes what we may call the "height due to the resilience," which, for each material, has a certain definite value, given in feet in Table II., Ch. XVIII., for various common materials.

Now in cases of impact where the limit of elasticity is not exceeded, the whole energy of the blow is spent in straining the material or structure, and hence that energy must not, in any case, exceed the resilience. Thus, on reference to the table, it will be seen that in ordinary wrought iron the height is given as 2 ft. 2 in., from whence it follows that in the most favourable case a piece of iron will not stand a blow of energy greater than that of its own weight falling through twenty-six inches, without being strained beyond the elastic limit. If the parts of the body are subject to torsion, about 50 per cent. may be added to these numbers, but, on the other hand, they are subject to large deductions on account of the inequality of distribution of stress within the body. Only a portion of the body is subjected to the maximum stress, the rest is strained to a less degree, and consequently has absorbed a less amount of the energy of the blow. Thus, for example, a beam or circular section, even though it be of "uniform strength" (Art. 161), has only one fourth the resilience of a stretched bar of the same weight, because it is only the particles on the upper and lower surfaces which are exposed to maximum stress, the central parts having their strength only partially developed.

---

* *Transactions of the Institution of Naval Architects*, vol. ii.

We now draw two very general and important conclusions.

(1) When a body or structure is exposed to a blow exceeding that represented by its own weight falling through a very moderate height, a part, or the whole, is strained beyond the elastic limit.

(2) When a body or structure is not of uniform strength throughout the excess of material is a cause of weakness.

On reference to Table II., Ch. XVIII., it will be seen that exceptions occur to the first principle in the case of the hardest and strongest steel, and in wood and some other substances of organic origin of low specific gravity; but, as a rule, the property of ductility or plasticity is essential to resistance to impact. Bodies which do not possess it are generally brittle. In good ductile iron and soft steel the non-elastic part of the resistance to impact will be seen hereafter to be at least 1,000 times the elastic part, assuming both equally developed through all parts of the material. These remarks apply to a single blow; the effect of repetition will be considered hereafter.

As an example of the application of the second principle we may mention the bolts for armour plates invented by the late Sir W. Palliser. In these bolts the shank is turned down to the diameter of the base of the thread so as to be of equal strength throughout. (See Ex. 4, p. 280.)

EXAMPLES.

1. A hammer weighing 2 lbs. strikes a nail with a velocity of 15 feet per 1 inch drivin $\frac{1}{8}$ inch, what is the mean pressure overcome by the nail? *Ans.* 673 lbs.

2. If the load on a stretched bar is suddenly reversed so as to produce compression, show that the stress will be trebled.*

Energy stored in stretched bar will on the release of the load be employed in compression, and in addition the load will be exerted through a distance = original extension + compression. The two together must be equal to the work done in compressing the bar.

Note—Such sudden reversal as is here supposed rarely if ever occurs in practice.

3. A load of 1000 lbs. falls through 1" before commencing to stretch a suspending rod by which it is carried. If the sectional area of the rod is 2 sq. in., length 100", and modulus of elasticity 30,000,000, find the stress produced.

Stress = 17,828 lbs. per sq. in.

4. A load of 5000 lbs. is carried by the rod of the preceding question, and an additional load of 2000 lbs. is suddenly applied; what is the stress produced?

Stress = 4500 lbs. per sq. in.

5. A beam will carry safely 1 ton with a deflection of 1 inch; from what height may a weight of 100 lbs. drop without injuring it, neglecting the effect of inertia? *Ans.* 10·2 inches.

6. The maximum stability of a vessel is 4000 foot-tons. The curve of stability is represented sufficiently approximately by a triangle, such that the angle of maximum stability is $1/n$ the angle of vanishing stability. Find the uniform moment which, applied suddenly to the ship upright and at rest, would just capsize her.

Ans. Capsizing moment $= \dfrac{4000\sqrt{n}}{1+\sqrt{n}}$.

7. A crane is observed to deflect through 1 inch when a load of 1 ton is suspended from it. A load of 2 tons is lowered at the rate of 2 f.s. and then suddenly stopped; in what ratio is the stress on the parts of the crane augmented? *Ans.* 87 per cent.

---

* This result which appears little known was pointed out to the writer by Mr. Hearson. Some examples on impact will be found in Prof. Alexander's treatise on Applied Mechanics, part I.

CHAPTER XVII.

STRESS, STRAIN, AND ELASTICITY.

SECTION I.—STRESS.

199. *Ellipse of Stress.*—Stress consists, as we have said (Art. 147), in a mutual action between two parts, into which we imagine a body divided by an ideal section. If the section be plane, and if the stress be uniform, the intensity and direction of the stress at each point of the section are the same at all points of a given section, and, for a given point, depend only on the position of the plane. In a fluid the intensity is the same for all planes, and the direction is normal to the plane. In simple tension and compression the direction of the stress is the same for all planes, but its intensity varies, becoming zero for planes parallel to the stress. In shearing the intensity is the same for all planes perpendicular to a third given plane, but the direction varies: on one pair of planes it is normal, on another tangential.

We now proceed to consider stress more generally, and we shall first examine the effect of combining together a pair of simple longitudinal stresses, the directions of which are at right angles and the intensities of which are given. Let the plane of the paper be parallel to the directions of the stresses, and let us consider a piece of material of thickness unity. If the stress be uniform, the size and shape of the piece are immaterial. Let us then imagine a rectangular block *ABCD* (Fig. 149) with sides perpendicular to the stresses p_1, p_2. On the faces *AB*, *CD* a stress, of intensity p_1, and of total amount $p_1 . AB$ will act; while on *BC* and *AD* there will be a stress of intensity p_2, and of total amount $p_2 . BC$. Divide now the rectangle by a diagonal plane *AC*; there will be a stress on that plane, which it is our object to determine in direction and magnitude. Let θ be the angle which the normal to the plane makes with the direction of p_1; by determining rightly the ratio of the sides of the rectangle this angle may be made what we please. Proceeding as in Art. 182, we find for the normal stress

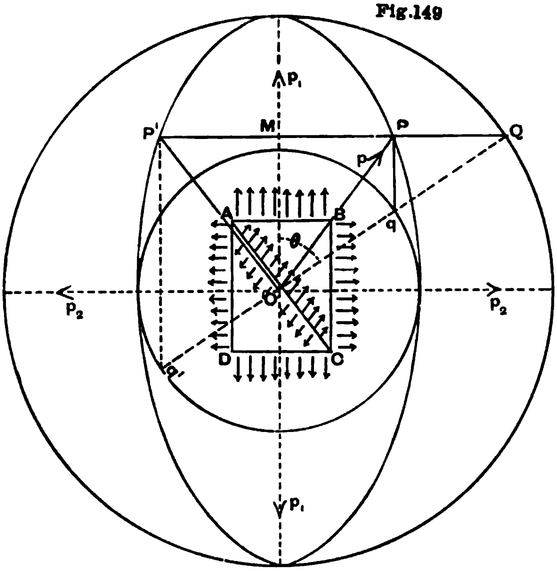
$$p_n = p_1 . \cos^2\theta + p_2 . \sin^2\theta,$$

and for the tangential stress

$$p_t = (p_1 - p_2) \sin \theta . \cos \theta .$$

The resultant stress might be found in direction and magnitude by compounding these results, but it is better to proceed by a graphical construction. On the perpendicular set off OQ to represent p_1 and Oq to represent p_2; also draw the ordinate QM and qP parallel to p_1 to meet it in P. Then

Fig. 149



$$OM = OQ . \cos \theta = p_1 . \frac{AB}{AC};$$

$$PM = Oq . \sin \theta = p_2 . \frac{BC}{AC}.$$

Whence it follows that the intensity of the stress on AC due to p_1 is represented by OM, and that due to p_2 by PM. If then we join OP we shall obtain the resultant stress on AC in direction and magnitude. It is easily seen that P lies on an ellipse of which p_1, p_2 are the semi-axes. This ellipse is called the Ellipse of Stress.

If the pair of stresses p_1, p_2 have opposite signs, then $Oq' = p_2$ must be set off on the opposite side of O, and OP' the radius vector of the ellipse lies on the other side of OM, but in other respects the construction is unaltered. When p_1, p_2 are equal the ellipse becomes a circle; if they have the same sign the stress is the same in all directions in magnitude and direction like fluid pressure; if they have opposite signs, as in the chapter on Torsion, the intensity is the same, but the angle of inclination $P'OQ$, called the "obliquity" of stress, is variable, being always twice QOM.

200. *Principal Stresses. Axes of Stress.*—We now propose to show that any state of stress in two dimensions (Art. 203) may always be reduced to a pair of simple stresses such as we have just considered.

For, drawing the same figure as in the last article, let us inquire the effect of replacing p_1, p_2 by other stresses of any magnitude and in any directions. Whatever they be, they evidently must have given tangential and normal components, of which, reasoning as in a former chapter, we know that the tangential must be equal and of opposite tendency.

Let the equal tangential components be p_t, and the normal components p_n and p'_n. Consider the equilibrium of the triangular portion ABC (Fig. 150), and let us determine under what conditions it is possible that the stress on AC should be a normal stress only, without any tangential component. Resolve parallel to BC; then, if p be that normal stress,

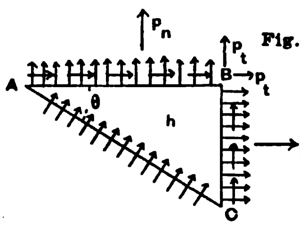


Fig. 150

$$p \,.\, AC \,.\, \cos\theta = p_t \,.\, BC + p_n \,.\, AB\,;$$

or
$$p - p_n = p_t \,.\, \tan\theta.$$

Similarly resolving parallel to AB,

$$p - p'_n = p_t \,.\, \cot\theta,$$

whence, subtracting one equation from the other,

$$p_n - p'_n = p_t \,.\, (\cot\theta - \tan\theta) = 2p_t \,.\, \cot 2\theta\,;$$

or
$$\tan 2\theta = \frac{2p_t}{p_n - p'_n}.$$

This equation always gives two values of θ at right angles, showing that two planes at right angles can always be found on which the stress is wholly normal. The magnitude of the stress on these planes is found by multiplying the equations together, when we get the quadratic

$$(p - p_n)(p - p'_n) = p_t^2,$$

the roots of which, p_1, p_2, are the stresses required. Having determined p_1, p_2, the ellipse of stress can now be constructed by the method of the last article.

Every state of stress in two dimensions then can always be represented by an ellipse, the semi-axes of which are called Principal Stresses and their directions the Axes of Stress.

The particular case in which p'_n is zero is one of constant occurrence in practical applications. If q be the shearing stress, the equations may then be written

$$p_n \tan 2\theta = 2q \quad (1)\,; \quad p(p - p_n) = q^2 \qquad (2).$$

Of the roots of the quadratic the greater has the same sign as that of p_n, and the other the opposite. Also, we find by dividing the two equations for p by one another,

$$\tan^2\theta = \frac{p - p_n}{p} = \frac{q^2}{p^2},$$

from which it appears that of the two values of θ furnished by (1) the one less than 45° must correspond to the greater value of p. Hence, the major principal stress is of the same kind as p_n, and inclined to it at an angle less than 45°.

201. *Varying Stress. Lines of Stress. Bending and Twisting of a Shaft.*—In proving the two very important propositions just given, we have assumed (1) that the stress was uniform, throughout the region including the portion of matter we have been considering; (2) that gravity or any other force acting not on the bounding surface, but on each particle of the interior, may be neglected. It is however to be observed that by taking the portion of matter small enough, both these suppositions may be made, in general, as nearly true as we please: the first, because any change of stress must be continuous, and therefore becomes smaller the less the distance between the points we consider; the second, because any internal force is proportional to the volume, while any force on the boundary of a piece of material is proportional to the surface of the piece. Now the volume of a body varies as the cube, and the surface as the square of its linear dimensions, and it follows that the internal force vanishes in comparison with the stress on the boundary when the dimensions diminish indefinitely. Hence these propositions are still true as respects the state of stress at any given point of a body, even though the stress be variable, and notwithstanding the action of gravity. When however we consider the variation of stress from point to point, gravity must be considered. Thus, for example, in the case of a fluid the action of gravity does not prevent the pressure from being the same in all directions, but it does cause the pressure to vary from point to point.

When the stress varies from point to point, both the intensity and the direction may vary; thus, for example, in a twisted shaft the intensity of the stress at any point varies as the distance from the axis, and the direction of the stress varies according to the position of the point, the principal stresses making an angle of 45° with the axis of the cylinder. The axes of stress in this case always touch certain lines which give, at each point they pass through, the direction of the stress at that point. These lines are called Lines of Stress; in a simple distorting stress, or, in other cases where the principal stresses are of opposite signs, one is a Line of Thrust, the other a Line of Tension.

In a twisted shaft of elastic material the lines of stress are spirals traced on a cylinder passing through the point considered, the spirals being inclined at 45° to the axis. If the shaft be bent as well as twisted, the maximum normal stress at any point of the transverse section is given by the equation

$$p_n = \frac{M}{\frac{1}{4}\pi r^3} \qquad \text{(Art. 155)},$$

where M is the bending moment and r the radius. The shearing stress at the external surface due to a twisting moment T is given by

$$q = \frac{T}{\frac{1}{2}\pi r^3} \qquad \text{(Art. 184)}.$$

Combining these two together we get, by solving the quadratic for the principal stresses,

$$p = \frac{M \pm \sqrt{M^2 + T^2}}{\frac{1}{2}\pi r^3},$$

which gives the principal stresses at that point of the shaft where the stress is greatest. The maximum stress is the same as would be given by a simple twisting moment equal to $M + \sqrt{M^2 + T^2}$, which is sometimes called the simple equivalent twisting moment. The minor principal stress ought, however, also to be considered in calculations respecting strength, as will be seen hereafter.

The lines of stress here are spirals of variable pitch angle.

202. *Straining Actions on the Web of an I Beam.*—Let us now return to the case of an I beam with a thin web, in which the web resists nearly the whole of the shearing force F, and the flanges nearly the whole of the bending moment M. The intensity of the shearing stress q is approximately

$$q = \frac{F}{ht},$$

where h is the depth and t the thickness. The intensity of the normal stress at a point distant y from the neutral axis is

$$p_n = \frac{M}{I} \cdot y.$$

The principal stresses and axes of stress are given by the equations

$$p(p - p_n) = q^2; \ \tan 2\theta = \frac{2q}{p_n}.$$

From this it appears that, even when the web is very thin so that it carries a very small fraction of the total bending moment. it cannot be treated as resisting shearing alone, and if it is so treated will be the most severely strained part of the beam. Let us, for example, suppose the flanges to be subject to a stress of 4 tons per sq. inch at a given section, and the web to a shearing stress also of 4 tons per sq. inch: then at points in the web near the flanges, say, for example, at a distance from the centre, of three fourths the half depth of the beam, the normal stress will be 3 tons per sq. inch. Putting these values in the formula, we get the quadratic equation

$$p(p-3)=16;$$

whence

$$p=5{\cdot}77, \text{ or } -2{\cdot}77,$$

a result which shows that the web is much more severely strained than the flanges. The lines of stress are found from the equation for θ. By a graphical method it is possible to draw the lines of stress approximately. As to this the reader is referred to a treatise by Mr. Chalmers, cited on page 73.

203. *Remarks on Stress in General.*—We have hitherto been considering only the stress on planes at right angles to a certain primary plane, to which we have supposed the stress on every plane to be parallel. In most practical questions relating to strength of materials this is sufficient, since, though stress frequently exists on the primary plane, it is usually normal and of relatively small intensity. Thus, for example, in a steam boiler there is stress on the internal and external surface of the boiler due to the pressure of the steam and the atmosphere; but it is of small amount compared to the stress on planes perpendicular to the surface. We therefore content ourselves with a statement without demonstration of corresponding propositions in three dimensions.

(1) Any state of stress at a point within a solid may always be reduced to three simple stresses on planes at right angles.

(2) The resultant stress on any plane due to the action of three simple stresses at right angles to each other is always represented in direction and magnitude by the radius vector of an ellipsoid.

The first of these propositions may be regarded as the last step in a process of analysis, by which we reduce all external forces acting on a structure of any kind: *first*, into a set of forces acting on each piece of the structure; and *second*, into forces acting on each of the small elements of which we may imagine that piece composed; and *lastly*, into three forces at right angles acting upon the element, of which one in practical cases is usually small. All questions in Strength of Materials, then, ultimately resolve themselves into a consideration of the effects of forces so applied.

One method of conceiving the effect of three such forces is to imagine each separated into two parts, one of which is the same for all, being the mean value of the three; while the other is compressive for one and tensile for the two others, or *vice versa*. In isotropic matter (Art. 207) the first set produces change of volume only, and may be called the "volume-stress," or, as no other stress can exist in fluid bodies at rest, a "fluid" stress. The second is a distorting stress, consisting of three simple distorting stresses tending to produce distortion in the three principal planes.

EXAMPLES.

1. A tube, 12 inches mean diameter and ½ inch thick, is acted on by a thrust of 20 tons and a twisting moment of 25 foot-tons. Find the principal stresses and lines of stress.

Taking a small rectangular piece with one side in the transverse section, we find one face acted on by a normal stress of 1·06 tons per square inch due to the thrust, and a tangential stress of 2·66 tons due to the twisting. Substituting these values for p_n, p_t, and observing that the stress on the other face is wholly tangential, we find from the quadratic

Major principal stress = 3·24 (thrust);
Minor principal stress = 2·18 (tension).

Lines of stress are spirals, the lines of tension inclined at 50½° to the axis, and the lines of thrust at 39½°.

2. A rivet is under the action of a shearing stress of 4 tons per square inch, and a tensile stress, due to the contraction of the rivet in its hole, of 3 tons per square inch. Find the principal stresses.

Ans. Major principal stress = 5·8 tons (tension).
Minor principal stress = 2·77 tons (thrust).

3. The thrust of a screw is 20 tons; the shaft is subject to a twisting moment of 100 foot-tons, and, in addition, to a bending moment of 25 foot-tons, due to the weight of the shaft and its inertia when the vessel pitches. Find the maximum stress and compare it with what it would have been if the twisting moment had acted alone. Shaft 14 inches diameter.

Ans. Major principal stress = 2·9, Ratio = 1·32.
Minor principal stress = 1·6.

4. A half-inch bolt, of dimensions given in Ex. 6, page 245, is screwed up to a tension of 1 ton per square inch of the gross sectional area. Assuming a co-efficient of friction of ·16, find the true maximum stress on the bolt while being screwed up. *Ans.* Principal stresses = 1·8 and ·2 tons.

5. It has been proposed to construct cylindrical boilers with seams placed diagonally instead of longitudinally and transversely. What is the object of this arrangement, and what is the theoretical gain of strength? *Ans.* Increase of strength = 26½ per cent.

6. A thick hollow cylinder is under the action of tangential stress, applied uniformly all over its internal surface in directions perpendicular to its axis, the cylinder being prevented from turning by a similar stress, applied at the external surface. Find the principal stresses and lines of stress. *Ans.* The principal stresses are equal and opposite, forming a simple distorting stress, of intensity varying inversely as the square of the distance from the centre. Lines of stress equiangular spirals of angle 45°.

7. In Ex. 9, page 338, suppose the beam so loaded that the maximum stress due to bending is 4 tons per square inch, and the total shearing force divided by the sectional area of the web 3 tons per square inch: find the principal stresses at points immediately below the flanges. *Ans.* Principal stresses 4½ and 1·9 tons per square inch.

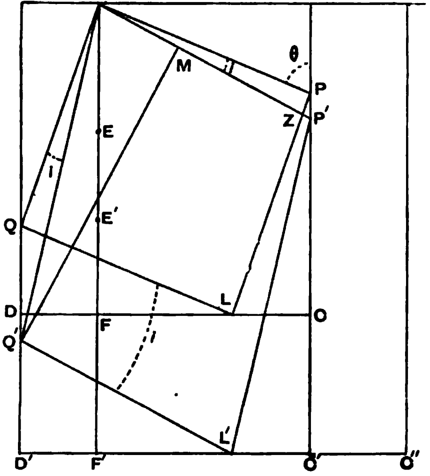
8. In any state of stress at a point in a body show that the sum of the normal stresses on three planes at right angles is the same however the planes be drawn.

SECTION II.—STRAIN.

204. *Simple Longitudinal Strain. Two Strains at Right Angles.*—We now go on to consider the changes of form and size which are produced by the action of stress. Such changes, it has already been said, are called Strains.

In uniform strain every set of particles lying in a straight line must still lie in a straight line, and two lines originally parallel must still be parallel. The lengths of all parallel lines are altered in a given ratio $1 + e : 1$, where e is a quantity, in practical cases very small, which measures the strain in the direction of the line considered. Two sets of parallel lines, however, will not in general remain at the same inclination to each other, nor will their lengths alter in the same ratio. Thus the sides of a cube remain plane, and opposite sides are parallel, but the parallelopiped is not generally rectangular, and its sides are not equal.

Fig.151



The simplest kind of strain is a simple longitudinal strain in which all lines parallel to a fixed plane in the body are unaltered in length, while all lines perpendicular to that plane remain so: that is to say, a simple change of length, the breadth, and thickness remaining unaltered.

Fig. 151 shows an extensible band *OBCD*, in which *OB* is fixed, while *CD* moves to *C′D′*, the breadth being in the first instance unaltered, and the length altered so that

$$CC' = e_1 . BC.$$

If any line *AEF* be traced in the band parallel to *BC*, the points *EF* will shift to *E′F′* positions in the same line, such that

$$EE' = e_1 . AE : FF' = e_1 . AF.$$

$$E'F = (1 + e_1)EF ;$$

for since the strain is uniform the change of length of all parts of the band is the same. If, however, we draw a line QL inclined at an angle θ to BC, that line will shift to $Q'L'$, a position such that QL has not increased in so great a ratio, and is not inclined to BC at the same angle as before. We are about to determine the actual change of length and angular position of QL by finding that of a parallel AP drawn through A. It has been already remarked that parallel lines in uniform strain must suffer the same strain. Now AP shifts to AP' such that

$$PP' = e_1 . BP = e_1 . AP . \cos\theta.$$

If now the angle PAP' $(=i)$ be so small that i^2 may be neglected compared with i, and i compared with unity,

$$AZ = AP : P'Z = PP' . \cos\theta;$$

and therefore

$$AP' - AP = PP' . \cos\theta = e_1 . AP . \cos^2\theta.$$

Thus the strain (e) in the direction of AP is

$$e = e_1 . \cos^2\theta.$$

Also, it is clear that

$$i = \frac{PZ}{AP} = \frac{PP'}{AP} . \sin\theta = e_1 . \sin\theta . \cos\theta.$$

By these formulæ the changes of length and angular position of all lines in the band are determined.

Next draw a line AQ perpendicular and equal to AP, and let AQ' be the position into which it moves in consequence of the strain; we find for e', the extension of AQ,

$$e' = e_1 . \sin^2\theta;$$

while the angle QAQ' is

$$i' = e_1 . \sin\theta . \cos\theta = i.$$

Imagine now the square AQL completed; this square, in consequence of the strain, will have its sides altered in length by the quantities e, e', and will have suffered a distortion given by

$$2i = 2e_1 . \sin\theta . \cos\theta.$$

In this way the effect of a simple logitudinal strain is completely determined, for we can calculate the changes taking place in any portion of the band we please.

Next suppose the band to suffer a second simple longitudinal strain e_2 in the direction of the breadth, and observe that since the strains are very small, the effect of e_1, e_2 taken together must be the sum of those due to each taken separately; then we find for the change of length and position of any line AP,

$$e = e_1 . \cos^2\theta + e_2 . \sin^2\theta;$$
$$i = (e_1 - e_2) \sin\theta . \cos\theta,$$

results which may be applied as before to show the changes of dimension and the distortion of a square traced anywhere in the band.

We have here regarded the angle i as a measure of the distortion a square suffers in consequence of the strain. If, however, we drop QM perpendicular to AP', we have

$$AQM = 2i = \frac{AM}{AQ}.$$

Now AM is the space through which the line $L'Q$ has shifted parallel to itself in consequence of the strain, and we see therefore that the angle i also gives a measure of the magnitude of this shifting. By some writers this is called "sliding." It is also called "shearing strain."

205. *Comparison between Stress and Strain.*—If we compare the equations we have just obtained for strain with those previously obtained in Art. 199 for stress, we find them identical; and hence it appears that, so long at least as the strains are very small, all propositions respecting stress must also be true, *mutatis mutandis*, with respect to strain. Thus, for example, a simple distortion must be equivalent to a longitudinal extension accompanied by an equal longitudinal contraction; and, again, every state of strain can be reduced to three simple longitudinal strains at right angles to each other, and represented by an ellipsoid of strain. The simple strains are called Principal Strains, and their directions Axes of Strain. Strain, like stress, generally varies from point to point of the body: but the relations here proved still hold good at each point, and we have Lines of Strain just as we previously had Lines of Stress.

SECTION III.—CONNECTION BETWEEN STRESS AND STRAIN.

206. *Equations connecting Stress and Strain in Isotropic Matter.*—So far we have merely been stating certain conditions which stress must satisfy in order that each element of a body may be in equilibrium, and certain other conditions which strain must satisfy if the body is continuous. We now connect the two by considering the way in which stress produces strain, which differs according to the nature of the material.

We first consider perfectly elastic material (see Art. 147), and suppose that material to have the same elastic properties in all directions, in which case it is said to be isotropic. Metallic bodies are often not isotropic, as will be seen hereafter (Ch. XVIII.). Suppose a rectangular bar under the action of a simple longitudinal stress p_1, then there results (Art. 148) a longitudinal strain e_1 given by

$$p_1 = Ee_1,$$

where E is the corresponding modulus of elasticity. Accompanying the longitudinal extension we find a contraction of breadth that is a lateral strain of opposite sign of magnitude $1/m^{\text{th}}$ the longitudinal strain where m is a co-efficient. The contraction in thickness will be equal, because the material is supposed isotropic. Hence the effect of the simple longitudinal stress p_1 is to produce three simple longitudinal strains at right angles,

$$e_1 = \frac{p_1}{E};\ e_2 = -\frac{p_1}{mE};\ e_3 = -\frac{p_1}{mE}.$$

Next remove p_1, and in its place suppose a simple stress p_2 applied in the direction of the breadth of the bar; we have by similar reasoning the three strains

$$e_1 = -\frac{p_2}{mE};\ e_2 = \frac{p_2}{E};\ e_3 = -\frac{p_2}{mE}.$$

And similarly removing p_2 and replacing it by p_3 acting in the direction of the thickness,

$$e_1 = -\frac{p_3}{mE};\ e_2 = -\frac{p_3}{mE};\ e = \frac{p_3}{E}.$$

These three sets of equations give the strains due to p_1, p_2, p_3, each acting alone; and we now conclude that if all three act together we must necessarily have

$$e_1 = \frac{p_1}{E} - \frac{p_2 + p_3}{mE},$$

with two other symmetrical equations.

Hence it appears that the effect of three principal stresses, and consequently of any state of stress whatever on isotropic matter, is to produce a strain, the axes of which coincide with the axes of stress, and in which the principal strains are connected with the principal stresses by the equations just written down.*

207. *Elasticity of Form and Volume.*—The value of the constant m may be found directly by experiment, though with some difficulty, on account of the smallness of the lateral contraction which it measures; but it may also be found indirectly, by connecting it with the co-efficient employed in a former chapter to measure the elasticity of torsion. For if we subtract the second of the three equations just obtained from the first, we get

$$e_1 - e_2 = (p_1 - p_2)\frac{m+1}{mE},$$

or

$$p_1 - p_2 = \frac{m}{m+1} \cdot E(e_1 - e_2).$$

---

* The form in which these equations are given is due to Grashof. For practical application it is more convenient than any other.

Now referring to Art. 199 we find

$$p_t = (p_1 - p_2) \sin \theta . \cos \theta,$$
$$2i = 2(e_1 - e_2) \sin \theta . \cos \theta,$$

where p_t is the tangential stress on a pair of planes inclined at angle θ to the axes, and $2i$ is the distortion of a square inclined at that angle to the axes of strain. Since now the axes of strain coincide with the axes of stress, we must have

$$\frac{p_t}{2i} = \frac{p_1 - p_2}{2(e_1 - e_2)} = \frac{1}{2}\frac{m}{m + 1} . E,$$

an equation which, compared with Art. 183, shows that the co-efficient of rigidity C must be

$$C = \frac{1}{2}\frac{m}{m + 1} . E.$$

Experiment shows that in metallic bodies C is generally about $\frac{2}{5}E$, whence it follows that m lies between 3 and 4. In the ordinary materials of construction the comparison cannot, however, be made with exactness, because such bodies are rarely exactly isotropic and homogeneous. The value of m for iron is supposed to be about $3\frac{1}{2}$.

Again, if we add together the three fundamental equations, we find

$$E(e_1 + e_2 + e_3) = \left(1 - \frac{2}{m}\right)(p_1 + p_2 + p_3).$$

Now the volume of a cube, the side of which is unity, becomes when strained $(1 + e_1)(1 + e_2)(1 + e_3)$, and therefore the volume strain is $e_1 + e_2 + e_3$ when the strains are very small. Hence, if we separate the stress into a fluid stress N and a distorting stress (Art. 204), we have

$$N = \frac{m}{3(m - 2)} . E \times \text{Volume Strain},$$

and the co-efficient

$$D = \frac{m}{3(m - 2)}E$$

measures the elasticity of volume. The two constants C and D, which measure elasticity of distinctly different kinds, may be regarded as the fundamental elastic constants of an isotropic body. The ordinary Young's modulus E involves both kinds of elasticity.

208. *Modulus of Elasticity under various circumstances. Elasticity of Flexion.*—When the sides of a bar are free the ratio of the longitudinal stress to the longitudinal strain is the ordinary modulus of elasticity E; but the equations above given show that, when the sides of the bar are subject to stress, the modulus will have a different value. For example, let the bar be forcibly prevented from contracting, either in breadth or thickness, by the application of a suitable lateral tension, $p_2 (= p_3)$, then e_2, e_3 are both zero, and

$$Ee_1 = p_1 - \frac{2p_2}{m}; \; 0 = p_2 - \frac{p_1 + p_2}{m},$$

whence we obtain for the magnitude of the necessary lateral stress

$$p_2 = \frac{p_1}{m-1},$$

and for the corresponding extension of the bar

$$Ee_1 = \frac{m^2 - m - 2}{m^2 - m} \cdot p_1.$$

Hence the modulus of elasticity is now

$$A = \frac{m(m-1)}{(m+1)(m-2)} \cdot E.$$

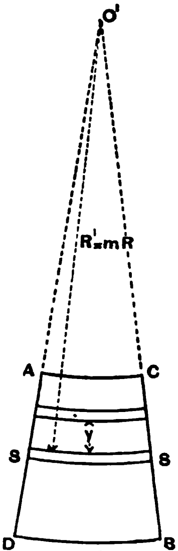


Fig.152

This constant A is what Rankine called the direct elasticity of the substance: it is of course always greater than E. For $m = 4$, $A = \frac{6}{5}E$; for $m = 3$, $A = \frac{3}{2}E$.

If the bar be free to contract in thickness, but not in breadth, we have p_3 and e_2 zero, and the equations become

$$Ee_1 = p_1 - \frac{p_2}{m}; \quad 0 = p_2 - \frac{p_1}{m}; \quad Ee_3 = 0 - \frac{p_1 + p_2}{m},$$

whence we find

$$Ee_1 = p_1 \cdot \frac{m^2 - 1}{m^2},$$

so that the value of the modulus of elasticity is $\frac{m^2}{m^2 - 1}E$. In a similar way if p_2, p_3 have any given values the modulus can be found.

It will now be convenient to examine an important point already referred to in the theory of simple bending, that is to say the assumption (Art. 153) that the modulus of elasticity E was the same as in the case of simple tension, notwithstanding the lateral connection of the elementary bars, into which we imagined the whole beam split up. If these elementary bars were prevented from contracting freely, as they would do if separated from each other, the modulus could not be the same. In fact, however, there is nothing in their lateral connection which prevents them from doing so. Figure 152 shows, on a very exaggerated scale, the form assumed by a transverse section $ACBD$ originally rectangular, cutting a series of longitudinal sections originally parallel to the plane of bending in the straight lines shown. Assuming the upper side stretched as in Fig.

122, page 232, these lines all radiate from a centre O' above the beam, which bends transversely, while the originally straight horizontal layers are cut in arcs of circles struck from the same centre. The upper side of the beam contracts and the lower side expands, and reasoning exactly in the same way as we did when we derived the principal formula,

$$p = \frac{Ey}{R}, \qquad \text{(Art. 153)}$$

we find a corresponding formula for the transverse curvature,

$$p = m\frac{Ey}{R'},$$

whence it follows immediately that

$$R' = mR.$$

In order that this transverse curvature of the originally horizontal layers shall not be inconsistent with the reasoning by which the formula for bending is obtained, all that is necessary is that the deviation from a straight line shall be small as compared with the distance of the layer from the neutral axis. Let x be that deviation, then (see Art. 163) if b be the breadth,

$$x = \frac{b^2}{8R^1} = \frac{b^2}{8mR} = \frac{b^2 \cdot p}{8mEy}.$$

Now the stress being within the elastic limit p/E is very small, for example, take the case of wrought iron, for which p/E is not more than $\frac{1}{1200}$th, and suppose $m = 4$,

$$x = \frac{b^2}{38{,}400 \cdot y_1},$$

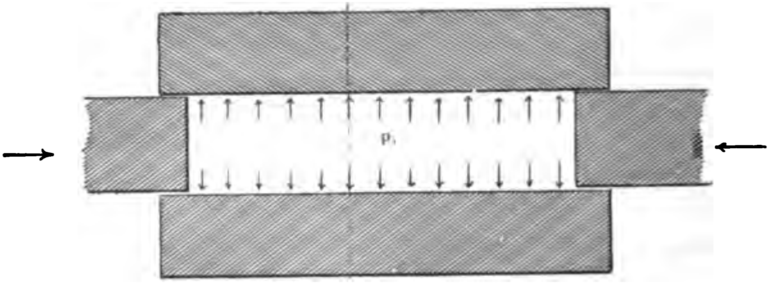
where y_1 is the greatest value of y, say $\frac{1}{2}h$, where h is the depth, thus

$$x = \frac{b^2}{19{,}200h}.$$

It is obvious that x must be always very small compared with y, except very near the neutral axis, and unless b be very large compared with h. When then a beam is bent *within* the limit of elasticity, the lateral connection of the parts cannot have any sensible influence on its resistance to bending, unless its breadth be great as compared with its depth. The case of a broad thin plate has not been hitherto dealt with theoretically. Beyond the limit of elasticity the lateral connection of the parts may greatly increase the resistance to bending, but this is a matter for subsequent consideration.

209. *Thick Hollow Cylinder under Internal Pressure.*—The equations connecting stress and strain in combination with suitable equations expressing the continuity of the body and the equilibrium of each of its elements are theoretically sufficient to determine the distribution of stress within an elastic body exposed to given forces, and in particular to determine the parts of the body exposed to the greatest stress, and the magnitude of such stress. The most important cases hitherto

Fig. 153a



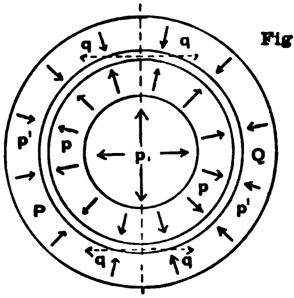


Fig. 153b

worked out, in addition to those considered in preceding chapters, are the torsion of non-circular prisms and the action of internal fluid pressure on thick hollow cylinders and spheres. For M. St. Venant's investigations on torsion we must refer to Art. 184, page 328, and the authorities there cited. We shall only consider the comparatively simple case of a homogeneous cylinder.

Fig. 153*a* shows a longitudinal section of a hollow cylinder open at the ends, which are flat: the cylinder contains fluid which is acted on by two plungers forced in by external pressure so as to produce an internal fluid pressure p_1. Fig. 153*b* shows the same cylinder in transverse section: imagine a cylindrical layer of thickness t, this thin cylinder will be acted on within and without by stress which symmetry shows must be normal; let these stresses be p and p', and the internal and external radii of the thin cylinder be r and r'. Now, if p' the external pressure had existed alone, a compressive stress q would have been produced on the material of the cylinder given by the equation (see Art. 150)

$$p'r' = qt;$$

and if the internal pressure had existed alone, we should have had a tensile stress given by

$$pr = qt;$$

hence when both exist together, we must have

$$p'r' - pr = qt,$$

where q is the stress on the material of the cylinder on a radial plane in the direction perpendicular to the radius reckoned positive when compressive. Clearly $t = r' - r$, and therefore proceeding to the limit we may write the equation

$$\frac{d}{dr}(pr) = q,$$

which is one relation between the principal stresses p, q at any point of the cylinder. We now require a second equation, to get which it is necessary to consider the way in which the cylinder yields under the application of the forces to which it is exposed. The simplest way to do this is to assume that the cylinder remains still a cylinder after the pressure has been applied: if so, it at once follows that points in a transverse section originally remain so, or, in other words, that the longitudinal strain is the same at all points. It is not to be supposed that there is anything arbitrary about this assumption: no other, apparently, can be made if the ends of the cylinder are free, the pressure on the internal surface exactly uniform, and the cylinder be homogeneous and free from initial strain. For when this is the case, there is no reason why the cylinder should be in a different condition in one part of its length than in another. If the ends are not free, or if the pressure is greater in the centre, the middle of the cylinder will bulge, but not otherwise.

It is also clear that the total pressure on a transverse section must be zero because the ends are free, and hence it is natural to suppose that it is also zero at every point of the transverse section, an assumption which we shall presently verify.

The equations connecting stress and strain therefore become

$$Ee_1 = p - \frac{q}{m};$$

$$Ee_2 = q \quad \frac{p}{m};$$

$$Ee_3 = 0 - \frac{p+q}{m},$$

where e_1, e_2, e_3 are the strains in, the direction of the radius, the direction perpendicular to the radius in the transverse section, and, the direction of the length, respectively. Of these the last is constant, as just stated, and therefore

$$p + q = \text{const.} = 2c_1$$

is the second equation connecting p, q. Substituting for q, we find

$$\frac{d}{dr}(pr) + p = 2c\,;$$

or

$$r\frac{dp}{dr} + 2p = 2c_1.$$

Multiply by r and integrate, then

$$p = \frac{c_2}{r^2} + c_1, \text{ and consequently } q = c_1 - \frac{c_2}{r^2},$$

where c_2 is a constant of integration. The two constants c_1, c_2 are now determined by consideration of the given pressures within and without the cylinder.

If n be the ratio of the external radius to the internal radius R, we have at the internal surface

$$\left.\begin{matrix} p = p_1 \\ r = R \end{matrix}\right\} \therefore p_1 = c_1 + \frac{c_2}{R^2}\,;$$

and at the external surface

$$\left.\begin{matrix} p = 0 \\ r = nR \end{matrix}\right\} \therefore 0 = c_1 + \frac{c_2}{n^2R^2}\,;$$

from which two equations we get

$$c_1 = -\frac{p_1}{n^2 - 1}, \text{ and } c_2 = \frac{p_1 n^2}{n^2 - 1} \cdot R^2.$$

Substituting these values in the equation for q,

$$q = -\frac{p_1}{n^2 - 1}\left\{1 + n^2 \cdot \frac{R^2}{r^2}\right\};$$

the negative sign in this formula indicates that the stress is tensile, as we might have anticipated. The formula shows that the stress decreases from $\dfrac{n^2 + 1}{n^2 - 1} \cdot p_1$ at the internal surface to $\dfrac{2p_1}{n^2 - 1}$ at the external surface. The mean stress is obtained from the equation (Art. 150)

$$q_0(nR - R) = p_1 R\,;$$

hence the maximum stress is greater than the mean in the ratio $n^2 + 1 : n + 1$, and it is clear that it can never be less than p_1.

Verification of Preceding Solution.—The radial strain (e_1) and the hoop strain (e_2) are given by the above equations in terms of the stress. Now these changes of dimension are not independent, but are connected by a certain geometrical relation which it is necessary to examine in order to see whether it is satisfied by the values we have found.

Returning to the diagram, suppose the internal radius of the elementary ring represented there to increase from r to s, and the external radius from r' to s'; then

$$2\pi s = 2\pi r(1 + e_2),$$

$$2\pi s' = 2\ r'\left(1 + e_2 + t\frac{de_2}{dr}\right),$$

$$\therefore\quad s' - s = (r' - r)(1 + e_2) + r't\frac{de}{dr};$$

or since the thickness of the ring changes from t to $(1 + e_1)t$,

$$1 + e_1 = 1 + e_2 + r\frac{de_2}{dr},$$

$$e_1 = \frac{d}{dr}(e_2 r).$$

This relation must always hold good, in order that the rings after strain may fit one another, and should therefore be satisfied by our results. On trial it will be found that it is satisfied, and we conclude that the solution we have obtained satisfies all the conditions of the problem, and is therefore the true and only solution, subject to the conditions already explained. For further remarks on this question, see Appendix.

210. *Strengthening of Cylinder by Rings. Effect of great Pressures.*—The stress within a thick hollow cylinder under internal fluid pressure may be equalized, and the cylinder thus strengthened by constructing it in rings, each shrunk on the next preceding in order of diameter. For a cylinder so constructed will be in tension at the outer surface and compression at the inner surface before the pressure is applied, and therefore after the pressure has been applied will be subjected to less tension at the inner and more tension at the outer surface than if it had been originally free from strain. It is theoretically possible to determine the diameters of the successive rings so that the pressure shall be uniform throughout. The principle is important, and frequently employed in the construction of heavy guns.

When the limit of elasticity is overpassed the formula fails, and the distribution of stress becomes different. If the pressure be imagined gradually to increase until the innermost layer of the cylinder begins to stretch beyond the limit, more of the pressure is transmitted into the interior of the cylinder, so that the stress becomes partially equalized. If the pressure increases still further, the tension of the innermost layer is little altered, and in soft materials longitudinal flow of the metal commences under the direct action of the fluid pressure. The internal diameter of the cylinder then increases perceptibly and permanently. This is well known to happen in the cylinders employed in the manufacture of lead piping, which are exposed to the severe pressure necessary to produce flow in the lead. The cylinder is not weakened but strengthened, having adapted itself to sustain the pressure. Cast-iron hydraulic press cylinders are often worked at the great pressure of 3 tons per sq. inch, a fact which may perhaps be explained by a similar equalization.

EXAMPLES.

1. When the sides of a bar are forcibly prevented from contracting, show that the necessary lateral stress is given by

$$p_1 = Be,$$

where $B = \dfrac{mE}{m^2 - m - 2}$. This constant B is what Rankine called the "lateral" elasticity of the substance.

2. With the notation of the preceding question and of Art. 208, prove that

$$C = \frac{A - B}{2}.$$

3. In a certain quality of steel $E = 30{,}000{,}000$; $C = 11{,}500{,}000$: find the elasticity of volume and the values of A and B, assuming the material to be isotropic. *Ans.* $m = 3\frac{2}{7}$; $D = 25{,}555{,}000$.

4. The cylinder of an hydraulic accumulator is 9 inches diameter. What thickness of metal would be required for a pressure of 700 lbs. per square inch, the maximum tensile stress being limited to 2,100 lbs. per square inch? Also, find the tensile stress on the metal of the cylinder at the outer surface. *Ans.* Thickness = 1·84″; Stress = 1,400 lbs. per square inch.

5. If the cylinder in the last question were of wrought iron, proof resistance to simple tension 21,000 lbs. per square inch, at what pressure would the limit of elasticity be overpassed? $m = 3·5$. (See Art. 223, p. 387.) *Ans.* 6,400.

6. Find the law of variation of the stress within a thick hollow sphere under internal fluid pressure. By a process exactly like that for the case of the cylinder (page 367) it is found that the equation of equilibrium is

$$\frac{d}{dr}(pr^2) = 2qr.$$

The equation of continuity is the same as that for a cylinder (Art. 209), and the equations connecting stress and strain are now

$$Ee_1 = p - \frac{2q}{m};$$

$$Ee_2 = q - \frac{p + q}{m}.$$

We can now by elimination of q, reduction, and integration obtain

$$p = c_1 + \frac{c_2}{r^3};$$

$$q = c_1 - \frac{c_2}{2r^3};$$

the constants being found as in the cylinder.

7. The cylinder of an hydraulic press is 8 inches internal and 16 inches external diameter. If the pressure be 3 tons per sq. inch find the principle stresses at the internal and external circumference.

Ans. At inner circumference $\begin{cases}\text{Major Stress} = 5 \text{ (Tension).}\\ \text{Minor Stress} = 3 \text{ (Thrust).}\end{cases}$

At outer ,, $\begin{cases}\text{Major Stress} = 2 \text{ (Tension).}\\ \text{Minor Stress} = 0.\end{cases}$

8. In the last question find the "equivalent simple tensile stress" (p. 388), assuming $m = 3·5$. *Ans.* 5·86 and 2 tons.

CHAPTER XVIII.

MATERIALS STRAINED BEYOND THE ELASTIC LIMIT. STRENGTH OF MATERIALS.

211. *Plastic Bodies.*—If the stress and strain to which a piece of material is exposed exceed certain limits its elasticity becomes imperfect, and ultimately separation into parts takes place. We proceed to consider what these limits are in different materials under different circumstances: it is to this part of the subject alone that the title "Strength of Materials" is, strictly speaking, appropriate.

Reference has already been made (Art. 147) to a certain condition in which matter may exist, called the Plastic state, which may be regarded as the opposite of the Elastic state, which has been the subject of preceding chapters. In this condition the changes of size of a body are very small, as before; but if the stress be not the same in all directions the difference, if sufficiently great, produces continuous change of shape of almost any extent. Some materials are not plastic at all under any known forces, but many of the most important materials of construction are so, more or less, under great inequality of pressure.

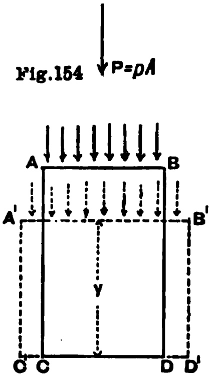


Fig.154

Fig. 154 shows a block of material which is being compressed by the action of a load P applied perfectly uniformly over the area AB. Let the intensity of the stress be p, then so long as p is small the compression is small and proportional to the stress; but when it reaches a certain limit the block becomes visibly shorter and thicker. This limit depends on the hardness of the material, and the value of p may be called the "co-efficient of hardness." In an actual experiment the friction of the surfaces between which the block is compressed holds the ends together, so that it bulges in the middle, as in Fig. 158, p. 380, which represents an experiment on a short cylinder of soft steel. In the ideal case the sectional area remains uniform, changing throughout inversely as the height, as expressed by the equation

$$Ay = A_1 y_1,$$

where A is the area and y the height of the block.

In a truly plastic body p the intensity of the stress remains constant, and therefore the crushing load P varies as A, that is inversely as y. This is the same law as that of the compression of an elastic fluid when the compression curve is an hyperbola, and we therefore conclude (Art. 90) that the work done in crushing is

$$U = Py \, . \log_\epsilon r = pAy \log_\epsilon r = pV \log_\epsilon r,$$

where r is the ratio of compression and V the volume. Certain qualities of iron and soft steel will endure a compression of one-fourth or even of one-half the original height, and amounts of energy are thus absorbed which are enormous compared with the resilience of the metal. To illustrate this, suppose that plasticity begins as soon as the limit of elasticity f is overpassed, then for p we must write f, and by Art. 149 the resilience for a volume V is

$$\text{Resilience} = \frac{f^2}{2E} \, . \, V \, .$$

The ratio which the work just found bears to the resilience is therefore

$$\text{Ratio} = \frac{2E}{f} \, . \log_\epsilon r.$$

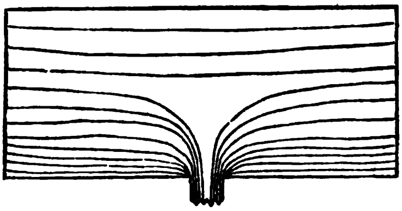
In wrought iron for a compression of one fourth the height ($r = 1{\cdot}333$) this is about 800. The actual ratio must be much greater, because, as we shall see presently, the hardness of the material increases under stress.

If lateral pressure of sufficient magnitude be applied to the sides of the block, the longitudinal force being removed, the effect is elongation instead of compression, contraction of area instead of expansion. The magnitude of the lateral pressure is found by imagining a tension applied both longitudinally and laterally of equal intensity. Such a tension has no tendency to alter the form of the block, being equivalent to fluid pressure, but it reduces the lateral pressure to zero, while it introduces a longitudinal tension of the same amount, which has the same value as the longitudinal compression of the preceding case. We see then that in every case a certain definite difference of pressure is required to produce change of shape in a plastic body, the direction of the change depending on the direction of the difference. The work done is found by the same formula as before, r meaning now the ratio of elongation.

In the process of drawing wire the lateral pressure is applied by the sides of the conical hole in the draw-plate, which are lubricated to reduce friction, and the force producing elongation in the wire is the sum of the tensile stress applied to draw the wire through the hole and the compressive stress on the sides. The work done is given by the same formula as before, p being now the sum in question.

212. *Flow of Solids.*—When a plastic body changes its form the process is exactly analogous to the flow of an incompressible fluid, which indeed may be regarded as a particular case. In the solid the distorting stress at each point at which the distortion is going on has a certain definite value which in the fluid is zero. The experimental proof of this is furnished by the experiments of M. Tresca, of which Fig. 155 shows an example. Twelve circular plates of lead are placed one upon another in a cylinder, which has a flat bottom with a small orifice at its centre. The pile of plates being forcibly compressed, the lead issues at the orifice in a jet, and the originally flat plates assume the forms shown in the figure. The lines of separation, indicating the position of particles of the metal originally in a transverse section, are quite analogous to the corresponding lines in the case of water issuing from a vessel through an orifice in the bottom. Tresca's experiments were very extensive, and showed that all non-rigid material flowed in the same way. Lead approaches the truly plastic condition; the difference of pressure necessary to make it flow being always about the same. Tresca ascribes to it the value of 400 kilogrammes per square centimetre, or about 5,700 lbs. per square inch;* but it is probably subject to considerable variations.

Fig. 155.



The manufacture of lead pipes, the drawing of wire, and all the processes of forging, rolling, &c., by which metals are manipulated in the arts, are examples of the Flow of Solids.

213. *Preliminary Remarks on Materials. Stretching of Wrought Iron and Steel.*—Materials employed in construction may roughly be divided into three classes. The first are capable of great changes of form without rupture, and, when possessing sufficient strength to resist the necessary tension, may be drawn into wire. This last property is called ductility, and this word may be used to describe the class which we shall therefore

---

* The co-efficient employed by Tresca, and called by him the "co-efficient of fluidity," is half that used in the text. It is the magnitude of the distorting stress necessary to produce flow. See also note in Appendix.

call Ductile Materials. The second, being incapable of enduring any considerable change of this kind, may be described as Rigid Materials. The third are in many cases not homogeneous, but may be regarded as consisting of bundles of fibres laid side by side, they may therefore be described as Fibrous Materials; they are generally of organic origin.

We shall commence with the consideration of ductile materials, and more epecially of

WROUGHT IRON AND STEEL.

Accurate experiments on the stretching of metal are difficult to make, the extensions being very small and the force required great. If levers are used to multiply the effect of a load or to magnify the extensions, errors are easily introduced. If the levers are dispensed with, a great length of rod is necessary and a heavy load the manipulation of which involves difficulties. In the best modern testing machines the load is usually applied at one end of the specimen by hydraulic pressure and measured at the other by a weight travelling along a long lever. The elongations are measured by micrometers. The experiment we select for description was made by Hodgkinson on a rod of wrought iron ·517 inch diameter, 49 feet 2 inches long, loaded by weights placed in a scale pan* suspended from one end. The load applied was increased by equal increments of 5 cwts. or 2667·5 lbs. per square inch of the original sectional area of the bar; each application of the load being made gradually, and the whole load removed between each. At each application and removal the elongation was measured so as to test the increment of elongation, both temporary and permanent, occasioned by each load. If the rod were perfectly elastic the temporary increments should be equal and the permanent elongations (usually called "sets") zero.

The annexed table shows part of the results of this experiment, the first column giving the load, the second the total elongation, the third the successive increments of the elongation, the fourth the total permanent set.

On examining the table we see that, after some slight irregularities at the commencement due to the material not being perfectly homogeneous, the increments of elongation are nearly constant till we reach the eighth load of 21,340 lbs. per square inch, after which the increments show a rapid increase. Further, the permanent set, which at the commencement is very minute and increases very slowly, at the same point shows a sudden increase indicating that the observed increase is almost wholly due to a permanent elongation of the bar, the temporary increase following approximately the same law as before. Notwithstanding this the bar is not torn asunder till a much greater load is applied. The table shows the results up to a load of 32,000 lbs. per square inch, but rupture did not occur till a load of 53,000 lbs. was applied. The extension at the same time increased to nearly 21 inches, being more than forty times its amount at the elastic limit.

---

* Being one of the best of its kind of old date this experiment has often been quoted. For the original description see the *Report of the Commissioners appointed to enquire into the Application of Iron to Railway Structures.*

STRETCHING OF A WROUGHT IRON ROD, 49 FEET 2 INCHES LONG.

LOAD.		ELONGATION IN INCHES.	INCREMENT OF ELONGATION.	PERMANENT SET.
2667·5 × 1	2667·5	·0485	·0485	
,, × 2	5335	·1095	·061	
,, × 3	8003	·1675	·058	·0015
,, × 4	10,670	·224	·0565	·002
,, × 5	13,338	·2805	·0565	·0027
,, × 6	16,005	·337	·0565	·003
,, × 7	18,673	·393	·056	·004
,, × 8	21,340	·452	·059	·0075
,, × 9	24,008	·5155	·0635	·0195
,, × 10	26,675	·598	·0825	·049
,, × 11	29,343	·760	·162	·1545
,, × 12	32,010	1·310	·550	·667

We conveniently represent the results graphically by setting off the elongations as abscissæ along a base line with corresponding ordinates to represent the stress, thus obtaining a curve of "Stress and Strain" (Fig. 156). The curve will be seen to be nearly straight up to a stress of 22,000 lbs. and then to bend sharply, becoming nearly straight in a different direction. A curve of permanent set may also be constructed which is seen to follow the same general law.

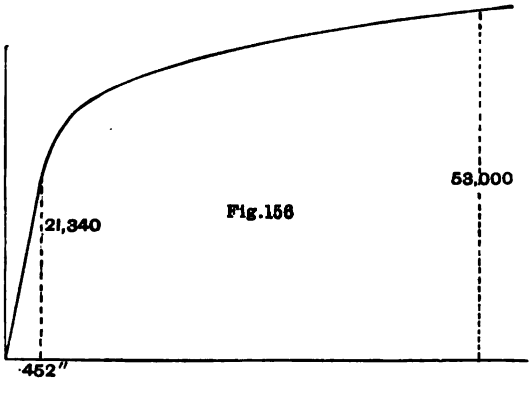


Fig. 156

This experiment may be taken as a type of many hundreds of such experiments which have been made on iron and the softer kinds of steel, showing that in these materials a tolerably well-defined limit exists, within which the material is nearly perfectly elastic (compare Art. 127); the small deviations are more due to the want of perfect homogeneity in the bar than to actual defect in the elasticity. They usually diminish greatly if the experiment be tried a second time on the same bar.* The position of the limit of elasticity and the value of the modulus of course vary. Some examples will be given presently.

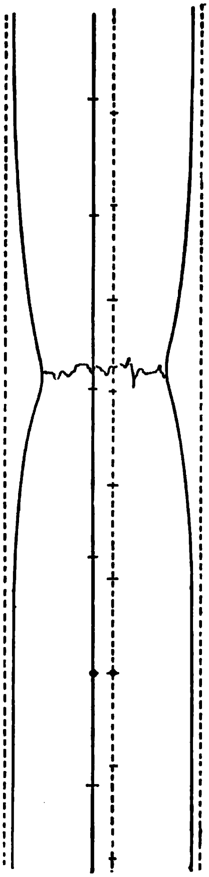


Fig.157

Accompanying the increase of length of the bar we find a contraction of area; within the elastic limit, however, this is so small as to escape observation. Outside the limit it becomes visible, consisting in the first instance of a more or less uniform contraction at all or nearly all points, followed by a much greater contraction at one or sometimes two points where there happens to be some local weakness.† Within the elastic limit the density of the bar diminishes, but by an amount so small that the fact is rather known by reasoning than determined by experiment. Outside the limit there is a permanent diminution which is perceptible, though still very small.

Thus beyond the elastic limit the bar draws out, changing its form like a plastic body without sensible change of volume. The bar finally tears asunder at the most contracted section, as shown by the annexed figure (Fig. 157) representing an experiment by Mr. Kirkaldy on a bar of iron 1 inch diameter, in which the contraction of area was 61 per cent., and the elongation 30 per cent., ultimate strength 58,000 lbs. per square inch of original area, 146,000 lbs. per square inch of fractured area. The contraction of section in good iron and soft steel is 50 or 60 per cent.

---

* These points have of late been studied much more thoroughly: some particulars will be found in the Appendix.

† On this point see *Preliminary Experiments on Steel by a Committee of Civil Engineers*, London, 1868. On account of the uncertainty of the amount of contraction at various points, the ultimate extension is an imperfect measure of the ductility of the iron, even when the pieces are of the same length and sectional area.

214. *Real and Apparent Tensile Strength of Ductile Metals.*—Thus the process of stretching an iron bar beyond the limit of elasticity till it breaks is an example of the "flow of solids," the iron behaving to a certain extent like a plastic body. There is, however, this difference, that a constantly increasing stress is necessary to produce continuous flow, which increase is supplied partly by increase of the stretching load, partly by the contraction of area. The actual stress at each instant on the contracted area is much greater than the apparent stress given in the table, which is merely the total load divided by the original sectional area. Hence when contraction has once begun at some point of local weakness it continues there, and the process goes on till the stress per square inch of the reduced area becomes greater than the metal will bear, when fracture takes place. This stress may to a certain extent be regarded as a measure—though a rough and imperfect one—of the true tenacity of the metal, as distinguished from the "apparent" tenacity which is reckoned per square inch of the original area.* For many purposes the "true" tenacity, in good iron more than double the apparent, is a better test of the quality of the iron than the actual breaking load, inferior iron often showing a high apparent tenacity but contracting far less.

Hence it follows that if the form of the piece be such as partly or wholly to prevent contraction the apparent strength will be increased. For example, if two pieces of the same bar be taken and one turned down to a certain diameter, while in the other narrow grooves are cut so as to reduce the diameter to the same amount at the bottom of the grooves, the strength of the grooved piece will be found to be much greater than that of the piece the diameter of which has been reduced throughout, and this can only be explained by observing that the length of the reduced part of the grooved bar is insufficient to permit contraction to any considerable extent. This is a point to be noticed in considering experimental results.† The form of the specimen tested may have much influence. Further, since the limit of elasticity is the point at which flow commences, and since the flow is due to difference of stress, it follows that the same causes must raise the limit of elasticity, and thus we are led to the conclusion that there are two elements constituting strength in a material, first, tenacity, and, secondly, rigidity.

---

* Some further remarks on this important point will be found in the Appendix.

† See *Experiments on Wrought Iron and Steel*, by Mr. Kirkaldy, p. 74. 1st edition. Glasgow, 1862.

In some materials, such as these we are now considering, the tenacity is much greater than the rigidity, and in them the limit of elasticity will depend on the rigidity, and will have different positions according to the way the stress is applied. It will lie much higher, and the apparent strength will be much greater when lateral stress is applied to prevent contraction.

215. *Increase of Hardness by Stress beyond the Elastic Limit.*—In clay and other completely plastic bodies a certain definite difference of pressure is sufficient to produce flow: in iron, copper, and probably other metals, however, as we have just seen, this is not the case, the metal acquiring increased rigidity in the act of yielding to the pressure. Thus the effect of stress exceeding the elastic limit is always to raise the limit, whether the stress be a simple tensile stress or whether it be accompanied by lateral pressure. All processes of hammering, cold rolling, wire drawing, and simple stretching have this effect. If a bar be stretched by a load exceeding the elastic limit and then removed, on re-application of a gradually increasing load we do not find a fresh drawing out to commence at the original elastic limit, but at or near the load originally applied.* If the load be further increased drawing out re-commences. Hence, whenever iron is mechanically "treated" in any way which exposes it to stress beyond the elastic limit, contraction is prevented and the apparent strength is increased: for example, iron wire is stronger than the rod from which it is drawn; when an iron rod is stretched to breaking the pieces are stronger than the original rod. It is not certain that the real strength of materials is always increased by such treatment; perhaps in some cases the contrary, for we know that the modulus of elasticity and specific gravity are somewhat diminished.† On the other hand there are cases in which the increase of strength is greater than can be accounted for in this way. On annealing the iron it is found to have resumed its original properties, a circumstance which indicates that the increased rigidity is due to a condition of constraint which is removed by heating the metal till it has assumed a completely plastic condition. In considering the effect of impact, the diminution of ductility occasioned by the application of stress beyond the elastic limit is a most important fact to be taken into account (see Art. 226). Working iron or steel hot has generally the effect of increasing both its strength and its ductility.

---

* Styffe *On Iron and Steel*, p. 68.

† The raising of the limit of elasticity by mechanical treatment of various kinds has long been known: in the case of simple stretching the effect appears to have been first noticed by Thälen in a paper, a translation of which will be found in the *Philosophical Magazine* for September, 1865.

216. *Compression of Ductile Material.*—In a perfectly elastic material compression is simply the reverse of tension, the same changes of dimension being produced by the same stress, but in the reverse direction. Also in a plastic body a given difference of stress produces flow, whether the stress be tensile or compressive; hence in ductile metals we should expect to find the modulus of elasticity and the limit of elasticity nearly the same in compression as in tension. These conclusions are borne out by experiment. In the case of wrought iron and steel, experiments on the direct compression of a bar are more difficult to carry out than experiments in tension, the bars are necessarily of limited length, and must be enclosed in a trough to prevent lateral bending; minute accuracy is therefore hardly attainable. A considerable number have, however, been made, from which it appears that the modulus of elasticity and the limit of elasticity are nearly the same in the two cases.*

EXPERIMENT BY SIR W. FAIRBAIRN ON A BLOCK ·72 INCH DIAMETER OF SOFT BESSEMER STEEL.

TOTAL LOAD $= P$.	HEIGHT OF BLOCK $= y$.	CRUSHING STRESS $p=\frac{Py}{Ay_1}$.
0	·997	0
16·7	·92	37·8
20·1	·865	42·9
23·3	·797	45·9
26·3	·731	47·4
29·5	·672	48·9
32·6	·613	49·4
35·8	·574	50·6
39·3	·535	51·9
41·0	·505	50·8

REMARKS.—The apparent ultimate tensile strength of this steel was 36 tons, its limit of elasticity 22 tons per square inch. Modulus of elasticity 30,300,000 lbs. Ratio of contraction ·41. Real tensile strength 88·5 tons.

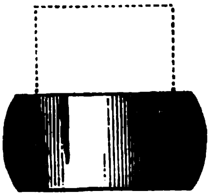
The metal yields beyond the limit by a process of flow of the same character as in tension, but expanding laterally instead of contracting. This is especially seen in experiments made by the late Sir W. Fairbairn in 1867, and somewhat earlier by Mr. Berkeley, on the compression of short blocks of steel. In both, the blocks were pieces of round bars, of height somewhat greater than the diameter, and the results were very similar.

---

* Perhaps the best set of experiments are those made by the "Committee of Civil Engineers." See their report already cited, pp. 7-13.

The annexed table gives the results of one of Sir W. Fairbairn's experiments. Column 1 gives the actual load laid on; column 2 the corresponding height of the block, both given directly by the experiments. Column 3 is calculated by dividing the product of load and height by the original sectional area and height, and represents the crushing stress per square inch of the mean sectional area. If the block did not bulge in the centre on account of friction holding its ends together (Fig. 158), this would be the actual crushing stress which, however, must in fact be less. The table shows that after a compression of about one-third the crushing stress remains nearly constant at about 50 tons per square inch. The experiment terminated at a compression of one-half. This kind of steel then is perfectly elastic up to 22 tons per square inch, is partially plastic between 22 and 50, and behaves as a plastic body under a difference of stress of 50 tons per square inch.

Fig. 158.



In ductile materials fracture takes place under compression by longitudinal cracks as shown in Fig. 158, which represents an experiment on a different quality of steel. The amount of compression which different materials will bear is very different according to their malleability; it is generally difficult to fix upon the ultimate strength, as it depends on the mode in which the experiment is made. In iron and steel it is somewhat less than the apparent tensile strength.

The compression of iron blocks has been less thoroughly studied than that of steel, but it is known that the results are similar although the strength and the ultimate ratio of compression are much less. Set becomes sensible at about 10 tons per square inch, and the ultimate strength is from 40 to 50,000 lbs. per square inch if lateral flexure be prevented.

217. *Bending beyond the Elastic Limit.*—Since wrought iron and steel are nearly perfectly elastic when the stress applied is not too great, it follows that the formula already obtained for the moment of resistance to bending of a bar must be true for these materials so long as the stress does not exceed the elastic limit determined by tension experiments of the kind just described. Experience fully confirms this conclusion, for the deflection obtained by experiment agrees well with that found from formulæ previously given with the same value of the modulus. As soon, however, as the maximum stress exceeds this limit, it is no longer true that the stress at different points of the transverse section varies as the distance from the neutral axis. It does not increase so fast, because the extension and compression at points near the surface is not accompanied by a proportional increase of stress. Hence, a partial equalization of stress takes place over the transverse section, and consequently the maximum stress for a given moment of resistance is not so great.

Again, it has been repeatedly explained in the earlier part of this book that the lateral connection of the several layers into which we imagine a beam divided has no influence on the stress produced by bending so long as the limit of elasticity is not exceeded. But when the limit is passed, the connection between those layers which are most stretched and compressed with those layers which have not yet lost their elasticity prevents their contraction and expansion, and so raises the limit of elasticity in accordance with the general principle explained in Art. 214. Thus, the limit of elasticity lies higher, and the apparent strength is greater in bending than in tension. In Fairbairn's experiment quoted above the same steel was tested in tension, compression, and bending. The elastic limit in bending was 30 tons, in tension 22 tons. The magnitude of the difference will depend on the form of transverse section, and on the ductility of the material. According to Mr. Barlow it may reach 50 per cent. in a rectangular section.* The case of cast iron will be referred to farther on.

Putting aside the effect of lateral connection, it may be interesting to make a calculation of the effect of equalization, by supposing that under a bending moment very slowly and steadily applied beyond the elastic limit, the metal behaves like a truly plastic material throughout the transverse section, so that the stress is uniform. Referring to the formula on page 284, we have

$$\Sigma pybt = M,$$

in which we must now, instead of assuming that p varies as y, suppose p a constant. Then

$$M = 2p \, . \, A\bar{y},$$

where A is the area of the part of the section which lies on either side of the neutral axis and $\bar{y}$ the distance of its centre of gravity from that axis. For the same value of the modulus this gives a moment of resistance in a rectangular section 50 per cent. greater than if the material had been elastic. How far any apparent increase of strength due to equalization or lateral connection may be regarded in practice is uncertain. A failure of elasticity must have taken place at certain points in order that there may be any increase at all, and in cases where the load is frequently reversed the bar must be weakened. (See Art. 225.)

CAST IRON AND OTHER RIGID MATERIALS.

218. *Stretching of Cast Iron.*—The phenomena attending rupture by tension of cast iron are essentially different from those described above for the case of ductile metals. This will be sufficiently shown by an experiment, also made by Hodgkinson, on a bar of this material 50 feet long, 1·159 inch diameter. The experiment was made in the same way as that already described on the wrought-iron rod,* and the results are shown in the annexed table. The first four loads were applied as before, by increments of 5 cwt., here equivalent to 531 lbs. per square inch; the whole load, after measurement of the elongation, being completely removed, and the permanent set measured. After the fourth load the increment was 10 cwt., and this was carried on till the bar broke at a stress of 16,000 lbs. per square inch. The third column as before shows the increments of elongation, which, after a stress of 5,308 lbs. per square inch, or $\frac{1}{3}$ the breaking load, has been reached, show a gradual increase till actual rupture occurs. The results of the experiments are graphically exhibited in the annexed diagram (Fig. 159) of stress, strain, and permanent set. The form of the curve is different from that of wrought iron, showing no point of maximum curvature, because in this material the bar does not draw out.

---

* *Phil. Trans.*, 1855-57.

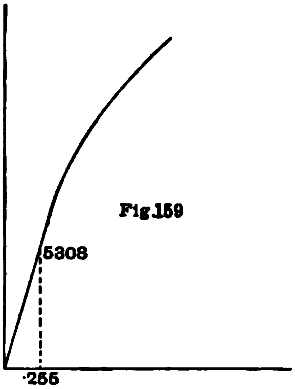


Fig. 159

Hodgkinson experimented on a large variety of different kinds of

STRETCHING OF A CAST-IRON BAR 50 FEET LONG, 1·159 INCH DIAMETER.

LOAD IN LBS. PER SQUARE INCH.	ELONGATION IN INCHES.	INCREMENT OF ELONGATION.	PERMANENT SET.
1. 531	·024	·024 × 2 = ·048	Perceptible.
2. 1,062	·0495	·0255 × 2 = ·051	·0015
3. 1,592	·0735	·024 × 2 = ·048	·002
4. 2,123	·09828	·0247 × 2 = ·0514	·0045
5. 3,185	·1485	·0503	·0105
6. 4,246	·200	·0515	·0155
7. 5,308	·255	·055	·022
8. 6,370	·313	·058	·028
9. 7,431	·374	·061	·037
10. 8,493	·435	·061	·046
11. 9,554	·504	·069	·056
12. 10,616	·572	·068	·067
13. 11,678	·648	·076	·0795
14. 12,739	·728	·080	·095
15. 13,801	·816	·088	·1115
16. 14,863	·912	·096	·132
17. 15,924	1·000	·088	—

---

* *Report of Commissioners on the Application to Railway Structures*, p. 51.

iron, and expressed his results by a formula, which may be written

$$p = Ee(1 - ke),$$

where, as before (Art. 148), p is the stress, e the extension per unit of length, E the ordinary modulus of elasticity, and k a constant. The term ke here expresses the defect of elasticity of the bar. From the results of his experiments we find the average values

$$E = 14{,}000{,}000\,;\ k = 209.$$

Cast iron, however, is a material of variable quality, and the value of these constants may have a considerable range. Up to one third the breaking load it may be regarded as approximately perfectly elastic, but the limit is by some authorities placed much higher.

219. *Crushing of Rigid Materials.*—In the ductile metals the effects of compression are nearly the reverse of those of extension, as has been sufficiently shown in previous articles, but in cast iron this is by no means the case. Hodgkinson experimented in this question with great care and accuracy, testing pieces of iron of exactly the same quality under compression and tension to enable a comparison to be made. The bars were enclosed in a frame and tested by direct compression. Hodgkinson expressed his results by a formula, which may be written

$$p = Ee(1 - ke),$$

the symbols having the same meanings as before, and the values may be taken as $E = 13{,}000{,}000\,;\ k = 40.$

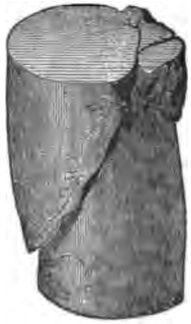
The smaller value of k indicates that the elasticity under compression is much less imperfect under the same stress. Short cylinders of the metal were also crushed, and the crushing load found to be five times the tensile strength or more.

It thus appears that in compression cast iron is six times stronger than in tension, and this is true not merely of the ultimate resistance, but in great measure also of the elastic resistance, for the elasticity of the metal is not sensibly impaired until one third the crushing load is reached.

The manner in which crushing occurs is shown in the accompanying figure; instead of bulging out like a ductile metal, oblique fracture takes place on a plane inclined at 45° or rather less to the axis, being (approximately) the plane on which the shearing stress is a maximum (Fig. 160).

Great resistance to compression, as compared with tension, and sudden fracture by shearing obliquely or by splitting longitudinally are characteristics of all non-ductile materials, of which cast iron may be taken

Fig. 160.



as a type. They are, in fact, materials the tenacity of which is much less than the rigidity.

220. *Breaking of Cast-Iron Beams.*—When a cast-iron bar is bent till the tensile stress at the stretched surface exceeds one third the tensile strength of the material, the defective elasticity of the metal causes a partial equalization of stress on the transverse section as in the case of wrought iron. Besides this, the elasticity being much more perfect under compression than under tension, the equalization is greater on the stretched side than on the compressed side, and the neutral axis moves towards the compressed edge of the beam. For both these reasons the moment of resistance to bending is greater for a given maximum tensile stress than it would be if the material were perfectly elastic. Thus it follows that if the co-efficient in the ordinary formula for bending be assumed equal to the tensile strength of the material, the calculated moment of resistance will be less than the actual moment of rupture of the beam by an amount which is greater for a rectangular section than for an I section. The discrepancy is found by experiment to be very great, the calculated moment for a rectangular section being less than one half, while for an I section it is about equal that found by experiment. The causes just pointed out only partially account for this, especially as Mr. Barlow's experiment cited above appears to show that no considerable deviation of the neutral axis takes place, and it is probable that the lateral connection of the several layers of the beam has (near the breaking point) a sensible influence on the strength of the parts of the beam exposed to tensile stress, a question we shall return to farther on.

SHEARING AND TORSION. COMPOUND STRENGTH.

221. *Shearing and Torsion.*—We now pass on to cases where the ultimate particles of the material are subject not to a simple longitudinal stress, but to stress of a more complex character. The simplest case is that of a simple distorting stress where the stress consists of a pair of shearing stresses (Fig. 140) on planes at right angles, or what is the same thing (Art. 183) of a pair of equal and opposite longitudinal stresses (Fig. 141) on planes at right angles. Examples of this kind of stress occur in shearing, punching, and twisting. Experiments on shearing are subject to many difficulties and are often not conducted in such a way as to satisfy the conditions necessary for uniformity of distribution of stress on the section. Moreover they necessarily give the ultimate resistance only without reference to the limit of elasticity. The whole process of shearing and punching is very complex, being at

the commencement of the operation usually accompanied by a flow of the metal similar to that already referred to. Thus, when a hole is punched in a thick plate the punch sinks deep into the plate before the actual punching takes place, the metal being displaced by lateral flow, and the piece ultimately punched out being of less height than the thickness of the plate.*

Separation takes place in the first instance by the formation of fine cracks inclined at 45° to the plane of shearing. In soft materials the surfaces slide past each other and separate, but in harder materials there is a strong tendency to the formation of an oblique fracture. In wrought iron and steel the ultimate resistance to shearing is probably about three fourths the ultimate resistance to tension of the same material. The question of a theoretical connection between the elastic strengths in the two cases is considered further on.

Experiments on torsion are not numerous, and many of those which exist are not experiments on simple twisting, but on a combination of bending and twisting. Such experiments would be of great value if accompanied by corresponding experiments on simple twisting and bending made on similar pieces of material. It is known however that in the ductile metals the elastic resistance to torsion is less than the resistance to tension. A series of experiments on torsion made by Prof. Thurston give some interesting results.† Curves are drawn the abscissæ of which represent angles and the ordinates twisting moments, and the form of these curves shows that in some cases defective homogeneity causes a great deficiency in the elasticity at small angles of torsion. In general, however, the curves closely resemble the ordinary curve of stress and strain, already given for a stretched bar, being nearly straight up to a certain point and then curving towards the axis.

In twisting, as in bending, after passing the elastic limit, the stress at each point of the section, instead of varying as the distance from the centre as it must do in perfectly elastic material, varies much more slowly so as to become partially equalized. Hence the twisting moment corresponding to a given maximum stress is greater than it would be if the elasticity were perfect. In the case where the equalization is perfect it is easy to show that the twisting moment is increased in the proportion 4 : 3, a result first given in 1849 by Prof.

---

* On this subject see M. Tresca's paper cited above, and two articles in the Journal of the Franklin Institute.

† See Paper on *Materials of Machine Construction*, read before the American Society of Civil Engineers, 1874. No diameters are given, except for the woods, so that the stress corresponding to the limit of elasticity cannot be found.

J. Thomson. The curves given by Thurston show that in many cases an approximately constant twisting moment was reached indicating that nearly complete equalization must have existed. On the case of cast iron see Art. 223.

222. *Theories of Compound Strength.*—A simple distorting stress is included in the more general case of three simple longitudinal stresses of any magnitudes acting on planes at right angles. To this, indeed, all cases of stress can be reduced, and if we knew the powers of resistance of a material to three such stresses simultaneously, all questions relating to strength of materials could (at least theoretically) at once be answered. Unfortunately experiments fitted to decide the question have not hitherto been made, and in consequence hypotheses have explicitly or implicitly been resorted to.

First, it is often tacitly supposed that the powers of resistance of a material to a simple longitudinal stress are unaffected by the existence of a lateral stress. For example, if a material bears 10 tons per square inch under a simple stretching force, it is assumed that when formed into a pipe and exposed to internal fluid pressure it would also bear 10 tons on the square inch if the pipe were homogeneous and free from joints, notwithstanding the fact that the material is exposed to stress (Art. 150) tending to tear it transversely as well as longitudinally. It is, however, far from probable that this can be the case, at any rate as regards the elastic strength. In ductile materials, the limit of elasticity of which depends as we have seen on rigidity, any lateral force must raise or lower the elastic limit according as it acts in the same direction as the longitudinal stress or in the opposite direction.

Secondly, it may be supposed that the maximum elongation or contraction of a material in a given direction must be a certain definite quantity, irrespective of any elongation or contraction in any other direction. This theory leads to results which are more probable than the preceding, and as it has been much employed by Continental writers we shall give some examples.

Let us take a piece of wrought iron and imagine that when exposed to a simple stretching force its limit of elasticity corresponds to a stress of 10 tons per square inch, accompanied by an elongation of $\frac{1}{1200}$th of its length. The second theory asserts that the maximum admissible elongation is still $\frac{1}{1200}$th, even though the sides of the bar be acted on by any force, the effect of which will be that quite a different longitudinal stress will be required to produce that elongation.

The relations between stress and strain are expressed by the equations (Art. 206)

$$Ee_1 = p_1 - \frac{p_2 + p_3}{m};$$

$$Ee_2 = p_2 - \frac{p_1 + p_3}{m};$$

$$Ee_3 = p_3 - \frac{p_2 + p_1}{m}.$$

The first theory supposes that p_1 can never exceed 10 tons, and the second that e_1 can never exceed $\frac{1}{1200}$th (or Ee_1 10 tons), whatever p_2, p_3 are. In the case of a thin pipe under internal fluid pressure $p_3 = 0$ (nearly), $p_2 = \frac{1}{2}p_1$ (Art. 150); thus assuming $m = 4$ we have on the second theory

$$10 = p_1 - \frac{p_1}{8}, \text{ or, } p_1 = 11 \cdot 43,$$

so that the material will bear under these circumstances a stress of 11·43 tons per square inch as safely as it bears 10 tons under simple tension, and this value, therefore, may be assumed for the co-efficient in the formula which gives the corresponding internal pressure. In like manner in the case of a thin sphere the material will bear a stress of $13\frac{1}{3}$ tons per square inch, being an increase of 30 per cent.

223. *Connection between the Co-efficients of Strength for Shearing and Tension.*—On either theory the resistance to a simple distorting stress may be found in terms of the resistance to simple tension, for such a stress consists (p. 326) of a pair of equal and opposite simple stresses of equal intensity. In the first case the resistances to tension and shearing ought to be equal, in the second, since, writing $p_2 = -p_1$, we find

$$Ee_1 = p_1 + \frac{p_1}{m},$$

$$\text{or, } p_1 = \frac{m}{m+1} Ee_1,$$

it follows that the resistance to shearing is $\frac{m}{m+1}$ or about four fifths the resistance to tension, a result on the whole borne out by experience. It should be remarked that the theory only professes to give a connection between the elastic resistances in the two cases, the equations only holding good for perfectly elastic material, which, moreover, must be supposed isotropic. The ultimate resistance to torsion of cast iron is much greater than its resistance to tension, which is probably due to the same causes as in the case of bending.

Now, rigid materials on this theory are imagined to give way to longitudinal compression, when the lateral expansion produced by the compression is the same as would be produced by a simple tensile stress; from which it appears that the elastic resistance to compression should be from three to four times the elastic resistance to tension, as may easily be supposed to be the case.

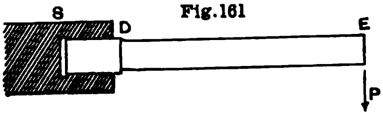
A third theory, more easily conceivable *a priori*, is to suppose that each material is capable of enduring, without injury to its elasticity, a certain definite change of volume and a certain definite change of shape. We thus have two co-efficients of elastic strength analogous to the two fundamental constants which express the other elastic properties of isotropic matter. On this theory, if the resistance to a simple distorting stress in any plane be independent of the existence of any other kind of stress whether fluid or otherwise, as in fact is the case before the limit is reached, it would follow that this resistance must be one half the resistance to a longitudinal stress. It is probable that some theory of this kind may ultimately prove to be the true one; but, in the absence of the necessary experimental data, the second theory may be provisionally assumed, as its results have not as yet been contradicted by experience. It is applied by first finding the principal stresses as in Ch. XVII., and then deducing the principal strains as just now explained. The greatest of these strains multiplied by E may be described as the "equivalent simple tensile stress," and should not exceed the limit prescribed by the strength of the material.

REPETITION AND IMPACT.

224. *Wöhler's Experiments on Alternate Bending and Twisting.*—In bodies which satisfy the definition of perfect elasticity a load within the elastic limit produces no permanent change, unless perhaps some thermodynamic effect, and it follows from this that after removal the body is completely uninjured, so that the load may be repeated indefinitely. Experience confirms this conclusion. The balance spring of a watch bends and unbends more than a million times a week for years together, and the parts of a machine if originally sufficiently strong, remain so to all appearance for an indefinite time. But, if the load be beyond the elastic limit, permanent changes are produced, and there is every reason to believe that a slow deterioration of strength, due perhaps to some kind of internal abrasion, is ultimately destructive. The most definite information on this point is furnished by the experiments of M. Wöhler published in 1870. Bars were loaded in various ways and the load wholly or partially removed: the process was repeated till the bar broke; the number of repetitions necessary for this purpose being counted was found to depend, first, on the maximum stress and, secondly, on the fluctuation of stress.

First suppose the stress alternately tensile and compressive of equal intensity. Wöhler tried this both in bending and in twisting. Figure 161 represents a round bar *DE*, with one end enlarged and fitted into a socket in a revolving shaft *S*. At the free end *E* a load *P* was applied, which produced at *D*, the point of maximum bending, a stress of intensity found by the usual formula. The shaft being set in motion the piece of material was bent alternately backwards and forwards once in each revolution. A number of pieces being tried successively with gradually diminishing loads, the revolutions necessary to produce fracture were found to increase as shown by the annexed table for the case of wrought iron. The pieces broken were exactly similar, and we therefore find a regular increase in the number of revolutions necessary to produce fracture as the stress diminishes. It is already very large at 18,700 lbs. per square inch, and at 16,600 the piece cannot be broken at all. We may therefore place the resistance to alternate bending of this kind of iron at about 17,000 lbs. per square inch, while for cast steel of various qualities it was found to range from 25,000 to 30,000, and for copper 10,400. These results do not differ much from the limit of elasticity of the materials in question as determined in the usual way by experiments on tension. Indeed we have here the most satisfactory definition of the limit of elasticity. If we attempt to define the elastic limit as the stress at which the material ceases to possess the properties of a perfectly elastic body, we are embarrassed by the small and variable deviations which we find under almost any load, which only gradually and at very

Fig. 161



ALTERNATE BENDING OF A BAR OF AXLE IRON FURNISHED BY THE PHŒNIX COMPANY IN 1857.

STRESS IN LBS. IN SQ. INCH.	REVOLUTIONS.	REMARKS.
33,300	56,430	The last of these pieces was unbroken after more than 132 million revolutions. The ultimate tensile strength of this iron was 47,000 lbs. per square inch and the elongation about 20 per cent.
31,200	99,000	
29,100	183,145	
27,000	479,490	
25,000	908,800	
23,000	3,632,588	
20,800	4,918,000	
18,700	19,187,000	
16,600	——	

different loads under different circumstances pass into the large differences characteristic of the non-elastic state. The resistance to unlimited alternate stress however is a definite quantity which, so far as we know, is independent of the causes which produce these variations.

Similar experiments were made with a different apparatus on alternate twisting. They were less extensive, but led to the important conclusion that the strength of the qualities of steel for which they were tried was four fifths that of the same steel under alternate bending. From this it is inferred that the proof resistance to shearing is four fifths the proof resistance to tension, as required by a theory of strength already referred to. (See Art. 223.)

225. *Influence of Fluctuation of Stress.*—It had already been shown by Prof. J. Thomson, in a paper published in 1848,* that twisting or bending a bar beyond its elastic limit in one direction must increase its powers of resistance to a second strain in the same direction, and diminish it to a strain in the opposite direction. Accordingly, we find that when a bar is strained in one direction only its powers of resistance to unlimited repetition are greatly increased. Wöhler made very extensive experiments on stretching, bending, and twisting of pieces of iron and steel to a given maximum stress, the load being wholly or partially removed at each repetition. The number of repetitions necessary for fracture was found to vary, not only according to the magnitude of the maximum stress, but also according to the fluctuation. It was greater when the load was only partly removed than when it was wholly removed. Some results are given in the annexed table, which shows the limits between which the stress varied when fracture was just not produced by unlimited repetition.

Resistance to Unlimited Repetition of Bending.

Nature of Fluctuation.	Fluctuation of Stress.			
	Iron.		Steel.	
Alternating,	+17,000 ;	- 17,000	+29,000 ;	- 29,000
Load wholly removed,	31,000 ;	0	50,000 ;	0
Load partially removed,	45,000 ;	25,000	83,000 ;	36,500

Remark.—The ultimate tensile strength of the iron was 47,000, and of the steel, 106,000.

Any greater fluctuations with the given maximum stress, or any greater maximum stress with the given fluctuation, produced fracture. Experiments on stretching and twisting led to similar results, and it should be especially noticed that in cases of unlimited repetition the resistance to stretching is the same as the resistance to bending, but the resistance to twisting less. In the case of cast iron the resistance to stretching with complete removal of load was found to be 10,400, but no experiments on bending or twisting were made.

---

* *Cambridge and Dublin Mathematical Journal.*

Thus it appears that the ultimate strength of a material is very different according to the fluctuation in the load to which it is exposed; the same iron, which will bear only 17,000 lbs. per square inch when bent alternately backwards and forwards, will bear 31,000 when bent in one direction only, and 45,000 when the stress varies between 25,000 and 45,000. Several formulæ have been devised to represent the results of the experiments, of which one will now be given.* Let p_0 be the ultimate tensile strength of a material and Δ the fluctuation, then the actual ultimate strength under unlimited repetition will be

$$p = \tfrac{1}{2}\Delta + \sqrt{p_0(p_0 - \tfrac{3}{2}\Delta)}.$$

When $\Delta = 2p$ we get the case of alternate stress with which we commenced, where $p = \frac{1}{3}p_0$, and when $\Delta = p$ we have the case of repeated stress in one direction with complete removal at each repetition. The formula gives the same results as the experiments in the extreme cases, and may be expected to be approximately correct in intermediate cases

226. *Impact.*—In Wöhler's experiments the load was applied. without shock. In cases of impact also there is reason to believe that within the limit of elasticity a material will bear unlimited repetition. Thus in Hodgkinson's experiments on beams struck by a pendulum weight, it was found that if the blow produced less than one third the ultimate deflection, the beam would sustain more than 4,000 blows without apparent injury, a plate of lead being introduced to prevent local damage.

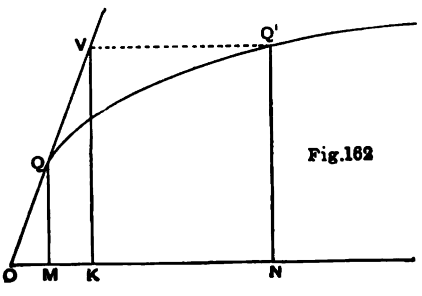


Fig.162

In most cases of impact, however, the elastic limit is exceeded, and the destructive effect of repetition is then much greater than when the load is gradually applied. In the ductile metals the resistance to impact is at first very great, as has already been sufficiently explained; but every time the limit of elasticity is overpassed the hardness of the metal is increased, so as to make it less able to resist the second blow. This may be illustrated by a diagram in which OQQ' is a curve of stress and strain, Q the original elastic-limit, $Q'N$ the stress produced by the first blow, so that the area $OQQ'N$ represents the energy of that blow. The effect of the blow is to raise the limit from the stress QM to the stress $Q'N$ nearly. Hence the curve of stress and strain now becomes OVS, where V is the new limit, and the material will only bear a blow the energy of which is the triangle OVK, without the original stress $Q'N$ being exceeded. Thus by constant repetition of blows, which originally only produced a stress not much exceeding the elastic limit, a much greater stress may be produced. It is believed that this is in the main the explanation of the destructive effect of repeated blows and continuous severe vibration: pieces of material exposed to which are found to have a short life. The effect may be further augmented by synchronism (Art. 197, p. 348).

---

* *Elements of Machine Design*, by Prof. W. C. Unwin, p. 25.

CO-EFFICIENTS OF STRENGTH AND FACTORS OF SAFETY.

227. *Factors of Safety and Co-efficients of Working Strength.*—Before we can apply theoretical formulæ to the determination of the dimensions of actual structures and machines, it is necessary to know the value of the co-efficients of strength to be used, and this is always a matter which requires great care and attention to the circumstances under which certain dimensions are found to be sufficient by long practical experience. In the first instance it depends on the ultimate strength of the material, and may be expressed by dividing that quantity by a Factor of Safety. But the ultimate strength varies as we have seen, and the word "factor of safety" is used with various meanings.

The primary meaning of the expression is the divisor necessary to provide a margin of strength for unknown contingencies such as the following.

(1.) The ultimate strength of a piece of material is uncertain, for two pieces of material of the same description and manufacture are not always equally strong. The liability to variation is much greater in some materials than others, for example, in cast iron than in wrought iron. The strength of stone varies so much that, in carrying out any important work, experiments are frequently made on the stone to be employed in it.

(2.) The piece of material may be subject to corrosion or other influence, which in course of time diminishes its strength.

(3) Errors of workmanship are unavoidable, and in some instances may greatly increase the stress to which the material is exposed. This,

for example, is the case in pillars, the factor of safety for which must always be greater than for other parts of a structure.

(4.) The magnitude of the load and its mode of application is generally more or less uncertain. This however may be provided for by assuming a maximum load.

The factor required to provide for contingencies such as these may be called the "real" factor of safety, but by an addition to its value it may be made to provide against contingencies which can if necessary be exactly foreseen and calculated. Assuming all the forces acting on a structure to be known it is possible to find the stress on each part of it, but the calculation may be too complex to be often used, or its result may be known approximately under similar circumstances. Hence it often happens that the dimensions of a piece are determined by a formula involving only part of the straining forces which act on it, and the rest are provided for by an increased factor of safety. Thus the real stress on the metal of a screw bolt, when the effect of screwing up is taken into account, may be double the total tension per square inch of the gross sectional area. If that bolt be used for a cylinder cover exposed to steam pressure the total tension will be much greater than that due to the pressure of the steam. These two circumstances taken together may be taken into account by the use of a factor of safety three or four times greater than the real one. Such cases are common in practice, but the factor to be used must then be determined by comparisons with good examples under similar circumstances.

Again, it is necessary that a piece should be stiff enough as well as strong enough, and when formulæ for strength are used in such cases it is often necessary to employ very large and very arbitrary factors of safety. Here however the difficulty arises from an erroneous method of calculation.

228. *Values of Co-efficients.*—In parts of machines subject to alternating straining actions we know by Wöhler's experiments that the ultimate strength is somewhat less than the elastic strength under simple tension, being for wrought iron and soft steel about one third the ultimate tensile strength. The load on such parts will rarely be applied without shock, the effect of which cannot precisely be determined. In ordinary cases it will be sufficient to treat this case as if the load were suddenly applied by using a further divisor of 2. We thus obtain the working strength by using a total factor of safety of 6. For wrought iron this gives a co-efficient of 4 tons, or 9,000 lbs. per square inch, which is known by experience to give sufficient strength where all the straining actions are taken into account. In long struts a factor of 8 or more must be used for reasons already sufficiently explained, and this is also necessary where, as has been the case, till lately with steel, the material is not completely reliable. For timber the usual factor is 10. The co-efficient for shearing and torsion is to be taken provisionally as four-fifths that for tension and bending, that is for wrought iron 3¼ tons per square inch ; but from the incompleteness of experimental data it is not certain that this value is not too large.

In structures the fluctuation of the straining actions is in general much less, and the ultimate strength by Wöhler's experiments is much greater. Yet the working strength employed is not very different. In the first place, it is rarely permissible to exceed the elastic limit on account of the permanent deformation which ensues. In the second place, the whole of the straining actions on each piece of the structure, especially the effect of imperfect joints, are rarely included in calculations. For example, the friction of pin joints may, under unfavourable circumstances, add 60 per cent. to the maximum stress on the links of a suspension chain (Ex. 4, p. 398). Hence the working strength for wrought iron rarely exceeds 4½ or 5 tons per square inch. In reckoning the load Rankine recommended that the "dead" load should be divided by 2 and added to the "live" load in order to obtain the effective live load. More recently the importance of Wöhler's experiments has been recognized, and it has been proposed to find the ultimate strength of each piece under the maximum stress and fluctuation of stress to which it is subject, and divide by a constant factor of safety. There can be no doubt that a smaller co-efficient is necessary the greater the fluctuation, bnt the principle of a constant factor is open to question : it appears to lead either to co-efficients which are smaller than are known to be safe, or else to values above the limit of elasticity.

229. *Fibrous Materials Ropes.*—Fibrous materials are those which may be regarded as made up of fibres, usually of organic origin, more or less closely united by cohesion or interlacing. The relative movements of the fibres are hindered by forces of the nature of friction, which are much less than the molecular forces to which the tenacity of a homogeneous solid body is due. Hence the strength and stiffness of a piece of material are much less than those of the fibres of which it is made up.

In most kinds of woods the fibres are arranged longitudinally, and the material is therefore especially characterized by its low resistance to division into parts longitudinally. Thus the resistance to longitudinal shearing of fir timber is only 600 lbs. per square inch, whereas its tenacity is about 20 times this amount, approaching that of cast iron.

So, again, crushing takes place by longitudinal splitting under a stress little more than half the tenacity. Further, the condition of the material greatly influences the lateral cohesion of the fibres and thus affects its strength and elasticity. In timber which has been artificially dried the elasticity is nearly perfect up to the breaking point, whereas in the green state the elasticity is imperfect and the strength greatly reduced. Hence the importance of seasoning timber so as to be moderately dry.

The ordinary formulæ, however, will apply in all cases where the stress is a simple longitudinal stress, the direction of which is that of the fibres; that is to say, in tension, compression, and ordinary cases of bending. They will only fail when the bending is accompanied by crushing and shearing of considerable intensity, as when short pieces are acted on by transverse forces.

In cloth and similar materials two sets of fibres at right angles are united by interlacing. Resistance to tension is thus obtained with almost complete flexibility.

In ropes of all kinds the fibres are ranged in spiral curves in the process of manufacture, and their tension then produces lateral pressure, the friction arising from which is sufficient for union. The strength of a rope, though very great compared with its weight, is only one third that of the yarn of which it is spun, and on a similar principle the strength of large cables is less than that of the smaller ropes called "hawsers" of which they are made up. The strength of a rope is usually expressed by the formula

$$T=\frac{C^2}{k},$$

where C is the girth of the rope in inches, T the tension in tons, and k a constant. The old rule in the navy was to take $k=5$ to obtain the breaking weight of a rope, but the table now employed gives $k=3{\cdot}3$, that is, a strength 50 per cent. greater. In small ropes k may be even less. The safe working load is not more than one sixth the breaking load. In iron wire ropes $k=1$, or for ropes above 6 inches girth somewhat more. The strength of wire ropes is more than doubled by the employment of steel. The safe working load may be taken as one fifth their breaking load.

230. *Tables of Strength.*—For a detailed account of the properties of materials the reader is referred to the authorities cited above and at the end of this chapter. A convenient summary is given in Rankine's *Useful Rules and Tables.* It will be here sufficient to give a few examples.

TABLE I.—WEIGHT AND WORKING STRENGTH OF VARIOUS MATERIALS.

MATERIAL.	WORKING STRENGTH.				WEIGHT PER YARD IN POUNDS.			Working Strength in Feet of Material = λ	
	Stress in Tons per Square Inch.		Area in Square Inches per Ton.		Per ton of Stress under Working Load.		Per Square Inch of Area.		
	T.	C.	T.	C.	T.	C.		T.	C.
Cast Iron, - -	1·5	4·5	·667	·222	6	2	9	1120	3360
Wrought Iron, -	4·5	4·5	·222	·222	2·22	2·22	10	3024	3024
Ordinary Soft Steel,	7	7	·143	·143	1·43	1·43	10	4700	4700
Ordinary Steel Wire,	13	—	·077	—	·77	—	10	9000	—
Copper Wire, -	4	—	·25	—	2·9	—	11·5	2320	—
Deal, - - - -	·5	·3	2	3·3	1·5	2·5	·75	4480	2700
Oak, - - - -	·75	·45	1·33	2·22	1·33	2·22	1·	5040	3024
Granite, - -	—	·3	—	3·33	—	11·66	3·5	—	576
Brickwork, - -	—	·06	—	15·6	—	39	2·5	—	160
Hemp Ropes, -	·6	—	1·67	—	2·5	—	1·5	2700	—
Iron Wire Ropes, -	2	—	·5	—	2·6	—	5·25	2600	—
Steel Wire Ropes,	5	—	·2	—	1·1	—	5·5	6000	—

TABLE II.—ELASTICITY AND RESILIENCE.

MATERIAL.	ELASTIC STRENGTH.						ELASTICITY (Tons and Inches).		RESILIENCE UNDER TENSION.	
	Stress in Tons per Square Inch.			Strain.			Young's Modulus.	Rigidity.	Foot Pounds per Cubic Foot.	Height in Feet and Inches.
	T.	C.	S.	T.	C.	S.				
Cast Iron, - -	3	9	—	·000375	·001125	—	8000	—	185	4″·5
Wrought Iron, -	9	9	7	·0007	·0007	·0014	13000	5000	1060	2′ 2″
Soft Steel, - -	15	15	12	·0012	·0012	·0024	13000	5200	2900	6′
Hard Steel, - -	25	25	20	·002	·002	·004	13000	5200	8000	16′ 6″
Tempered Steel, -	50	—	—	—	—	—	15000(?)	—	34500	72′
Strongest Steel Wire,	150	—	—	·0115	—	—	13000	—	276000	577′
Fir, - - - -	1½	—	—	·0021	—	—	700	35	2150	58′
Oak, - - - -	2	—	—	·0028	—	—	700	35	4300	86′

TABLE III.—ULTIMATE STRENGTH AND DUCTILITY.

MATERIAL.	Ultimate Strength in Tons per Square Inch.			Elongation per Cent.	$\frac{\text{Tension} \times \text{Elongation}}{2}$
	T.	C.	S.		
Iron Bars, - - - - - - -	25	22	18	20	250
Iron Plates, - - - - - -	22	19	16	10	110
Soft Steel (·15 to ·3 per cent. of carbon),	30	—	22½	25	375
Medium Steel (·3 to ·5 per cent. of carbon)	35	—	27	15	262
Hard Steel (·5 to ·75 per cent. of carbon),	45	—	—	8	180
Cast Iron, - - - - - - -	7½	45	12	—	—
Lead, - - - - - - - -	1½	—	—	—	—
Sheet Copper, - - - - - -	13½	—	—	—	—
Cast Copper, - - - - - -	8½	—	—	—	—
Oak, - - - - - - - -	5½	—	1	—	—
Fir, - - - - - - - -	5½	—	·27	—	—

Table I. gives the weight and working strength of a variety of materials. From what has been said in preceding articles it appears that the working strength varies according to circumstances. Hence the values given in the table may be exceeded, and sometimes greatly exceeded when special care is taken in the selection of material, in the estimation of strains, and in the execution of the work. On the other hand cases occur in which they are too large, and, it may be, greatly too large if due care is not exercised. The first two columns give the safe load in tons per square inch of sectional area, the second two the area necessary to sustain a load of 1 ton, in tension (T) and compression (C) respectively. The next two give the weight of 1 yard length of a bar which will sustain 1 ton, and the numbers therein given are therefore the comparative weights of bars of equal strength. The same comparison is effected in a different way in the last two columns, which give the length in feet of a bar or column the weight of which is equal to the working load on its transverse section. It is on this quantity, which is denoted by λ in Arts. 40, 41, pp. 80, 81, that the limiting dimensions of a structure depend. It is used for this purpose in Ex. 13, p. 294, and Ex. 11, p. 338. It will be observed that weight for weight timber is stronger than wrought iron.

Table II. gives the elastic properties of certain materials in tension (T), compression (C), and shearing (S). It has been sufficiently explained in proceding chapters, and it need only be added that the elastic strength as well as the working strength varies. It may be very different according to the exact definition of the limit of elasticity adopted and according to the condition of the material. This point has already been discussed, but some further remarks will be found in the Appendix.

Table III. shows the ultimate strength and ductility of materials in common use. The first three columns give the ultimate resistance to tension, compression, and shearing. The fourth gives the elongation expressed as a percentage of the original length, which, if the length of the pieces experimented on be a constant multiple of the diameter, forms a measure of the ductility. The ultimate strength and ductility of steel vary according to the amount of carbon it contains in such a way that the sum of the two remains nearly constant, other things being equal. Thus in the examples given in the table the sum is about 53. In steel compressed in a fluid state by Sir J. Whitworth's process the constant sum is about one third greater. The last column gives half the product of the ultimate tensile stress and the elongation, a quantity which is sometimes used as a measure of the powers of resistance to impact. The actual amount of work done in stretching a bar till it breaks is much greater than this, as is seen on considering the form of the curve of stress and strain. A more exact measure of the resistance to impact would be furnished by an experiment on a short block such as that described on page 380.

EXAMPLES.

1. Show that the modulus of rupture of a material is 18 times the load which will break a bar of the material 1 inch square and 1 foot long: the bar being supported at the ends and the load applied at the centre.

N.B.—The modulus of rupture is the value of the co-efficient in the ordinary formula for bending when the load is that found by experiment to break the beam.

2. A balcony, 6 feet long and 4 feet broad, is supported by a pair of cast-iron beams fixed in the wall at one end. The beams are of rectangular section, 2 inches broad, and depth near the wall 4 inches. What load per square foot will the balcony bear, the stress on the iron being limited to 1 ton per square inch? Also, how should the depth vary for uniform strength along the length of the beam?

Ans. Equating the greatest bending moment to the maximum moment of resistance to bending we find the load which the balcony will bear

$$= 41 \cdot 5 \text{ lbs. per square foot.}$$

As to the depth of the beam: for uniform strength $\frac{M}{I}y$ must be constant from which we find that the depth at any point of the beam must be proportional to the distance from the outer end of the beam; so that the lower side of the beam should be a sloping plane.

3. A paddle shaft is worked by a pair of engines with cranks at right angles. Supposing the steam pressure constant, and the resistance of each wheel equal and uniform, and obliquity of connecting rod neglected; compare the co-efficients of strength to be used in calculating the diameter of the paddle and intermediate shafts.

Ans. The uniform moment of resistance of the paddle wheel = ½ the mean turning moment of the two engines. The twisting moment of the paddle shaft, when either crank is on the dead centre, = ½ maximum twisting moment of one engine. At the same instant this is the twisting moment on the intermediate shaft. When the other crank is on the dead centre the twisting moment on intermediate shaft is the same in magnitude, but reversed in direction, and when the two cranks make angles of 45° with the dead centres the twisting of the paddle shaft = ½ the maximum combined twisting moment of the two engines, that is $\sqrt{2}$ times its amount when either crank is on the dead centre; but the twist is in the same direction always. Therefore on the paddle shafts the stress alternates between x and $x\sqrt{2}$, and on the intermediate shaft between x and $-x$.

Hence applying formula

$$p = \tfrac{1}{2}p + \sqrt{p_0(p_0 - \tfrac{3}{2}\bar{p})},$$

we have for paddle shaft,

$$\bar{p} = \cdot 414x ; \ p = 1 \cdot 414x ; \ \therefore \bar{p} = \cdot 292p ;$$

substituting, we obtain

$$p = \cdot 9 \, p_0.$$

For intermediate shaft, $\bar{p} = 2x$; $p = x$; $\bar{p} = 2p$; and $p = \tfrac{1}{3}p_0$.

If the stress on the paddle shaft alternates to zero, by the wheels rolling out of the water, or by the stopping of the engine, then $p = \cdot 6p_0$.

4. A suspension chain is constructed with bar links united by pin joints; the diameter of the pins is two-thirds the breadth of the link (p. 337). If the bridge vibrate show that the maximum stress on the links may be increased by deviation (p. 309) due to friction of pins (p. 223) in the ratio $1 + 2f : 1$, where f is the co-efficient of friction.

mpact
that

h will
ed at

mals

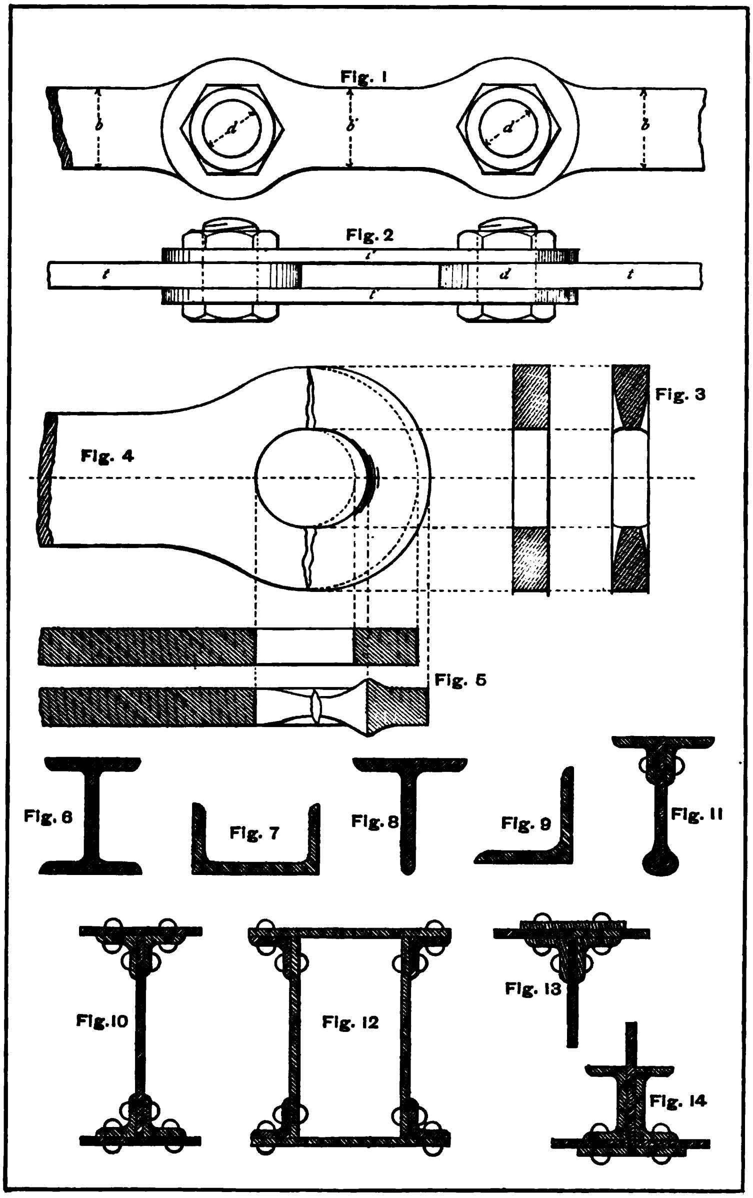
ame
and
ress
for

ace

re

e
g

PLATE VIII.



To face page 309.

AUTHORITIES ON STRENGTH OF MATERIALS.

In addition to the works expressly cited in this chapter may be mentioned—

HODGKINSON. *Experimental Researches on the Strength and other Properties of Cast Iron.* Weale. 1846.
WEYRAUCH. *Iron and Steel.* New York. 1877.
BARLOW. *Strength of Materials.* Lockwood.
REED. *Shipbuilding in Iron and Steel.* Murray.

The literature of the subject is however very extensive, much information being scattered in various memoirs, of which two need only be mentioned here as having been much employed in the preparation of this treatise—

FAIRBAIRN. *Mechanical Properties of Steel.* Report of the British Association for 1867.
WÖHLER. *Die Festigkeits-Versuche.* Berlin. 1870.

DESCRIPTION OF PLATE VIII.

To illustrate various questions considered in Chaps. XII. and XV. Plate VIII. has been drawn.

Figs. 1, 2 represent the pin joint connecting two bars in tension, discussed in Art. 191, p. 337. Figs. 3, 4, 5 show the way in which the joint yields when the pins are too small. In Fig. 4 the original dimensions of the eye and eyehole are shown by dotted lines, while the full lines show what they become after yielding. Fig. 3 gives transverse sections of the eye before and after failure, showing the thinning out due to lateral contraction during stretching beyond the elastic limit. After this contraction has reached a certain limit the metal tears asunder, as shown in Fig. 4. The longitudinal section (Fig. 5) shows the corresponding spreading out at the top of the hole due to compression beyond the elastic limit. This lateral expansion is partially prevented in rivetted joints, and (p. 387) this may be the reason why direct stress in them is of less importance. The failure of pin joints in this way furnishes a good example of the "flow of solids."

The remaining figures of this plate are intended to give some idea of the manner in which iron girders are constructed. Figs. 6, 7, 8, 9 are transverse sections of "H iron," "channel iron," "tee iron," and "angle iron": these are rolled in one piece and, in combination with plates, form the materials from which large girders are built up. For small beams such as floor joists H iron or tee iron of the requisite depth and sectional area may be used. Figs. 10, 12 are sections of two of the commonest forms of built-up girders. In the first the web is a single plate to which angle irons are rivetted to form the flanges, further strength being obtained by an additional covering plate. The second is similar, but the web consists of a pair of plates, a form known as a "box beam." Fig. 11 is commonly used in shipbuilding as a deck beam or otherwise: a "bulb iron" here forms the web and lower flange, while the upper flange is formed by a pair of angle irons as before. Figs. 13, 14 give examples of girders of more complex construction employed where greater strength is necessary: one flange only is shown in section in each case. For further details the reader is referred to the treatises by Mr. Hutchinson and Sir E. Reed, cited on page 53 and above.

PART V.

TRANSMISSION AND CONVERSION OF ENERGY BY FLUIDS.

PART V.—TRANSMISSION AND CONVERSION OF ENERGY BY FLUIDS.

231. *Introductory Remarks.*—We now return to the subject of Machines with the object of studying those machines in which fluids are employed as links in a kinematic chain for the purpose of transmitting energy, or as a means by which energy is supplied, stored, or converted.

A fluid is a body in which change of form is produced by the action of any distorting stress, however small, if sufficient time is allowed. In a perfect fluid a sensible change would be produced by a stress of sensible magnitude in an indefinitely short time, but in all actual fluids a time is required which is inversely as the stress—that is, the stress is proportional to the rate of change. This property of fluids is called Viscosity, and is measured by a co-efficient, as will be seen hereafter. The viscosity of a fluid varies greatly in different fluids, and, in the same fluid, is dependent on the temperature. At high temperatures it is much less than at low temperatures. The viscosity of water is exceedingly small.

Fluids are either liquid or gaseous. In liquids the changes of volume are in general small, and no diminution of pressure on the bounding surface will cause their volume to increase beyond a certain limit. Gases, on the other hand, expand indefinitely as the external pressure diminishes.

Liquids are employed in machines either as a simple link in a kinematic chain transmitting energy from some source independent of the liquid, or as a medium by means of which the force of gravity exerts energy. Such machines are called Hydraulic Machines, the fluid employed being in most cases water. On the other hand, gases in general serve as the means by which that form of energy which we call Heat is converted into mechanical energy, capable of being utilized for any required purpose. They may, however, also be employed for the storage and transmission of energy.

The motions of fluids may be studied in two different ways. In the first the Principles of Work and Momentum are applied to the whole mass of fluid under consideration, or to portions which, though small, are yet of visible magnitude; but no attempt is made to conceive, much less to determine, the movements of the smallest particles of which the fluid may be imagined to be made up. This method may be described as the experimental theory, and, as applied to water, forms that part of the subject which is called "Hydraulics." It is based directly on experiment, and requires continual recourse to experiment, just as is the case in questions relating to the friction of solids. Nevertheless, being continually verified by the large-scale experiments of the hydraulic engineer, its results, as far as they go, are as certain as those of any purely experimental subject. On the other hand, an analytical theory has been constructed, by means of which the motions of fluids are determined directly from the laws of motion, without reference to experience. This theory is usually called Hydrodynamics in treatises on mechanics. In the cases in which it is applicable it completely determines the motion of all particles of the fluid, and not merely that of the fluid as a whole.

The first two chapters of this division of our work will be devoted to Hydraulics and Hydraulic Machines, and the third to a brief discussion of the various applications of Elastic Fluids. The transmission and storage of mechanical energy by elastic fluids is often considered as part of hydraulics, because the method of treatment is in many respects similar. In this treatise it will be called "Pneumatics." The relations between heat and mechanical energy from a distinct science called "Thermodynamics," the principles of which will only be referred to when absolutely necessary.

CHAPTER XIX.

ELEMENTARY PRINCIPLES OF HYDRAULICS.

232. *Velocity due to a Given Head.*—When the level of the surface of the water in a reservoir is above surrounding objects, a HEAD of water is said to exist, the magnitude of which is measured, relatively to any point, by the depth (h) of the point below the surface. If the water extend to this point a pressure is produced there which, so long as the water is at rest, is given in lbs. per sq. ft. by the formula

$$p = wh,$$

where w is the weight of a cubic foot of water, that is to say, about 62½ lbs. for fresh water, or 64 lbs. for salt. Since the above formula may be written

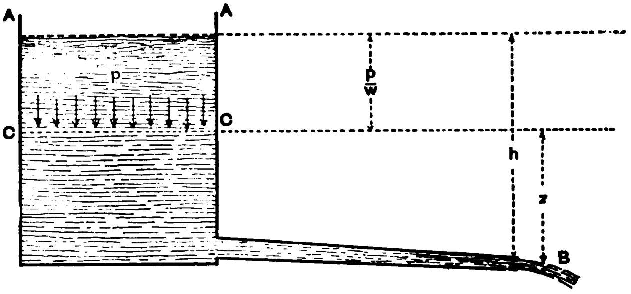
$$h = \frac{p}{w},$$

it appears that a pressure may be measured in terms of the head which would produce it. The fluid is usually water, for which h is reckoned in feet; and 1 lb. per sq. inch is equivalent to 2·3 feet of fresh, or 2·25 feet of salt water. For some purposes, however, mercury is employed, in which case the unit is generally 1 inch. One inch of mercury is equivalent to about ·49 lb. per sq. inch, that is, to a head of 1·1 feet of sea water, or 1·135 of fresh water.

If the surface of the water be exposed to the atmosphere, the pressure p will be in excess of the atmospheric pressure, which must be added to obtain the absolute pressure. The mean value of the atmospheric pressure is 14·7 lbs. per sq. inch, which corresponds to a head of about 33 feet for salt, or 34 feet for fresh water.

A head of water is a source of energy which may be employed in doing work of various kinds, or in simply transferring the water from one place to another. Let us take the second case, and imagine that, by means of a pipe, channel, or passage of any description, the water is delivered at B (Fig. 163), while at the same time, by a stream or otherwise, the surface of the water in the reservoir is kept constantly at the same level AA, so that the head h remains unchanged. The motion is then described as Steady, and consists simply in the transfer in each second of a certain weight of water from the stream to the reservoir, while an equal weight traverses the passage, and is delivered at B, the whole mass of water between AA and B remaining constantly in the same condition. The delivery at B may be supposed found by actual measurement; it is usually estimated in gallons per minute or cubic feet per second, as to which it need only be remarked that the gallon weighs 10 lbs., so that a cubic foot per second is about 375 gallons per minute. For large quantities, however, the cubic metre, which weighs about 1 ton, is also employed.

Fig.163.



On delivery the water is moving with a certain velocity, but the definition and measurement of this quantity is not so simple. We must now suppose that the centre of gravity of the water delivered in some given time is observed and its velocity noted. This velocity will be the same whatever the time be, and will be a measure of the velocity of the mass of water considered as a whole. In some cases all particles of the water may be moving with this velocity, but in general this is not the case: it is then the mean velocity, and may be described as the "Velocity of Delivery." If the water be discharged by a channel which, near the exit, is of uniform transverse section A, this velocity may also be defined by the equation

$$v=\frac{Q}{A}=\frac{W}{wA},$$

where Q is the discharge in cubic feet per second, and W the weight of this quantity.

The energy of motion of the water may now be separated into two parts, one external and the other internal (Art. 134, page 250), of which the first is

$$\text{Energy of Translation}=\frac{Wv^2}{2g},$$

while the second is due to the motions of the particles of water amongst themselves, and will be further considered as we proceed.

The whole energy of motion has been generated by the exertion of an amount of energy Wh due to the descent of the water from the level AA to the level B; and, in cases where the internal energy may be neglected, we have, neglecting also friction,

$$\frac{v^2}{2g}=h,$$

where h the head is measured to the centre of gravity of the issuing water (page 177).

It has been here supposed that the surface of the water in the reservoir, and after delivery at B, is exposed to the atmosphere, but this is not always the case. Suppose in the figure the reservoir filled to the level CC only, but that the pressure on the surface has any value p, instead of being simply that of the atmosphere. This pressure p may be produced by filling up the reservoir to the level AA where

$$h=z+\frac{p}{w};$$

and as the reservoir is supposed large, so that the water is sensibly at rest, except very near the exit, this can produce no change in the motion, which as before is given by

$$v^2=2gh=2g\left(z+\frac{p}{w}\right).$$

In other words, in addition to the actual head z, we have a *virtual* head p/w, due to the difference of pressure p, thus giving a total head h.

The jet of water has been supposed to issue into the atmosphere, but the nature of the medium into which the discharge takes place has little influence, provided its pressure be duly taken into account. It has been proved by experiment that if the pressure of the atmosphere be artificially increased or diminished, the velocity is given by the same formula, modified as explained in the next article. This is also true if the efflux take place into a vessel of water.

233. *Hydraulic Resistances in General.*—The actual velocity v' with which the water is delivered is less than the value v just found, because a certain part of the available energy is always employed in overcoming certain resistances of the nature of friction, the origin of which we shall see gradually as we proceed. They are measured in two ways: (1) by comparing the actual velocity of delivery with that due to the head; (2) by considering how much energy is employed in overcoming them.

In the first method we have only to introduce a co-efficient c given by

$$v' = cv,$$

which is called the Co-efficient of Velocity. It is of course always less than unity, and its value is found by experiment in each special case. In the second we write

$$h - h' = \frac{v'^2}{2g},$$

where h' is the "loss of head" due to the resistance. The value of h' is most conveniently expressed by connecting it with the *actual* velocity v' with which the water issues. For this purpose we replace h by $v^2/2g$ and v by v'/c, and thus obtain

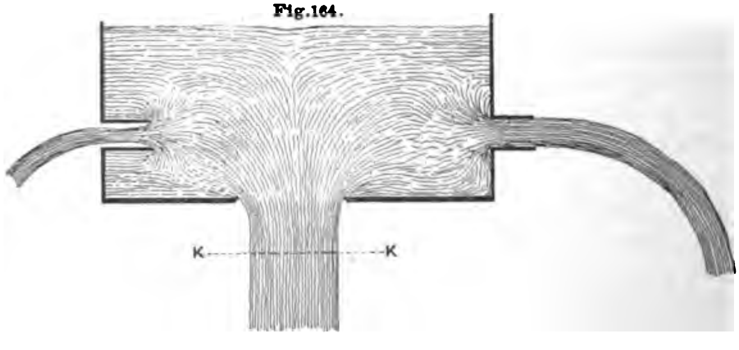
$$h' = \left(\frac{1}{c^2} - 1\right)\frac{v'^2}{2g} = F\frac{v'^2}{2g},$$

where F is a new co-efficient called the Co-efficient of Resistance connected with the previous one by the equation

$$F = \frac{1}{c^2} - 1.$$

It is found by experience that the values of these co-efficients depend mainly on the form and nature of the bounding surfaces within which the water moves, and, subject to proper limitations, not on the pressure or velocity of the water—a fact which may be expressed by the following law of hydraulic resistance: *the energy lost by resistances is a fixed multiple of the energy of motion of the water.* This multiple is the co-efficient F which is sometimes fractional, but is often very large, as we shall see farther on. The physical meaning of this law will be seen hereafter, and the apparent deviations from it which frequently occur will be accounted for

Fig. 164.



234. *Discharge from Small Orifices.*—Fig. 164 shows a vessel of water discharging through a circular hole in the bottom which is flat. The hole is small, and its circumference is chamfered below to a sharp edge at the upper surface.

On observing the jet of water which issues we see that it is nearly cylindrical but of diameter less than the diameter of the hole. The contraction is complete, so far as can be judged by the eye, at a distance of $d/2$ from the vessel; and by measurement is found to be in the ratio 4 : 5, that is, the sectional area of the jet is to the sectional area of the hole in the ratio 16 : 25.

If the hole be made in the vertical side of the vessel a contracted jet issues in the same way, but under the action of gravity it forms a curve which is very approximately parabolic in form, each particle moving nearly in the same way as a projectile *in vacuo.* This enables us to find the velocity of the efflux (v') by observing a point through which the jet passes, and we thus obtain experimentally the value of the co-efficient c, which appears to be about ·97. The discharge is now given by the formula

$$Q = A_0 \,.\, v' = ckA\sqrt{2gh},$$

where A_0, A are the contracted and actual areas of the orifice, and k is their ratio which is a fraction called the Co-efficient of Contraction. The discharge therefore depends on the product of the two co-efficients c and k, which may be replaced by

$$C = ck,$$

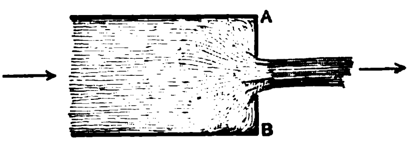
a quantity called the Co-efficient of Discharge.

The value of C can also be determined by direct measurement of the discharge, an observation which can be made with much greater accuracy than those of contraction amd velocity on which it depends. In the present case it is found to be ·62, agreeing well with the product ·97 × ·64 of the values given above.

With other forms of orifice the same co-efficients are used, but their numerical values are quite different. In the figure two cases are represented: on the right side of the vessel the water issues through a short pipe the entrance to which from the vessel is square-edged; on the left a similar pipe is employed but it projects inwards instead of outwards. When the pipe projects outwards the water is found to issue in a jet the full diameter of the pipe, that is, k is unity; while, on the other hand, the velocity is much diminished, the value of c being only ·815. When it projects inwards the jet contracts greatly, the value of k being ·5 while the velocity is about the same as in a simple orifice. Thus C instead of being ·62 is ·815 and ·5 in the two cases. The causes of these remarkable differences will be seen hereafter, the results are only given here to illustrate the meaning of the co-efficients under consideration.*

235. *Incomplete Contraction.*—The contraction of the issuing jet depends on the average angle at which the moving particles converge towards the orifice before reaching it, and this is the reason why it is so great in the case of a short pipe projecting inwards. If the circumstances be such that the convergence is small the contraction diminishes. Fig. 165 shows a pipe of some size through an orifice in the flat end *AB* of which water is being forced, issuing into the atmosphere. The co-efficient k is found to depend on the proportion which the area of the original orifice A bears to that of the pipe S, because the smaller S is, the less is the angle of the convergence. This has been expressed by an empirical formula due to Rankine which may be written

Fig.165.



$$\frac{1}{k} = \sqrt{2\cdot618 - 1\cdot618\frac{A^2}{S^2}},$$

which will be found to give $k = \cdot618$ when S is infinite, as is nearly the case for a simple orifice as explained above, while for smaller values k increases, becoming unity as it should when $S = A$.

In a similar way if an orifice be near a corner of the vessel the contraction will be diminished. In these cases the contraction is usually described as "incomplete."

236. *Discharge from Large Orifices in a Vertical Plane.*—When the orifices are large, compared with the head and the vessel from which the discharge takes place, the question is more complicated.

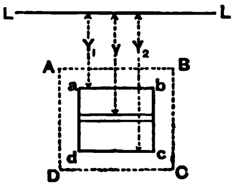
If the plane of the orifice be vertical the velocities of the several parts of the fluid are not the same as is the case, so far as can be judged by the eye, when the orifice is small. On the contrary the velocity of that part of the stream which issues from the lower part of the orifice is visibly greater than that proceeding from the upper part. Hence it follows that the centre of gravity of the fluid issuing in a given time, to which the head is measured, is not on the same level as the centre of gravity of the contracted section, but lies below it. The corresponding point on the section may be described as the Centre of Energy. Also the internal energy of motion of the jet is of sensible magnitude and cannot be neglected.

---

* Readers to whom the subject is new are recommended to pass at once to Art. 241, p. 417.

By supposing that each part flows independently of the rest the discharge can be found for an orifice of any shape. For example, take the case of a rectangular orifice *ABCD* (Fig. 166) from which water is being discharged from a reservoir, the level from which the head is measured being *LL*. The jet contracts on efflux, and the contracted section may be supposed rectangular. The position and dimensions of this section it will be necessary to suppose known by experiment; let its breadth be b, and let its upper and lower sides be at depths Y_1, Y_2 below *LL*. Divide the area into strips, and consider any one at depth y, then the velocity will be given by the formula (neglecting friction),

Fig.166.



$$v^2 = 2gy.$$

The quantity discharged per second will be given by

$$Q = \int_{Y_1}^{Y_2} bv\,.\,dy = b\sqrt{2g}\int_{Y_1}^{Y_2} \sqrt{y}\,.\,dy,$$

which by integration gives

$$Q = \tfrac{2}{3}b\,.\,\sqrt{2g}\,.\,(Y_2^{\frac{3}{2}} - Y_1^{\frac{3}{2}}),$$

which determines the discharge.

The energy of motion of the water discharge per second will be

$$U = w\int_{Y_1}^{Y_2} bv\,.\,\frac{v^2}{2g}\,.\,dy = bw\sqrt{2g}\int_{Y_1}^{Y_2} y^{\frac{3}{2}}\,.\,dy,$$

which by integration gives

$$U = \tfrac{2}{5}wb\sqrt{2g}\,.\,(Y_2^{\frac{5}{2}} - Y_1^{\frac{5}{2}}).$$

By dividing U by wQ we get the depth of the centre of gravity of the fluid discharged per second below *LL*, that is to say, the head h is given by the formula

$$h = \tfrac{3}{5}\,.\,\frac{Y_2^{\frac{5}{2}} - Y_1^{\frac{5}{2}}}{Y_2^{\frac{3}{2}} - Y_1^{\frac{3}{2}}}.$$

The velocity of delivery is

$$V = \frac{Q}{A} = \sqrt{2g}\,.\,\tfrac{2}{3}\,.\,\frac{Y_2^{\frac{3}{2}} - Y_1^{\frac{3}{2}}}{Y_2 - Y_1},$$

and the energy of translation on delivery is

$$U_0 = \frac{wQV^2}{2g},$$

a quantity less than the whole energy wQh by the energy due to internal motions. A common method of treating the question is to measure the head to the centre of the section and then employ the formula

$$V^2 = 2gh$$

with a proper co-efficient of discharge. This method is not exact, for it underestimates both the head and the energy of motion of the water; but its errors partially compensate one another, and its results are made to agree approximately with experiment by the employment of a variable co-efficient. To apply the exact formulæ it is necessary to know the dimensions and position of the contracted section for which the existing experimental data are insufficient. For further particulars on this subject, the reader is referred to Professor Unwin's work cited at the end of this chapter.

Again, if the dimensions of the orifice be not small compared with the surface of the water in the vessel from which the discharge takes place, this surface will sink with a velocity V which is of sensible magnitude.

If the area of the surface be S and that of the contracted section A_0, the discharge will be

$$Q = A_0 v = SV,$$

an equation which determines V. The water will now have a velocity V before descending through the height h, and the equation of energy is therefore,

$$v^2 - V^2 = 2gh,$$

This may be written if we please

$$\frac{v^2}{2g} = h + \frac{V^2}{2g},$$

showing that in addition to the actual head h we must consider the *virtual* head $V^2/2g$ due to the initial velocity of the water. In many hydraulic questions it is inconvenient or impossible to measure the head from still water. It is then measured from some point where the water is approaching the orifice with a velocity V determined by observation. The actual head h must then be increased by the height due to this velocity.

237. *Head relatively to Moving Orifices.*—The passages through which the water is moving may be attached to a ship, locomotive, or other moving structure, in which case the velocity must be reckoned relatively to the structure, and the height due to the velocity must be reckoned as part of the head. If for example in the bow of a vessel moving through the water with velocity V an orifice be opened at the surface level, the water will enter through it, and if unacted on will move within the vessel with velocity V and will possess relatively to the vessel the energy $V^2/2g$ per unit of weight. If it be acted on during entrance by the head due to any difference of level or pressure, so that its velocity is changed from V to v, the corresponding change of energy will measure the work which is done, and therefore the equation $v^2 - V^2 = 2gh$ applies as before.

238. *Steady Flow through Pipes. Conservation of Energy.*—Fig. 167 represents a vessel of water discharging through a large pipe, the section of which varies according to any law. If the pipe "runs full," that is, if it be always completely filled with water, the discharge is

$$Q = A_1 u_1 = A_2 u_2,$$

where u_1, u_2 are the velocities through two sections the areas of which are A_1, A_2. Hence the velocity is always inversely as the sectional area, and in an ordinary pipe in which the section is uniform must be the same throughout. Let the pressures be p_1, p_2, and the actual head, that is to say, the depths below the water surface *CC*, h_1, h_2, then it appears from Art. 232 that

$$\frac{u_1^2}{2g} = h_1 + \frac{p_0 - p_1}{w}\,;\quad \frac{u_2^2}{2g} = h_2 + \frac{p_0 - p_2}{w},$$

where p_0 is the pressure on the surface *CC*.

Take now some convenient line *DD* at a depth *Z* below the water surface *CC*, and let z_1, z_2 be the elevation of the section above this datum level so that

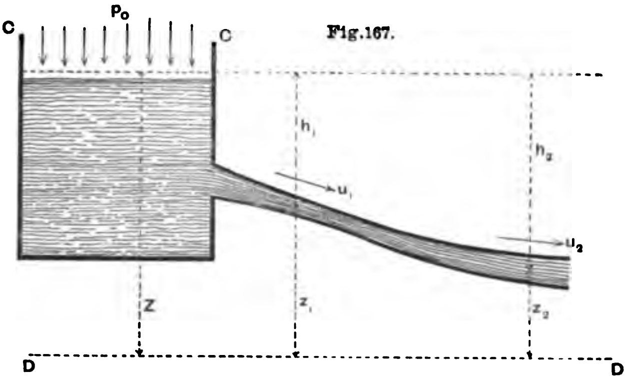


Fig. 167.

$$z_1 + h_1 = Z = z_2 + h_2,$$

then the above equations may be written

$$\frac{u_1^2}{2g} + \frac{p_1}{w} + z_1 = Z + \frac{p_0}{w} = \frac{u_2^2}{2g} + \frac{p_2}{w} + z_2.$$

This result shows that if u, p, z be the velocity, pressure, and elevation for any section of the pipe,

$$\frac{u^2}{2g} + \frac{p}{w} + z = \text{Constant}.$$

Each of the terms of this equation represents a particular kind of energy: the first is energy of motion, the third energy of position, the second is energy due to pressure, the origin of which will be further explained in the next chapter. The equation therefore shows that the total energy of the water remains constant as it traverses the pipe, and is accordingly the algebraical expression of the Principle of the Conservation of Energy. It supposes that no energy is lost by frictional resistances, and that any change in the internal motions of the particles amongst themselves may be disregarded. The word "head," the origin of which we have already seen, is frequently employed for the energy per unit of weight. (See Appendix.)

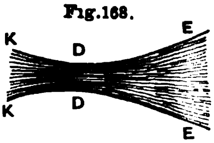
An important consequence of this principle is that where the sectional area of the pipe is least, and consequently the velocity greatest, there the pressure is least. Hence it follows that the velocity cannot exceed a certain limiting value u, found by putting $p=0$. At an elevation z above datum level

$$u^2 = 2g\left(Z - z + \frac{p_0}{w}\right).$$

At a greater velocity a negative pressure would be required to preserve the continuity of the fluid mass, and under these circumstances the water breaks up with consequences to be hereafter considered.

It further appears that water can flow through a closed passage against a difference of pressure, provided the area of the passage vary so as to permit a corresponding reduction of velocity. An example of this occurs in the case of the discharge through a trumpet-shaped mouthpiece. In Fig. 168 water enters from a vessel at KK, an orifice provided with a mouthpiece, which first contracts to DD, and then expands to EE where the jet enters the atmosphere. The pressure at EE is that of the atmosphere, and therefore at DD is less than that of the atmosphere, that is, less than it would be if the trumpet were cut off at the neck. Hence the discharge is increased by the addition of the expanded portion. If the water issued into a vacuum the jet would not expand to fill the wide mouth of the trumpet, which would not in that case have any influence on the discharge. The increased discharge and partial vacuum at DD have been verified by experiment.

Fig.168.



239. *Distribution of Energy in an Undisturbed Stream. Vortex Motion.*—If the reservoir in the last article be imagined to supply a stream running in a channel of any size either closed or open, that stream, if undisturbed by any of the causes mentioned hereafter, may be supposed made up of an indefinite number of elementary streams, each of which moves as it would do in a closed pipe, as just described, without in any way intermingling with the rest. The forms of these ideal pipes depend solely on the form of the channel in which the stream is confined. The equation

$$\frac{u^2}{2g} + \frac{p}{w} + z = Z + \frac{p_0}{w}$$

applies to the motion in every pipe, and from it we may draw two important conclusions. In the first place, it may be written in the form

$$\frac{p - p_0}{w} = Z - z - \frac{u^2}{2g};$$

and therefore, *the pressure at any point is less than if the water were at rest by the height due to the velocity at that point.* Again, the equation interpreted as in the last article shows that the energy of all parts of the fluid is the same, or, as we may otherwise express it, *the energy of the fluid is uniformly distributed.*

From either way of stating the result it appears that the pressure is greatest where the velocity is least, and conversely. Now, if the water move in curved lines in a horizontal plane, each particle of water is at the instant moving in a circle, and to balance its centrifugal force (Art. 132) the pressure on its outer surface must be greater than that on its inner. It follows therefore that, if a channel is curved so as to alter the direction of the stream, the pressure increases as we go from the inner side of the channel to the outer; while, on the other hand, the velocity is greatest at the inner side and least at the outer. The change is the greater the sharper the bend, for the centrifugal force is greater. In open channels the change at the surface where the pressure is constant is in elevation instead of in pressure.

The magnitude of the change can be calculated in certain cases (see Appendix), of which we can only here consider one which is of special importance. If the particles of water describe circles about a common vertical axis, the elementary streams will form uniform rings, the centrifugal force of which can be calculated as in Art. 145, p. 264. The resultant force on the half ring is—employing the notation of the article cited—given by

$$P = w \,.\, 2A \,.\, \frac{V^2}{g}.$$

This is balanced by an excess pressure on the outer surface of the half ring, and if that excess be Δp the corresponding resultant force is $\Delta p \,.\, 2r$, as shown on p. 278. Equating this to P

$$\Delta p = \frac{w}{g} \,.\, \frac{A}{r} \,.\, V^2.$$

The ring is supposed of breadth unity, and for A we may write the thickness of the ring, which may be called Δr. Dividing by this, and proceeding to the limit

$$\frac{dp}{dr} = \frac{w}{g} \,.\, \frac{V^2}{r},$$

an equation from which the pressure can be found if the law of velocity be given. If the fluid rotated about the axis like a solid mass, V would vary as r; but the case now to be examined is that in which V varies inversely as r, as expressed by the equation

$$Vr = \text{Constant} = k.$$

Substitute and integrate, then replacing k by Vr, it will be found that

$$\frac{p}{w}+\frac{V^2}{2g}=\frac{p_0}{w}+\frac{V_0^2}{2g},$$

where the suffix refers to a given point where the pressure is p_0 and the velocity V_0. This result shows that the energy is uniformly distributed, and we infer that if the direction of a moving current is changed so that the particles of water describe concentric circles, the velocity varies inversely as the distance from the centre.

A mass of rotating fluid is called a "vortex," and in the case just considered the vortex is described as "free," because the motion is that which is naturally produced (comp. Art. 264). A free vortex is necessarily hollow, for to hold the water together a negative pressure would be required near the axis of rotation, but the hollow may be filled up by water moving according to a different law.

240. *Viscosity.*—When the motion of a mass of water is free from sudden changes of direction, loss of energy takes place only through the direct action of viscosity, a property of fluids which it will now be necessary briefly to consider. In Fig. 154, p. 371, a block of plastic material is represented, and it was explained that to produce change of form a certain difference of pressure was necessary, depending on the hardness of the material. In a fluid a similar difference of pressure is necessary to produce a change of form at a *given rate*, and the magnitude of the difference is proportionate to the rate. If u be the rate at which the height of the block is diminishing and the breadth increasing, each reckoned per unit of dimension, the thickness remaining constant,

$$p=2cu,$$

where c is a co-efficient called the "co-efficient of viscosity." Or to express the same thing differently, if ω be the *rate* at which a small rectangular portion of the fluid is distorting, as in Figure 140, p. 325, q the corresponding distorting stress,

$$q=c\,.\,\omega.$$

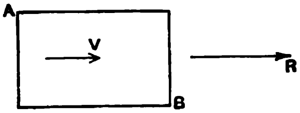
Hence, when a fluid moves, any change of form requires an amount of work to be done which is proportionate to the speed at which the change takes place. In a free vortex the rate of distortion is twice the angular velocity of the particles round the axis, and varies inversely as the square of the distance; the changes of shape are therefore very rapid near the centre, and energy is consequently dissipated much more rapidly than in the stream from which the vortex is produced.

In the case of water the viscosity is so small that such changes of form as occur in an undisturbed stream are not rapid enough to absorb any large amount of energy. For example, in the discharge from orifices in a thin plate the loss of head is only 5 or 6 per cent. It is only when the water is disturbed by the neighbourhood of a rough surface over which it moves, or otherwise, that large quantities of energy are dissipated and frictional resistances of great magnitude produced.

241. *Surface Friction in General.*—We now proceed to study experimentally some of the more important causes of hydraulic resistance.

Fig. 169 shows a thin flat plate AB with sharp edges completely immersed in the water. The plate is moving edgeways through the water with velocity V, then a certain resistance R is experienced which must be overcome by an external force. This resistance consists in a tangential action between the plate and the water, and so far is analogous to the friction between solid surfaces but it follows quite different laws, which may be stated as follows :—

Fig. 169.



(1) The friction is independent of the pressure on the plate.
(2) It varies as the area of the surface in contact with the water.
(3) It varies as the square of the velocity.

These laws are expressed by the formula

$$R = fSV^2,$$

where f is a co-efficient which, as in the friction of solid surfaces, is described as the "co-efficient of friction." The value of this co-efficient depends on the degree of smoothness of the plate. Thus, for example, in some experiments, to be described presently, on thin boards moving through water it was found that the co-efficient was ·004 for a clean varnished surface, and ·009 for a surface resembling medium sand paper, the units being pounds, feet, and seconds.

There are certain limitations to the truth of these laws, as in the case of solid surfaces. In the first place, if the velocity be below a certain limit the water adheres to the surface, and its velocity relatively to the surface is some continuous function of the distance from the surface so that the stream does not break up. This will be further referred to hereafter; for the present it is sufficient to say that the resistance then follows an entirely different law, varying nearly as the velocity instead of the (velocity)^2. The limiting velocity, however, at which this is sensibly the case is so low that in most practical applications the effect may be disregarded. In the second place, it is supposed that the water glides over all parts of the surface, with the same velocity; but if the surface be any considerable length the friction of the front portion of the surface on the water furnishes a force which drags the water forward along with the surface and so diminishes the velocity with which it moves over the rear portion. The friction is thus diminished, and in large surfaces very considerably diminished. Thus Mr. Froude experimenting on a surface 4 feet long, moving at 10 feet per second, found the value of f given above, but when the length was 20 feet and upwards, those values were diminished to ·0025 and ·005 respectively. Increasing the length beyond a certain amount produces very little change, and within a certain limiting length the effect is insensible. These limits must depend on the speed, but no exact observations have been made on this point. The power of the speed to which the friction is proportional has, however, been found to be diminished on long smooth surfaces, as shown below. The skin friction of vessels on which, as we shall see hereafter, the resistance chiefly depends at low speeds, is much diminished by the effect of length.

Experiments on surface friction were made by Colonel Beaufoy. They formed part of an elaborate series of experiments on the resistance of bodies moving through water, carried out during many years in the Greenland Dock, Deptford. Beaufoy employed the formula

$$R=f.SV^n$$

to represent his results, and for the index n obtained the values 1·66, 1·71, 1·9 in three series of experiments. The standard experiments on the subject are however due to the late Mr. Froude: they were made on boards $\frac{3}{16}$ inch thick, 19 inches deep, towed edgeways through the water. The boards were coated with various substances so as to form the surface to be experimented on.

The following table gives a general statement of Froude's results. In all the experiments in this table, the boards had a fine cutwater and a fine stern end or run, so that the resistance was entirely due to the surface. The table gives the resistances per square foot in pounds, at the standard speed of 600 feet per minute, and the power of the speed to which the friction is proportional, so that the resistance at other speeds is easily calculated.

Nature of Surface.	Length of Surface, or Distance from Cutwater, in Feet.											
	2 Feet.			8 Feet.			20 Feet.			50 Feet.		
	A	B	C	A	B	C	A	B	C	A	B	C
Varnish, - -	2·00	·41	·390	1·85	·325	·264	1·85	·278	·240	1·83	·250	·226
Paraffin, - -	1·95	·38	·370	1·94	·314	·260	1·93	·271	·237	—	—	—
Tinfoil, - -	2·16	·30	·295	1·99	·278	·263	1·90	·262	·244	1·83	·246	·232
Calico, - -	1·93	·87	·725	1·92	·626	·504	1·89	·531	·447	1·87	·474	·423
Fine Sand, -	2·00	·81	·690	2·00	·583	·450	2·00	·480	·384	2·06	·405	·337
Medium Sand,	2·00	·90	·730	2·00	·625	·488	2·00	·534	·465	2·00	·488	·456
Coarse Sand, -	2·00	1·10	·880	2·00	·714	·520	2·00	·588	·490	—	—	—

Columns A give the power of the speed to which the resistance is approximately proportional.

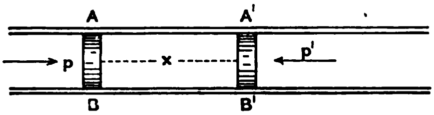
Columns B give the mean resistance per square foot of the whole surface of a board of the lengths stated in the table.

Columns C give the resistance in pounds of a square foot of surface at the distance sternward from the cutwater stated in the heading.

242. *Surface Friction of Pipes.*—When water moves through a pipe the friction of the internal surface causes a great resistance to the flow.

Fig. 170 shows a pipe of uniform transverse section (not necessarily circular) provided with two pistons, AB, $A'B'$, at a distance x, enclosing between them a mass of water. The pistons and included water move forward together with velocity v under the action of a force R, required on account of the friction of the pistons and of the water on the pipe. Omitting piston friction the force R will be given by

Fig.170.



$$R = f \,.\, S \,.\, v^2 = f \,.\, sx \, v^2,$$

where S is the wetted surface and s the perimeter.

If we imagine the pipe full of water moving through it with velocity v, the force R is supplied by the difference of the pressures p, p' on the pistons, and, therefore, if A be the sectional area

$$p - p' = f \,.\, \frac{s}{A} \,.\, xv^2.$$

The quantity A/s may be replaced by m and is described as the "hydraulic mean depth" of the pipe, a term derived from the case of an open channel to be considered hereafter. In the ordinary case of a cylindrical pipe $m = \frac{1}{4}d$. Further, we may reduce the pressures to feet of water by dividing by w, and thus obtain for the difference of pressure h'

$$h = f \,.\, \frac{x}{m} \,.\, \frac{v^2}{w} = f' \,.\, \frac{x}{m} \,.\, \frac{v^2}{2g},$$

where f' is a co-efficient connected with f by the equation

$$f = f' \,.\, \frac{w}{2g}.$$

The value of w, the weight of a cubic foot of water, differs so little from $2g$ that it is unnecessary, for our present purpose (Art. 283), to distinguish between f and f', especially as the value of f is always determined by special experiment on pipes.

This formula for the head necessary to overcome surface friction is continually in use. The formula gives directly the head necessary for a length x of the pipe, when the water, by being enclosed between pistons, is constrained to move over the surface with a given velocity : when the pistons are removed and the water flows freely it represents the facts very imperfectly. The central parts of the steam move quicker than the parts in immediate contact with the pipe, and besides, though the circumstances are different, we cannot be sure that the velocity over the internal surface is not affected in the same way as in the case of a moving surface. The value of f has therefore to be obtained by special experiment, and the result of such experiments are by no means always in accordance with each other. It is found, however, that f lies between the limits ·005 and ·01 according to the condition of the internal surface, and partly also on the diameter and velocity, the value being greater in small pipes than large ones, and at low velocities than high ones. For the present we assume ·0075 as roughly representing the facts when there is no special cause for increased resistance. For a pipe of circular section, length l, we have therefore

$$h' = 4f \cdot \frac{l}{d} \cdot \frac{v^2}{2g},$$

where for $4f$ we commonly assume the value ·03.

243. *Discharge of Pipes.*—The velocity v is the actual velocity with which the water moves, so that $v^2/2g$ is the energy of motion of each pound of the water. The loss of energy by friction is the same as that of raising the water through a height h', and is therefore equal to the energy of motion when

$$\frac{l}{d} = \frac{1}{4f} = 33 \text{ nearly},$$

that is, a length of pipe equal to 33 diameters absorbs an amount of energy equivalent to the whole energy of motion of the water. In pipes of any length, therefore, the effect of friction is very great, so much so that the size of a pipe is principally fixed by the loss of head which can be permitted. It is easily seen that to deliver water with a given velocity the loss varies inversely as the diameter, and that to deliver a given quantity it varies inversely as the fifth power of the diameter; thus, the smallest permissible diameter is fixed almost entirely by the value of h', which may be supposed already known.

The quantity discharged per second is given us by the formula

$$Q = Av = \frac{\pi}{4} d^2 v,$$

and on substitution this becomes

$$Q = \frac{\pi}{4}\sqrt{\frac{g}{2f}} \cdot \sqrt{\frac{h'}{l}} \cdot d^{\frac{5}{2}}.$$

All dimensions are here in feet and Q is in cubic feet per second. If we require gallons per minute for a diameter of d inches, the formula will be

$$G = C \,.\, \sqrt{\frac{h'}{l}} \,.\, d^{\frac{5}{2}} \,;$$

where C is a constant connected with $4f$ by the equation

$$C = \frac{4{\cdot}736}{\sqrt{4f}}.$$

For $4f = {\cdot}03$, this gives $C = 27{\cdot}3$, but for clean iron pipes not less than 4 inches in diameter the value 30 may be employed.

244. *Open Channels.*—Returning to Fig. 170, suppose the pipe, instead of being horizontal, is laid at an angle θ (see Fig. 171 next page), so that the difference of level of the two ends is $y = l \,.\, \sin\theta$, then the difference of pressure-head is

$$\frac{p - p'}{w} = f \,.\, \frac{l}{m} \,.\, \frac{v^2}{2g} - y,$$

and therefore may be made zero if the slope of the pipe be

$$\sin\theta = f \,.\, \frac{1}{m} \,.\, \frac{v^2}{2g} = \frac{h'}{l}.$$

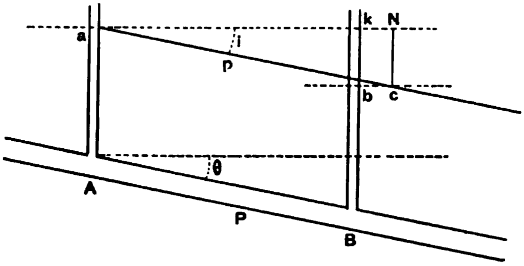
But if the pressure be constant we may remove the upper surface of the pipe and thus obtain the case of an open channel. The quantity m is now the sectional area of the channel divided by the wetted perimeter, and is therefore the actual depth in a very broad shallow channel, but in other cases less in a ratio dependent on the form of section. As before stated it is described as the "hydraulic mean depth" of the channel.

We can now find the velocity and discharge of a stream of given dimensions and fall, provided that we know the value of f, or conversely the size of channel for a given discharge and fall. The value of f, however, varies for the same reasons as in pipes which indeed apply with still greater force, so that the limits of variation are wider. The average value does not differ very widely from ·0075, already adopted for pipes; but to obtain results of even moderate accuracy a special study of the experiments on the subject is necessary, which will not be attempted in this treatise.

245. *Virtual Slope of a Pipe.*—If the pipe be laid at any other angle the pressure will not be constant, and the mode in which it varies is best seen by a graphical construction.

Suppose small vertical pipes Aa, Bb to be placed at points A, B of the pipe we are considering (Fig. 171), then (if they enter the water square, without being bent towards the direction of motion) the water will rise in them to a level representing the pressure in feet of water at these points. If there were no friction the level would be the same in both, and the difference (*bk* in the figure) therefore represents the loss by friction. Now draw a horizontal line through *b*, and take *c* on it, so that $ac = AB = l$, then the angle *caN* is given by the equation

Fig. 171.



$$\sin i = \frac{h'}{l},$$

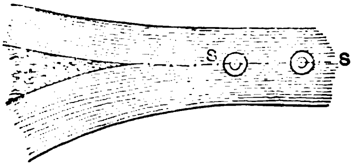
and is therefore the slope of a channel of the same length and hydraulic mean depth which would give the same discharge. This angle is therefore called the VIRTUAL SLOPE of the pipe. At any point *P* in the pipe, the water would rise to the level of the corresponding point *p* in the virtual channel, found by taking $ap = AP$. The construction would of course fail if h' were equal to, or greater than l, but this case does not occur in practice; on the contrary, in pipes as in channels the angle i is nearly always small. The virtual slope is frequently one of the data of the question. The line *ac* is variously described as the "pressure line," "line of virtual slope," or "hydraulic gradient."

The pipe need not be straight; it may be curved or be laid in sections at different slopes, there will still be a continuous hydraulic gradient, provided the diameter be the same throughout; but if the sections be of different diameters each section will have its own slope. In practice care must be taken that the pipe does not rise above its hydraulic gradient, for otherwise there will be a partial vacuum: the pipe then acts as a syphon, which is liable to fail on account of leakage and the presence of air in the water.

246. *Loss of Energy by Eddies and by Broken Water.*—We now proceed to consider other causes of frictional resistance.

In Fig. 172 two streams of water, moving with different velocities, converge towards each other and unite into one. Each stream, so far as can be judged by the eye, moves originally without disturbance in the manner described in Art. 241. On union, however, near the junction indicated by the dotted line *SS* in the figure, small depressions are observed, which move for some distance along with the stream, and then disappear. On examination these depressions are found to consist of small portions of the fluid in a state of rotation, the speed of rotation being greatest at the centre and gradually dying away towards the circumference. A motion of this kind was called a "vortex" in Art. 241, and in the present case is also described as an "eddy"; it is independent of the general motion of the stream, and its energy is therefore of the internal kind. The disappearance of the eddies thus formed is due to viscosity, the effect of which is much greater in the eddy than in the stream as already explained. After the eddies have disappeared the two streams are found to have become a single one, moving with a velocity intermediate between those of the streams which form it, but possessing less energy. Theoretically there is nothing to prevent two streams of a perfect fluid from moving side by side with different velocities, but such a motion is always unstable, and will not long continue without the formation of eddies by a sudden change of direction (Art. 239) in small portions of the fluid which separate from the rest. The instability is greater the more nearly perfect the fluid is. Whenever the water in motion intermingles with water at rest, or moving with a different velocity, internal motions of a complex kind are produced, representing a considerable amount of energy of the internal kind which is virtually lost even before its final dissipation by fluid friction.

Fig.172.



Again, in order that a mass of water may form a continuous whole, sufficient pressure must exist on the bounding surface to prevent the pressure at any point within the mass from becoming zero, as explained in Art. 240. If this condition is not satisfied the water breaks up more or less completely, and the result is a confused mass with complex internal motions rapidly disappearing as before by fluid friction. When waves break on a beach, or when paddles strike the water and drive it upwards in a mass of foam, the process takes place on a large scale before our eyes; but the same thing occurs in most cases where the velocity of a mass of water is suddenly changed, and of this we will now consider some examples.

Fig. 173*a* shows a jet of water filling a tank. Here the water pouring in possesses the kinetic energy $Wv^2/2g$ due to the original velocity of the water, and the height from which it falls into the tank. If it be of some size as compared with the tank the water will be completely broken up; if it be small it will penetrate the water in the tank without much apparent disturbance at the surface: in either case the result is a mass of water at rest as a whole, so that its energy is all of the internal kind. If the jet be shut off the water rapidly settles down to rest, the whole energy is then dissipated by fluid friction.

Fig.173a

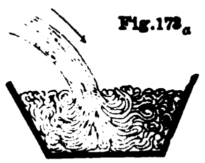


Fig. 173*b* shows a bucket moving horizontally, bottom foremost, with velocity *V*, while a horizontal jet moving with greater velocity strikes it centrally: the bucket is then filled with broken water which pours out under the action of gravity. In water-wheels a series of buckets are filled in succession, and the broken water carried on with the wheel. Here if the bucket were at rest the loss of energy would be, as before, $Wv^2/2g$; but as it is moving with velocity *V*, the striking velocity on which the breaking depends will be $v - V$, and the loss of energy is

$$U = W\frac{(v-V)^2}{2g},$$

where *W* is the weight of water acted on in the time considered. Both these cases may be treated as examples of the collision of two bodies considered on page 252, one of the bodies being indefinitely great. The energy of collision is employed in breaking up the water. It is represented in the first instance by internal motions, and subsequently dissipated by fluid friction.

Fig.173b

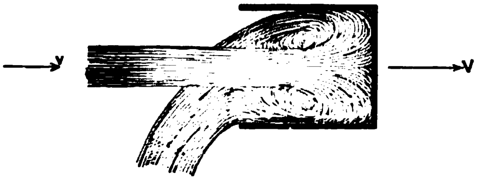
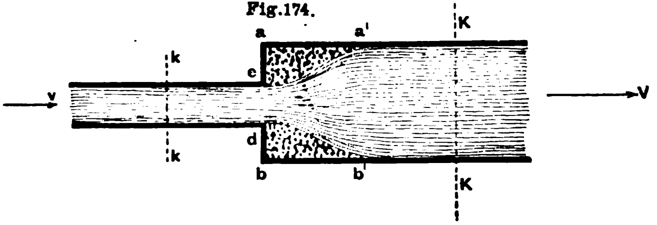


Fig. 174 represents a pipe which is suddenly enlarged from the diameter *cd* to the diameter *ab*. The water is moving through the small part of the pipe with velocity *v*, and, on passing through *cd* spreads out so as to fill the larger part. At some distance from the enlargement it moves in a continuous mass with velocity *V*, but in its immediate neighbourhood we have broken water, as in the case of the bucket, from which it only differs in the enclosure of the water in a

Fig.174.



casing. The loss of energy per unit of weight may be expected to be the same (Art. 252) as before, and is therefore

$$h' = \frac{(v - V)^2}{2g},$$

a formula which gives us the "loss of head." If the sectional areas of the two parts of the pipe be A, a the discharge is

$$Q = AV = av,$$

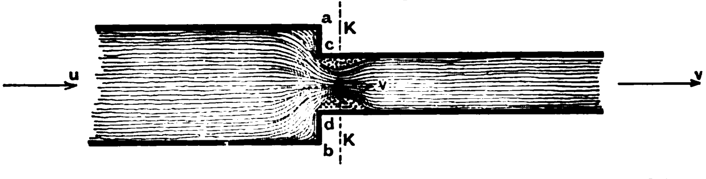
so that if m be the ratio of areas,

$$h' = (m - 1)^2 \frac{V^2}{2g} = \left(1 - \frac{1}{m}\right)^2 \frac{v^2}{2g}.$$

The co-efficient of resistance is therefore $(m - 1)^2$ or $(1 - 1/m)^2$, according as the velocity to which it is referred is that in the large pipe or that in the small one.

Instead of the water moving from a small pipe into a large one, we may have the converse case of a suddenly contracted pipe as in Fig. 175. The loss here is due to precisely the same cause, namely a sudden enlargement, which is produced as follows. In the figure the stream of water moving with velocity *u* contracts on passing through *cd* nearly as it would if the small part of the pipe were removed, as in Fig. 165, p. 410, until it reaches a contracted section *KK*, and is then moving with a velocity *v* which is greater than *u* in the ratio of the area of the large pipe to the *contracted* area *KK*. The loss of head in this part of the process is not large. After passing *KK*, however, an expansion takes place to the area of the small pipe, and this is accompanied by breaking up, the space between the contracted jet and the pipe being filled up with broken water.

Fig.175.



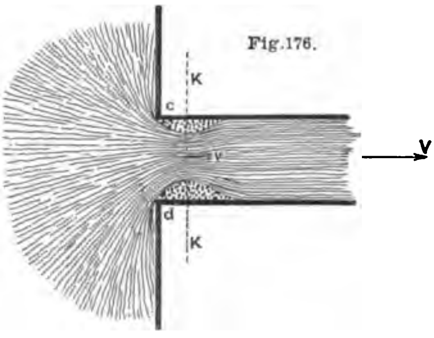


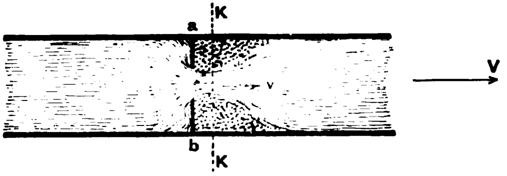
Fig.176.

In Fig. 176 we have the extreme case, in which the large pipe is a vessel of any size. We thus obtain the case of a pipe with square edged entrance which has already been referred to in Art.

236. Another modification is that of a diaphragm in a pipe, as in Fig. 177. The small pipe is here larger than the orifice through which the water enters, and in the figure we have simply a single pipe divided into parts by a diaphragm with an orifice in the centre. The stream of water, after passing the contracted section *KK*, expands to fill the pipe. In cocks when partially closed, a loss of head of the same kind occurs, which may be increased to any extent by closing the cock further.

In all these cases the loss of head may be calculated approximately by means of the formula for a sudden enlargement, but the ratio of enlargement is not known exactly, on account of the uncertainty of the value of the co-efficient of contraction to be assumed. Losses of head of this kind are indeed always subject to variation within certain limits from accidental causes; in general and on the average the quantity of water broken up will bear a certain proportion to the whole quantity passing, and in consequence we have the general law of hydraulic resistance stated on page 408, but the ratio may vary from time to time, and cannot be stated with precise accuracy. The causes of this uncertainty will be clearly understood on considering somewhat more closely the manner in which the loss takes place.

Fig. 177.



In Figs. 175, 177 two plane surfaces at right angles meet at *a*, forming an internal angle, through which water is flowing. The particles of water there describe curves which are all convex towards *a*, and in conformity with the general principle explained in Art. 239, the pressure must increase and the velocity diminish on going towards *a*. The water then moves slowly and quietly round the angle without disturbance. But when compelled by the general movement of the stream to move round an external angle such as *kea* in Fig. 174, the case is very different; the particles then describe curves which are concave round *e*; and consequently the pressure diminishes in going towards *e*, while the velocity increases. To hold the particles of water in contact with the surface, an infinite pressure would be required in the other parts of the fluid. The particles of water therefore leave the surface at *e*, and describe a path *ea'*, regaining the surface farther on; *ea'* is then described as a "surface of separation," as it separates the moving mass of water from a portion enclosed within it which is in a state of violent disturbance. Such are the surfaces shown in Figs. 174-178. It is not, however, to be supposed that these surfaces are sharply defined, and that they permanently separate different masses of water. On the contrary, no such equilibrium is possible; the surfaces are continually fluctuating, and a constant interchange takes place between the so-called "dead" water and the stream. In this intermingling eddies are produced nearly as in the comparatively simple case of two streams given on page 423. The process is always essentially the same, and consists in sudden changes of direction being communicated to parts of the stream which become detached from the rest.

247. *Bends in a Pipe. Surface Friction.*—In some other cases the process of breaking up by which energy is lost is less obvious, and the ratio is subject to greater variations.

When a pipe has a bend in it, if the internal surface of the pipe were perfectly smooth and free from discontinuity of curvature, there would be no disturbance of the current of water, which would flow as described in Art. 241. These conditions, however, are not satisfied by actual bends in pipes, and there is always a loss of head due to them in addition to the loss by surface friction. This loss can only be determined by experiment, but it is easy to conjecture that the loss will be proportional to the angle through which the pipe is bent, and that it will be greater the quicker the bend, that is, the smaller the radius of the bend is as compared with the diameter of the pipe. The extreme case of a bend is a knee, but the loss is not in this case proportional to the angle of the knee, but follows a complex law. For details respecting bends and knees the reader is referred to the treatises cited at the end of this chapter, but some common examples are given in the table on the next page.

In the case of surface friction the loss of energy is represented in the first instance by eddies formed at the surface and thrown off. In almost all practical cases of the motion of water in pipes and channels, even when to all outward appearance quite undisturbed, the fluid is in fact in a state of eddy motion throughout, and dissipation of energy at every point is going on much more rapidly than would be the case if the motion were of the simple kind described in Art. 241. The quantity of water broken up, however, is not generally in a fixed proportion to the quantity passing, for reasons which are sufficiently indicated in Art. 242.

248. *Summation of Losses of Head.*—The total loss of energy due to a number of hydraulic resistances of various kinds is found by adding together the losses of head due to each cause taken separately. The velocity of the water past each obstacle will not generally be the same for all, and it is then necessary to select some one velocity from which all the rest can be found by multiplication by a suitable factor for each obstacle. If n be this multiplier the loss of head will be

$$h' = \Sigma F n^2 \frac{V^2}{2g},$$

where V is the velocity selected for reference. The value of V is then found for motion under a given head H by the formula

$$(1 + \Sigma F n^2)\frac{V^2}{2g} = H.$$

The various values of F already given are collected with some additions in the annexed table:—

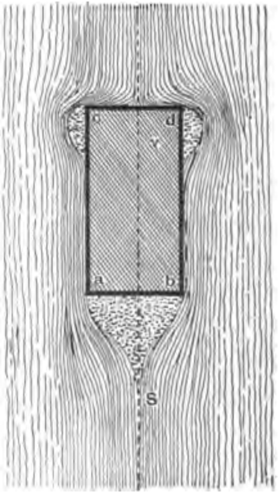
CO-EFFICIENTS OF HYDRAULIC RESISTANCE.

NATURE OF OBSTACLE.	VALUE OF F.	REMARKS.
Orifice in a Thin Plate.	·06	
Square-edged Entrance of a Pipe.	·5	
Sudden Enlargement of a Pipe in the ratio $m : 1$.	$(m - 1)^2$	Referred to Velocity through large part of Pipe.
Bend at right angles in a Pipe.	·14	Radius of Bend = 3 × Diameter of Pipe.
Quick Bend at Right Angles.	·3	Radius of Bend = Diameter of Pipe.
Common Cock partially closed.	·75, 5·5, 31	Handle turned through 15°, 30°, 45° from position when fully open.
Surface Friction of a Pipe the length of which is n times the diameter.	$4f.n.$	For a clean Iron Pipe d inches diameter, according to Darcy. $4f = \cdot 02\left(1 + \frac{1}{d}\right)$
Knee in a Pipe at Right Angles.	Unity.	In Bends the co-efficient is proportional to the Angle of the Bend, but in Knees the law is much more complex.

249. *Resistance of deeply Immersed Bodies. Ships at Low Speeds.*—The subject of the resistance of ships is outside the limits of this treatise, for the ship moves on the surface of water, exposed to the atmosphere, on which waves are produced; whereas in the branch of mechanics now under consideration, the water is supposed to move within fixed boundaries. A certain part of the subject, however, may properly be considered as belonging to Hydraulics. If a body be deeply immersed in a fluid, that part of the fluid alone which is in its immediate neighbourhood will be affected by its motion, and the question is not essentially different from the cases already considered of the movement of water in pipes and channels.

Fig. 178 shows a parallelopiped *abcd* moving through water in the direction of its length, the face *cd* being foremost. To an observer whose eye travels along with the body the water will appear to move past the solid in a stream of indefinite extent. At some distance away the action of the solid is insensible, but it becomes increasingly great as the solid is approached, and is greatest for that part of the water which moves in immediate contact with it. At *c* and *d* eddies are formed in passing round the corners exactly as is the case at the same points in Figs. 175, 176—the stream in fact is suddenly contracted in the same way as in passing from a large pipe to a small one, the diminution of area in this case being the transverse section of the solid. After this the water moves in actual contact with the solid until it reaches the corners *ab*, when it describes the curves *aS*, *bS*, meeting in *S*, after which it forms a continuous stream as before. The two curves enclose between them a mass of eddying water exactly similar to the eddies at *a* and *b* in Fig. 174—the stream, in fact, suddenly expands, just as in passing from a small pipe to a large one, the increase of area being in this case the sectional area of the solid. The eddies thus formed during the passage of the solid through the water absorb energy, which must be supplied by means of an external force, which drags the body through the water. The eddies at *cd* represent an increased pressure on the front face *cd* of the solid, while those at *aS*, *bS* diminish that at the rear. This kind of resistance to the movement of a body through water is called Eddy Resistance, and may be almost entirely avoided by employing "fair" forms, that is, by avoiding all discontinuity of curvature in the solid itself, and in the junction of its surface with the direction of motion.

Fig.178.



A general formula for eddy resistance is derived thus. As already stated the water suffers no sensible disturbance at a certain distance from the solid. If then we imagine a certain plane area A attached transversely to the solid, and moving with it, all the water affected by the solid will pass through this plane, and its quantity will be

$$Q = AV,$$

where V is the velocity. In similar solids this area must be proportioned to the sectional area S of the solid, so that we write $A = cS$, where c is a constant depending on the form. Of this water a certain fraction will be disturbed by eddies, and the velocity of each particle of water will be some fraction of the velocity of the solid. Hence it follows that the energy U generated per second in the production of eddies must be

$$U = c'wQ \cdot \frac{V^2}{2g} = cc'wS \cdot \frac{V^3}{2g},$$

where c' is a co-efficient. Now this amount of energy is generated by means of a force which drags the solid through the water, at the rate of V feet per second, notwithstanding an equal and opposite resistance R. We have then

$$RV = cc'wS \cdot \frac{V^3}{2g},$$

or dividing by V, and replacing cc' by a single constant k,

$$R = k \cdot wS \cdot \frac{V^2}{2g}.$$

The co-efficient k is to be determined by experiment for each form of solid. In the case of the parallelopiped shown in the figure, the value of k depends little on the length, unless it be so short that the eddies at the corners cd coalesce with those in the rear of the solid, and it then becomes the same as that of a plate moved flatwise. Further it is nearly the same, if the transverse section be circular instead of square, and does not greatly differ from unity. For the flat plate it is greater and may be taken as 1·25. It must be remarked, however, that resistance of this kind is very irregular, and may vary considerably in the course of the same experiment. Different results are therefore obtained by different experimentalists. By some authorities much larger values are given. The same remarks apply to the case of a sphere for which the value may be taken as about ·4.

In all cases the value of k is independent of the units employed. It is also to a great extent independent of the kind of fluid, being approximately the same for example in air as in water; but this would not hold good for fluids of very different viscosity; nor is it true for high speeds in air, because the compressibility of the air affects the question. The same remarks apply to the co-efficient (F) of hydraulic resistance employed above. It has been found that co-efficients of surface friction are greater in salt water than in fresh in the ratio of the densities of these fluids, as we might anticipate, since surface friction is a kind of eddy resistance.

In well-formed ships the eddy resistance (apart from certain special resistances to be mentioned presently) should not be more than 10 per cent. of the total resistance at low speeds, and is frequently less; the principal cause of resistance here is surface friction, which is given by the formula stated in Art. 241. The surface to be considered is the wetted surface, which can be found by direct measurement. It is convenient, however, to have a formula which gives the resistance in terms of the displacement (Δ) of the vessel. If R be the resistance, V the speed, the formula will be

$$R = K \,.\, \Delta^{\frac{2}{3}} V^2,$$

where K is a co-efficient which for speed in knots, displacement in tons, and resistance in lbs., may be taken from ·55 to ·85 according to the type of vessel, if the bottom be in good condition. The speed, however, must not exceed a certain limit on account of the wave resistance, which increases at a much more rapid rate. At low speeds the value of this resistance is small, and it may approximately be considered as compensating for the somewhat slower rate at which the surface friction increases (Art. 241), but at high speeds it becomes a principal part of the resistance, and has to be separately considered. The speed of a wave is proportional to the square root of its length, and the magnitude of the resistance in similar ships depends on the proportion between the length L of the ship, and the length of waves which travel at the same speed. The limit in question is therefore given by the equation

$$V_0 = K' \,.\, \sqrt{L}$$

where for length in feet and speeds in knots the co-efficient K' may be taken from ·6 to ·7. (See Appendix.)

Not only is the speed limited to which the resistance formula given above applies, but it must be further remarked that it supposes that the vessel is towed by an external force. If the vessel be propelled by steam power on board, the effective resistance is much greater, because the action of the propeller has always the effect of increasing the resistance. In screw propulsion this augmentation is very great, being at the rate of 20 to 40 per cent.; the larger value is of common occurrence, but it should be remarked that the resistance of stern posts and appendages is included in this estimate. In reckoning the engine power required, the resistance must be taken at its augmented value, and the formula of Art. 128, p. 243, employed for the efficiency of the mechanism, which is much less at low speeds than at high speeds, as the formula shows.

250. *Direct Impulse and Reaction.*—The generalized form of the second and third laws of motion, described as the Principle of Momentum in Chapter XI. of this work, may be employed with great advantage when the motion of water in large masses is under consideration, because the total momentum of a fluid mass depends solely on the motion of the centre of gravity (p. 250), and not on the very intricate motions of the parts of the fluid amongst themselves. Further, the energy dissipated by frictional resistances is accounted for by these internal motions, or by the mutual actions of the fluid particles, and the total momentum is therefore independent of these resistances. Hence it follows that results may be obtained which are true notwithstanding any frictional resistances, and in some cases the loss of energy by them may be determined *a priori.* Also the pressures on fixed surfaces may be found which do no work, and to which therefore the principle of work does not directly apply.

Fig. 179.

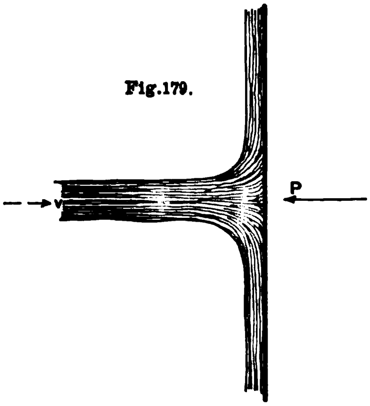


Fig. 179 shows a jet of water striking perpendicularly a fixed plane of infinite extent, and exerting on it a pressure P. The magnitude of this pressure is found by considering that the plane exerts an equal and opposite pressure on the water, which changes its velocity. The water, originally moving with velocity v, spreads out laterally, and any motion which it possesses is parallel to the plane. In time t the impulse is Pt, and the change of momentum is Mvt, where M is the mass of water delivered per second. Equating these we have

$$P = Mv = \frac{W}{g} \cdot v,$$

where W is the weight of water delivered per second.

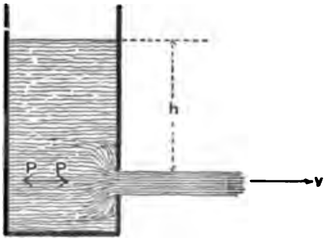
If the plane be smooth, and gravity be neglected, the motion of the water will be continuous; but if it be rough to any extent, so that breaking-up occurs, the result will still be correct, provided only the roughness be symmetrical about the axis of the jet. And the action of gravity parallel to the plane does not affect the question.

In Fig. 180 we have the converse case of water issuing from a vessel with a lateral orifice. Here the water, which originally was at rest, issues with velocity v, and the momentum generated in time t is Mvt. To produce this momentum a corresponding impulse is required, which is derived from the resultant horizontal pressure P of the sides of the vessel upon the water. We have as before

$$P = Mv = \frac{Wv}{g}.$$

A pressure equal and opposite to P is exerted by the water on the vessel: this is described as the "reaction" of the water; and, if the vessel is to remain at rest, must be balanced by an external force supplied by the supports on which it rests.

Fig. 180.



A remarkable connection exists between the change of pressure on the sides of the vessel consequent on the motion and the co-efficients of contraction and resistance.

First, suppose the water at rest, the orifice being closed, then the value of P is zero, and the pressure on the area of the orifice is $w \,.\, A \,.\, h$, the notation being as in Art. 234). When the orifice is opened the pressure on that side is diminished, first, by the quantity $w \,.\, A \,.\, h$; secondly, by an unknown diminution S due to the motion of the water (p. 415) over the surface near the orifice. Now

$$P = S + w \,.\, A \,.\, h = \frac{wA_0v'^2}{g} = 2wA_0(h - h'),$$

the notation still being as in the article cited. Replacing A_0 by kA we obtain

$$S = wA\left\{2k(h - h') - h\right\} = wAh\left(\frac{2k}{1+F} - 1\right).$$

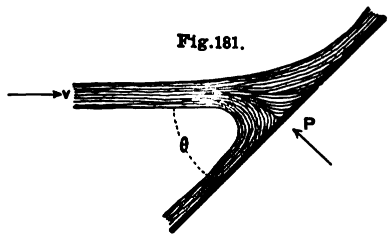
Since S is always positive the least value of k is

$$k = \frac{1+F}{2}.$$

If there be no frictional resistance $k = \cdot 5$, and this is the smallest value k can have under any circumstances. For a small pipe projecting inwards as in Fig. 164, p. 408, these conditions are approximately realized, the water being at rest over the whole internal surface of the vessel.

251. *Oblique Action. Curved Surfaces.*—When a jet impinges obliquely on an indefinite plane (Fig. 181), the water spreads out laterally as before, but the quantity varies according to the direction. In the absence of friction the velocity of individual particles is the same as that of the jet in whatever direction the water passes. At the same time the velocity of the whole mass of water parallel to the plane cannot

Fig. 181.



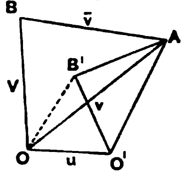
be altered by the action of the plane, and is therefore $v \cdot \cos \theta$, where θ is the angle the jet makes with the plane. It immediately follows that any small portion of water diverging from the centre of the jet at an angle ϕ with the jet must be balanced by another portion diverging in the direction immediately opposite, and the quantities so diverging must be in the ratio $1 - \cos \phi : 1 + \cos \phi$, being inversely as the changes of velocity parallel to the plane. But if the circumstances be such that breaking-up takes place, the motion of the water parallel to the plane will be undetermined, and in general there will be a tangential action on the plane of the nature of friction.

The normal pressure on the plane is in all cases the same, being given by the formula

$$P = Mv \cdot \sin \theta = \frac{W}{g} \cdot v \cdot \sin \theta.$$

If the surface on which the water impinges be curved it is necessary to know the average direction and magnitude of the velocity with which the water leaves the surface. In the absence of friction, as already noticed, the velocity of the individual particles is unaltered unless the water be enclosed in a pipe so that the pressure can be varied—a case for subsequent consideration; the direction, however, will depend on the way in which the water is guided. In cases which occur in practice it will generally be found either that the whole of the water is guided in some one direction, or that it leaves the surface in all directions symmetrically.

Fig. 182.



Taking the first case, suppose the original velocity (v) of the water to be represented by OA (Fig. 182), and the final velocity to be diminished to V by friction, and altered in direction so as to be represented by OB. Then the change of velocity in the most general sense of the word (p. 248) is represented by AB. If this be denoted by $\bar{v}$ the change of momentum per second is

$$P = \frac{W\bar{v}}{g}.$$

The resultant pressure on the surface is parallel to AB and numerically equal to P.

In applications to machines the curved surface is frequently a vane which is not fixed, but moves with a given velocity; the pressure can then be found by a simple addition to the diagram. Through O draw OO', representing the velocity (u) of the moving surface in direction and magnitude, then $O'A$ represents the velocity with which the water strikes the surface. Considering the vane as fixed, the velocity is now estimated with which the water would leave it, and $O'B'$ drawn to represent it: the change is now AB' instead of AB. If the absolute velocity is required with which the water leaves the surface, it may be found simply by joining OB', which will completely represent it: the change of velocity being AB', whether the velocities are absolute or relative to the moving surface.

The cup vane ACA (Fig. 183), against which a small jet of water impinges centrally, may be taken as an example where the water spreads in all directions symmetrically. If OA be tangent to the vane at A, making an angle θ with the centre line of the jet, the water leaves the vane in the direction OA with unaltered velocity (neglecting friction). The resultant pressure P is in the direction of the jet, and the velocity in that direction is altered from v to $v \cos \theta$ in the opposite direction, so that the change of velocity is $v(1 + \cos \theta)$. Thus we have

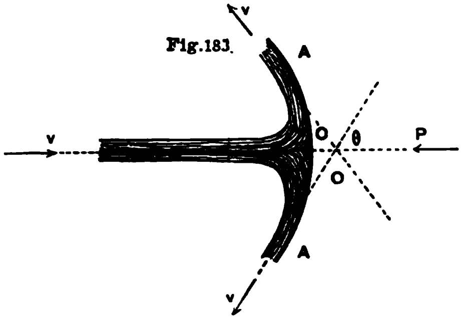


Fig. 183.

$$P = \frac{Wv}{g}(1 + \cos \theta).$$

252. *Impulse and Reaction of Water in a Closed Passage.*—When the water is moving in a closed passage the resultant pressure to be considered in applying the principle is not merely that on the sides of the passage, but also that on the ideal surfaces which separate the mass of water we are considering from the complete current. In the previous cases the pressure of the atmosphere on the free surface bounding the fluid was the same throughout, and was balanced by an equal pressure of the surface against which it impinges, which is not included in the preceding results. This is now no longer the case.

An important example is that of the sudden enlargement in a pipe already referred to in Art. 246. In Fig. 174, page 424, take ideal sections KK, kk of the large and small portions of the pipe, and consider the whole mass of water between them. This mass is acted on (1) by the pressure (p) on the transverse section kk, (2) by the pressure (P) on the transverse section KK, and (3) by the pressure of the sides of the pipe. If we resolve in the direction of the length of the pipe, the only part of (3) which we need consider is the pressure (p') on the annular surface ae, bd, the area of which is $A - a$, and the whole resultant pressure is therefore $PA - pa - p'(A - a)$ in the direction opposite to the motion of the water. Now let W be the weight of water delivered in one second, then in that space of time W passes from the small pipe, where its velocity is v, to the large pipe, where it has a velocity V, so that if we equate the resultant pressure to diminution of momentum

$$PA - pa - p'(A - a) = \frac{W}{g}(v - V) = \frac{wAV(v - V)}{g},$$

a formula may be written

$$\frac{P}{w} - \frac{p}{w} = \frac{V(v - V)}{g} + \frac{p - p'}{w}\left(1 - \frac{1}{m}\right),$$

m being as in Art. 246 the ratio of enlargement. Let now H be the total head in the large pipe and h in the small one, then subtracting $(v^2 - V^2)/2g$ from both sides and re-arranging the terms

$$h - H = \frac{(v - V)^2}{2g} + \frac{p - p'}{w}\left(1 - \frac{1}{m}\right).$$

Comparing this result with that obtained in the article cited, it appears that the value of the loss of head there given is a necessary consequence of supposing $p = p'$, but cannot otherwise be correct. That the pressure in the broken water at *ac*, *bd* is nearly equal to the pressure, in the small pipe may be considered probable *a priori*, independently of the experimental verification which the formula has received.

EXAMPLES.

1. The injection orifices of the jet condenser of a marine engine are 5 feet below the surface of the sea, and the vacuum is 27 inches of mercury: with what velocity will the water enter the condenser, supposing three-fourths the head lost by frictional resistances? Also find the co-efficients of velocity and resistance and the effective area of the orifices to deliver 100,000 gallons per hour. *Ans.* Velocity = 23·6′ per second; Area = 27 sq. inches.

2. Water is discharged under a head of 25′ through a short pipe 1″ diameter with square-edged entrance; find the discharge in gallons per minute. *Ans.* 66½.

3. Water issues from an orifice the area of which is ·01 sq. feet in a horizontal direction and strikes a point distant 4′ horizontally and 3′ vertically from the orifices. The head is 2′ and the discharge 25 gallons per min.; find the co-efficients of velocity, resistance, contraction, and discharge. *Ans.* $c = \cdot 816$, $F = \cdot 5$, $k = \cdot 72$, $C = \cdot 57$.

4. The wetted surface of a vessel is 7,500 sq. feet, find her skin resistance at 8 knots and the H.P. required to propel her, taking the resistance to vary as V^2 with a co-efficient of ·004. *Ans.* Resistance = 5,500 lbs., H.P. = 135.

5. The diameter of a screw propeller is 18′, the pitch 18′, and the revolutions 91 per min. Neglecting slip find the H.P. lost by friction per square foot of blade at the tips, taking a co-efficient ·008 to include both faces of the blade. *Ans.* Friction = 65 lbs. per square foot. H.P. = 10·6.

6. Two pipes of the same length are 3″ and 4″ diameter respectively: compare the losses of head by skin friction (1) when they deliver the same quantity of water, (2) when the velocity is the same. *Ans.* Ratio = 4·21 and 1·33.

7. Water is to be raised to a height of 20′ by a pipe 30′ long 6″ diameter: what is the greatest admissible velocity of the water if not more than 10 per cent. additional power is to be required in consequence of the friction of the pipe? *Ans.* 8½′ per sec.

8. Two reservoirs are connected by a pipe 6″ diameter and three-fourths of a mile long. For the first quarter mile the pipe slopes at 1 in 50, for the second at 1 in 100, while in the third it is level. The head of water over the inlet is 20 feet and that over the outlet 9 feet. Neglecting all loss except that due to surface friction, find the discharge in gallons per min., assuming $f = \cdot 0087$. *Ans.* $v = 3\cdot 43'$ per sec. Discharge = 253 gallons per min.

9. A river is 1000′ wide at the surface of the water, the sides slope at 45°, and the depth is 20′; find the discharge in cubic feet per sec. with a fall of 2′ to the mile, assuming $f = \cdot 0075$. *Ans.* 154,000.

10. A tank of 250 gallons capacity is 50′ above the street. It is connected with the street main, the head in which is 52′ by a service pipe 100′ long: find the diameter of the pipe that the tank may be filled in 20 min. What must the head in the main be to fill the tank in 5 min. with this service pipe? *Ans.* $d = 1\cdot 6''$. Head in main = 82′.

11. Water is discharged from a vessel by a long pipe: show that the discharge is the same for all pipes of the same length with the discharging extremity in the same horizontal line. Draw the hydraulic gradient and examine the case of a syphon.

12. In question 2 suppose the pipe instead of being short to be 25″ long, find the discharge, assuming for surface friction $f = \cdot 01$. *Ans.* 52.

13. A horizontal pipe is reduced in diameter from 3″ to ½″ in the middle, the reduction being very gradual. The pressure head in the pipe is 40′, what would be the greatest velocity with which water could flow through it, all losses of head being neglected? *Ans.* 1·4′ per sec.

14. A pipe 2″ diameter is suddenly enlarged to 3″. If it discharge 100 gallons per min., the water flowing from the small pipe into the large one, find the loss of total head and the gain of pressure head at the sudden enlargement. State the two values of the coefficient of resistance.

Ans. Loss of head = 8½″. $F = 1\cdot 56$ or ·31.
Gain of pressure = 1′ 2″.

15. In the last question suppose the water to move in the reverse direction. Find the loss of head and the change of pressure consequent on the sudden contraction, assuming the co-efficient of contraction to be ·66.

Ans. Loss of head = 7½″,
Diminution of pressure = 2′ 5¾″.

16. A horizontal pipe 30′ long is suddenly enlarged from 2″ to 3″ and then suddenly returns to its original diameter. Length of each section = 10′. Draw the hydraulic gradient when the pipe is discharging 100 gallons per min. into the atmosphere, assuming as co-efficient of surface friction $4f = \cdot 03$. Find the total loss of head. *Ans.* Total loss of head = 10′ 2½″.

17. A pipe contains a diaphragm with an orifice in it the area of which is one-fifth the sectional area of the pipe. Find the co-efficient of resistance of the diaphragm, assuming the contraction on passing through the orifice the same as that on efflux from a vessel through a small orifice in a thin plate. *Ans.* $F = 46$.

18. Find the loss of head in inches due to a bend through 45° of radius 6″ in a pipe 2″ diameter, the velocity of the water being 12′ per sec. *Ans.* 2″.

19. A plane area moves perpendicularly through water in which it is deeply immersed find the resistance per sq. foot at a speed of 10 miles per hour. Deduce the pressure of a wind of 20 miles per hour using the same co-efficient. *Ans.* Resistance = 269 lbs. Wind pressure = 1·312 lbs.

20. Compare the resistance of an area moving flatwise through the water with its resistance moving edgewise so far as due to surface friction, the co-efficient for which is ·004. *Ans.* Ratio = 312.

21. In question 1 suppose the ship moving at 10 knots and the orifice of entry so arranged as to cause no additional resistance : find the velocity of delivery. *Ans.* Additional head = 4·42′; velocity = 25′ per sec.

22. Water is supplied by a scoop to a locomotive tender at a height of 7′ above the trough. Assuming half the head lost by frictional resistances, what will be the velocity of delivery when the train is running at 40 miles per hour, and what will be the lowest speed of train at which the operation is possible? *Ans.* 36′ per sec. ; 14½ miles per hour.

23. A stream of water delivering 500 gallons per min. at a velocity of 15 feet per sec. strikes an indefinite plane (1) direct, (2) at an angle of 30° ; find the pressure on the plane.

24. Employ the principle of momentum to prove the formula on page 415 for the resultant centrifugal force of one-half a rotating ring of fluid.

REFERENCES.

For further information on subjects connected with the present chapter, the reader is referred to a treatise on Hydraulics by Professor W. C. Unwin, M.I.C.E., forming part of the article Hydro-Mechanics in the edition of the "Encyclopædia Britannica" now (1883) in course of publication.

CHAPTER XX.

HYDRAULIC MACHINES.

253. *Hydraulic Motors in General.*—Hitherto the energy exerted by means of a head of water has been supposed to be wholly employed in overcoming frictional resistances, and in generating the velocity with which the water is delivered at some given point. We now proceed to consider the cases in which only a fraction of the head is required for these purposes; the remainder then becomes a source of energy at the point of delivery by means of which useful work may be done. A machine for utilizing such a source is called an Hydraulic Motor.

Hydraulic energy may exist in three forms, according as it is due to motion, elevation, or pressure. In the first two cases it is inherent in the water itself, being a consequence of its motion or its position as in the case of any other heavy body. In the third it is due to the action of gravity or some other reversible force, sometimes on the water itself, but oftener on other bodies, as, for example, the load on an accumulator ram. The water is then only a transmitter of energy and not directly the source of it. As, however, the energy transmitted is proportional to the weight of water delivered, just as in the two other cases, the water is as before described as possessing energy. The energy per unit of weight is called "head," as sufficiently explained in the preceding chapter, and the "total head" is the sum of the "velocity head," the "actual head," and the "pressure head."

Hydraulic motors are classed according to the mode in which the water operates upon them, which may be either by weight, or by pressure, or by impulse, including in the last term also "reaction."

254. *Weight Machines.*—To utilize a head of water, consisting of an actual elevation (h) above a datum level at which the water can be delivered and disposed of, a machine may be employed in which the direct action of the weight of the water, while falling through the height h, is the principal motive force.

The common overshot water-wheel (Fig. 2, plate III., p. 137) may be taken as a type. Here the driving pair is a simple turning pair, and the driving link is the force of gravity upon the falling water, which acts directly on buckets open to the atmosphere. If G be the delivery in gallons per minute, the energy exerted in foot-pounds per minute is

$$E = 10Gh.$$

The head h is here measured from the level of still water in a reservoir which supplies the wheel. If v be the velocity of delivery to the wheel, the portion $v^2/2g$ is converted into energy of motion before reaching the buckets and operates by impulse. In a wheel of this class, therefore, the water does not operate wholly by weight. The speed of the wheel is limited to about 5 feet per second by the centrifugal force on the water, which, if too great, causes it to spill from the buckets. It will be seen hereafter that the velocity of the water should be about double this, so that v is about 10 feet per second, and the part of the fall operating by impulse is therefore about 1·5 feet. The remainder operates by gravitation, but a certain fraction is wasted by spilling from the buckets, and emptying them before reaching the bottom of the fall. More than one half the head operating by impulse is always wasted (Art 260), and this class of wheels is therefore only suitable for falls exceeding 10 feet. The great diameter of wheel required for very high falls is inconvenient, but examples may be found of wheels 60 feet diameter and more. The efficiency of these wheels under favourable circumstances is ·75, and is generally about ·65.

In "breast wheels" the buckets are replaced by vanes which move in a channel of masonry partially surrounding the wheel. The water is admitted by a moveable sluice through a grating of fixed blades in the upper part of the channel. The channel is thus filled with water, the weight of which rests on the vanes and furnishes the motive force on the wheel. There is a certain amount of leakage between the vanes and the sides of the channel, but this loss is not so great as that by spilling from the buckets of the overshot wheel. The efficiency is found by experience to be as much as ·75. As the diameter of the wheel is greater than the fall a breast wheel can only be employed for moderate falls.

In both these machines the water virtually forms part of the piece on which it acts. This link of the kinematic chain forms one element of the driving pair, while that attached to the earth forms the other. In the overshot wheel the water is contained in open buckets, in the breast wheel it is contained in a closed chamber or channel. A third class of weight machines is referred to farther on under the head of pumps.

255. *Hydraulic Pressure Machines in Steady Motion.*—A water wheel of great diameter is a slow-moving cumbrous machine, and for heads of 100 feet and upwards it is therefore necessary to employ a pressure or an impulse machine. Such machines are also often more convenient for low falls.

In pressure machines the driving link is compressed water, which is forced between the elements of the driving pair by some source of energy which supplies the necessary head. The head is sometimes an actual elevation either natural or artificial: in the docks at Great Grimsby the hydraulic machinery is operated from a tank placed on a tower 200 feet high. It is however difficult to get a considerable pressure in this way, and an apparatus called an Hydraulic Accumulator is therefore generally resorted to. Two forms occur, of which one is shown in Pl. IX. In the first a plunger or ram is forced into a cylinder by heavy weights placed in a plate-iron cage suspended from it and stayed by iron rods. The accumulator is supplied by pumps generally worked by steam, which is the ultimate source of the energy, the accumulator merely serving the purpose of a store of energy which can be drawn on at pleasure. For ordinary hydraulic machinery the pressure is limited to 750 lbs. per square inch from the difficulty of obtaining pipes of sufficient strength and of working slide valves under heavy pressures. In machines for rivetting and other special purposes, however, pressures of 1,500 lbs. per square inch and upwards are employed. The accumulator then consists of a cylinder *B* (Fig. 1, Pl. IX., p. 453), loaded with ring weights *EE*, sliding on a fixed spindle *F*, divided into two lengths, of which the upper portion is of smaller diameter than the lower.

In either form the accumulator provides a store of compressed water which can be supplied by suitable pipes to any number of machines, placed often at considerable distances. A head of 1,700 feet is thus readily obtained, and for special purposes much more: differences of level may therefore be disregarded as of small importance, and the water considered as operating wholly by pressure.

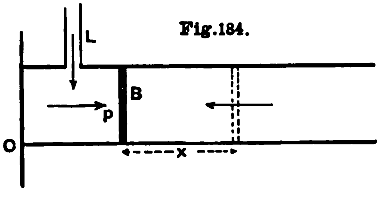


Fig. 184.

The driving pair of the machine forms a chamber of variable size which is alternately enlarged by the pressure of the water, and contracted to expel it. In most cases it is a simple cylinder *C* and piston *B* (Fig. 184): the water is admitted by a port from a pipe *L*, transmitting it from the accumulator at pressure p. Let the piston move through a space x, let A be its area, then

$$\text{Energy exerted} = pAx = p \, . \, X,$$

where X is the volume swept through by the piston. If w be as usual the weight of a cubic foot, $w . X$ is the weight of water which enters the cylinder as the piston moves through the distance x, and therefore

$$\text{Energy exerted per lb. of water} = \frac{p}{w} = \text{pressure head in cylinder.}$$

This might have been anticipated from what was said in the last chapter as to the meaning of the term "head," and in fact it is equally true if the driving pair be not a simple piston and cylinder, but of any other kind.

The head in the cylinder is less than that in the accumulator, on account of the friction in the supply pipe and other frictional resistances, and it is on the action of these resistances that the working of the machine depends. Let V be the velocity of the piston in its cylinder, p_0 pressure in accumulator, F the co-efficient of hydraulic resistance *referred to the velocity of the piston* (Art. 248), then, neglecting differences of level, also the heights due to velocities of working and accumulator pistons,

$$\frac{p_0 - p}{w} = F . \frac{V^2}{2g}.$$

If the machine be moving steadily the pressure p will be equal to the useful resistance which the piston is overcoming, increased by the friction of the piston in its cylinder. Thus p and p_0 will be known quantities, a certain definite velocity V_0 will then be determined, which may be described as the "speed of steady motion": it is given by the equation

$$V_0^2 = \frac{2g}{wF}(p_0 - p).$$

Since the hydraulic resistances may be increased to any extent at pleasure by the turning of a cock, it follows, that the speed of an hydraulic pressure machine can be regulated at pleasure. Further, if the resistance to the movement of the piston be diminished, the speed will increase only by a limited amount, and can, under no circumstances, be greater than is given by

$$V_0^2 = p_0 \frac{2g}{wF},$$

which can be regulated as before. The surplus energy is here absorbed by the frictional resistances, and an hydraulic pressure machine therefore possesses the very important, and, for many purposes, valuable characteristic that *it contains within it brakes which work automatically.*

256. *Hydraulic Pressure Machines in Unsteady Motion.*—Although the speed of a pressure engine cannot exceed a certain limit, which is easily found, yet it does not follow that that limit will ever be reached. When the engine starts, the piston and the water in the pipes have to be set in motion, the force required to do this is so much subtracted from that available to overcome resistances. A considerable time therefore elapses before a condition approaching steady motion can be obtained.

In Fig. 170, p. 419, water is supposed flowing through a pipe with a velocity u. Two pistons at a distance x enclose water between them, as in Art. 242, then the difference of pressure $p_1 - p_2$ in the case of steady motion is simply balanced by the surface friction, but in unsteady motion is partially employed in accelerating the flow of the water. Neglecting friction the acceleration g' will be given by the formula

$$(p_1 - p_2)A = W \cdot \frac{g'}{g},$$

where A is the sectional area of the pipe and W is the weight of the water between the pistons. Replacing W by $Ax \cdot w$, as in the preceding article,

$$\frac{p_1 - p_2}{w} = x \cdot \frac{g'}{g},$$

which gives a simple formula for the change of pressure head due to inertia. Now if nA be the area of the working piston, the velocity of the water in the pipe is n times the velocity of the piston, and the accelerations are necessarily in the same ratio; and hence it follows that the difference of pressure head between cylinder and accumulator due to an acceleration g' of the piston is for a length of pipe l

$$\frac{p_1 - p_2}{w} = nl \cdot \frac{g'}{g}.$$

The inertia of the piston itself requires a certain pressure to accelerate it. Let q_0 be the "pressure equivalent to that weight" found, as in Art. 109, page 212, then the pressure due to inertia is

$$q = q_0 \cdot \frac{g'}{g};$$

then, dividing by w,

$$\frac{p_1 - p_2}{w} + \frac{q}{w} = \left(nl + \frac{q_0}{w}\right)\frac{g'}{g} = \lambda\frac{g'}{g},$$

where λ is a certain length. This may be described as the "length of working cylinder equivalent to the inertia of the moving parts," and may always be readily calculated for any given engine. (See Appendix.) The pressure in feet of water necessary to overcome inertia will then always be given by the simple formula

$$\text{Pressure due to inertia} = \lambda\frac{g'}{g}.$$

It will now be seen that the weight of water in the pipes and cylinders is so much added to the weight of the piston, that in the pipes being multiplied by the square of the ratio of areas of cylinder and pipe. A water-pressure engine is therefore a machine with very heavy moving parts, a circumstance which greatly limits its speed irrespectively of frictional resistances. The smaller the pipes the heavier the parts virtually are, and this must be considered as well as friction (p. 420) in fixing their diameter.

It will be advisable to consider a particular case more in detail. Suppose, as is sometimes the case in practice, that a water-pressure engine is employed to turn a crank, and let us suppose that the crank shaft rotates nearly uniformly as in Ch. IX., then the difference between the pressure in the accumulator and that transmitted to the crank pin may be represented graphically thus :—

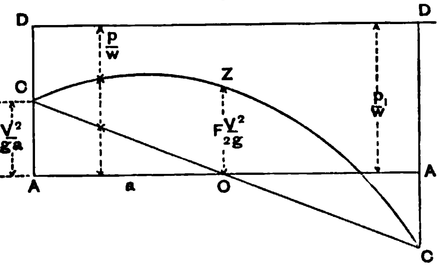
Let V be the velocity of the crank pin and let the stroke be $2a$ or AA in the diagram (Fig. 185). Set up

$$CA = \lambda \cdot \frac{V^2}{ga},$$

and draw the sloping line COC. Then, as in Art. 109, already cited, the ordinate of that line represents the pressure necessary to overcome the inertia of the piston and the water connected with it. Again, set up

$$OZ = F\frac{V^2}{2g} = CA \cdot \frac{Fa}{2\lambda},$$

Fig.185.



and on the oblique base COC draw the parabola CZC, then (comp. Arts. 20, 109) the ordinate of this parabola will represent the pressure necessary to overcome the hydraulic resistances at every point. If then the horizontal line DD be drawn at a height representing the pressure in the accumulator, the intercept between that line and the parabola will represent the pressure transmitted to the crank pin at each point of the stroke. The slope of CC and the height of the parabola increase rapidly with the speed, which must never be great enough to cause the parabola to touch DD, otherwise a violent shock will occur. The same effect will be produced by any falling off in the useful resistance: the angular acceleration of the crank shaft then raises the central part of the line CC and with it the line of frictional resistances. It should be observed that the curve of frictional resistances may also be taken to represent the kinetic energy of the piston, both these quantities being proportional to the square of the velocity of the piston. It is therefore the graphical integral of the curve of acceleration (Ch. IX.).

The simple example here given will serve as an illustration of the great variations of pressure which occur in water-pressure engines and their consequent liability to shocks. For which reason escape valves or air chambers must be provided to relieve the pressure when it becomes excessive. Unless the resistance be very uniform an additional accumulator is required as near as possible to the machine.

257. *Examples of Hydraulic Pressure Machines.*—Water-pressure engines form a large and interesting class of hydraulic motors of which a few examples will now be given.

(1) In direct-acting lifts a weight is raised by the direct action of fluid pressure on a ram the stroke of which is equal to the height lifted. The weight here rests on a cage or platform fixed to the upper end of the ram and sliding in guides. The water is frequently supplied from a tank at a moderate elevation, so that the pressure head diminishes as the lift rises. This is a very convenient arrangement for the purpose, as it supplies an additional pressure at the bottom of the stroke where it is required to overcome inertia at starting, and a diminished pressure at the top where the lift requires to be stopped. The useful resistance is here constant and the pressure head would be represented by the ordinates of a sloping line. A diagram of speed and acceleration may be constructed by a process similar to that given in the last article.

(2) A direct acting lift necessarily occupies a great space, and the stroke of the working cylinder is therefore often multiplied by the use of blocks and tackle as shown in Fig. 2, Plate IX. The cylinder may be placed in any convenient position, and the chain passes from the blocks over fixed pulleys to the cage which is suspended from it. The friction of the pulleys is here considerable, and there is a liability to breakage; but for convenience the arrangement is one which is frequently employed.

(3) In hydraulic cranes the working cylinder is sometimes placed below and sometimes occupies the crane post which is tubular. The stroke is multiplied by tackle as in the previous case, the chain passing through the crane post and over fixed pulleys to the extremity of the jib. An example is shown in Fig. 2, Plate IX., p. 453.

(4) A water-pressure engine may be employed to turn a crank. Three working cylinders inclined at 120° are frequently used as shown in Fig. 1, Plate X., p. 453. They are single-acting and drive the same crank as in the small steam engines of the same type employed where

great speed is required. The water is admitted to the outer ends of the cylinders, so that the piston rods are always in compression.

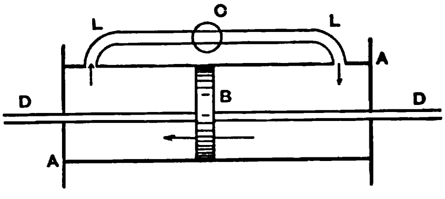
(5) The hydraulic mechanism applied to work heavy guns on board ship consists of a cylinder in which works a piston attached to a rod, the sectional area of which is one-half that of the cylinder. If water be admitted at both ends of the cylinder the piston moves outwards, but if to the inner end only, it moves inwards. The motive force in either case is the same, being due to the difference of areas. This apparatus serves also as a brake of the kind described in the next article. For details and illustrations the reader is referred to the *Gunnery Manual.*

258. *Hydraulic Brakes.*—It has been sufficiently explained that hydraulic resistances absorb an amount of energy which varies as the square of the speed. A hydraulic machine therefore may be employed as a brake, and it is in this way that large amounts of surplus energy are most easily disposed of. Moreover, by its use the speed of any machine to which it is applied is readily controlled.

An hydraulic brake is constructed by interposing a mass of fluid between the elements of a pair so that any motion of the pair causes a breaking-up of the fluid with a corresponding resistance.

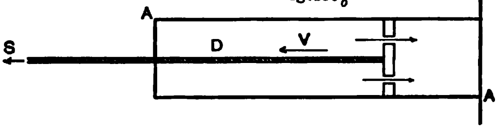
A common case is that of a sliding pair consisting of a piston and cylinder filled with water or oil, which passes from one side of the piston to the other whenever the piston moves. Two examples of this apparatus are shown in skeleton in Figs. 186*a*, 186*b*. In the first (Fig. 186*a*) the piston rod *DD* projects through both cylinder covers, and communication is made between the two ends of the cylinder by a pipe *LL* provided with a cock *C*, which can be closed at pleasure. At *D* the rod is attached to the piston rod of a steam cylinder employed to obtain the very considerable force necessary to work the starting and reversing gear of large marine engines. The resistance of this brake is zero when the piston begins to move, but increases as the square of the speed, and thus effectually prevents it from moving too rapidly. The maximum speed is controlled by turning the cock. For a detailed description of this gear the reader is referred to a treatise on the *Marine Engine*, by Mr. Sennett.

Fig. 186$_a$



In the second (Fig. 186b) the water passes from one end of the cylinder through orifices in the piston itself. This is the common "compresser" or Service Buffer.* The piston rod in this case passes out at one end only of the working cylinder, and is attached to the gun, the recoil of which is to be checked. The theory of this apparatus is of some interest, and will now be briefly considered.

Fig.186$_b$



Let n be the ratio of the area of the piston to the *effective* area of the orifices, then the loss of head must be

$$\frac{p_1-p_2}{w}=(n-1)^2\frac{V^2}{2g},$$

where V is the speed of piston and p_1, p_2 are the pressures on the two sides of the piston. Hence the pull

$$S=wA(n-1)^2\frac{V^2}{2g}$$

on the piston rod is necessary to overcome the hydraulic resistance at this speed. The gun is gradually brought to rest by this resistance, aided by the friction of the slide.

At the instant of firing a certain amount of kinetic energy is generated in the gun given by the formula

$$\text{Energy of Recoil}=\frac{WV_0^2}{2g}, \text{ (Art. 135, p. 251)}$$

where V_0 is the maximum velocity of recoil. As the gun recoils its velocity diminishes, and if P_0 be the friction of the slide the retarding force will be

$$S+P_0=wA(n-1)^2\frac{V^2}{2g}+P_0.$$

The maximum value of S will be found by writing V_0 for V, and may be denoted by S_0.

To represent this graphically, in Fig. 187 draw a curve in which the ordinate KN at any point N represents the retarding force after the gun has recoiled through the space ON from the point O, at which the action of the powder pressure ceases, and the gun has its maximum velocity V_0. This curve will start from a point A such that

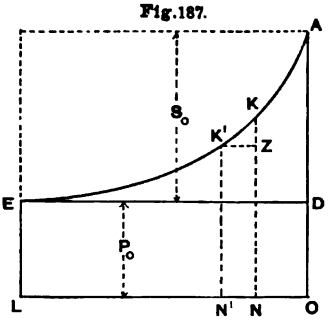
$$OA=S_0+P_0,$$

---

* *Manual of Gunnery for Her Majesty's Fleet*, p. 68.

and will reach the horizontal DE at a height P_0 above the base line at a point E, such that OL is the complete recoil. The area $OAEL$ of this curve represents the energy of recoil which has all been absorbed by the frictional resistance of the slide and the hydraulic resistance of the compressor. Further, the area $KNN'K'$ between two ordinates will represent the diminution of energy as the gun recoils through the space NN' between them, a circumstance which enables us to construct the curve, for if VV' be the velocities of the recoiling gun at NN' respectively,

Fig. 187.



$$\text{Area } KNN'K' = \frac{W(V^2 - V'^2)}{2g}.$$

But if SS' be the corresponding values of S,

$$KZ = S - S' = wA(n-1)^2\frac{V^2 - V'^2}{2g};$$

and if the ordinates be taken near together the area in question will be nearly $KN \, . \, NN'$. We have therefore, by division,

$$\frac{KZ}{KN} = NN' \, . \, \frac{wA(n-1)^2}{kW}.$$

That is, if a number of equidistant ordinates be drawn near together the ratio of consecutive ordinates is constant. The curve may be roughly traced from this property; it is identical with the curve already drawn in Art. 123, p. 236, except that it is a linear instead of a polar curve.

The mean resistance to recoil is given by the equation

$$(\bar{S} + P_0)l = \text{Energy of Recoil},$$

where l is the distance traversed. It would, of course, be advantageous to have a uniform resistance to recoil, because the maximum pressure in the compressor would be diminished and less strain thrown on the gear. This is the object of the various modified forms of the compressor, in which the orifices are not of constant area, but become smaller as the recoil proceeds. In order that the resistance may be constant we must have

$$\bar{S} = wA(n-1)^2\frac{V^2}{2g},$$

so that $(n-1)\ V$ is constant. Further, since the retardation is uniform

$$V^2 - 2g \, . \, \frac{\bar{S} + P_0}{W}x,$$

where x is the distance from the end of the recoil. It appears therefore that the orifices should vary in such a way that $(n-1)^2x$ should be constant. Descriptions of two forms of compressor, with varying orifices, will be found in the *Gunnery Manual.*

Instead of a sliding pair we may employ a turning pair. This is the common "fan" or "fly" brake used to control the speed and absorb the surplus energy of the striking movement of a clock, or in other similar cases. A friction dynamometer (p. 261) was designed by the late Mr. Froude for the purpose of measuring the power of large marine engines, in which the ordinary block or strap surrounding a shaft or drum is replaced by a casing in which a wheel works. Vanes attached to the wheel and the fixed casing thoroughly break up a stream of water passing through the casing. Any amount of energy may thus be absorbed without occasioning any considerable rise of temperature. Siemens' combined brake and regulator has been mentioned already on page 259.

259. *Transmission of Energy by Hydraulic Pressure.*—Energy may be distributed from a central source, and transmitted to considerable distances with economy by hydraulic pressure. The delivery in gallons per minute of a pipe d'' diameter is

$$G = 27\sqrt{\frac{h'}{l}} \cdot d^{\frac{5}{2}} \qquad \text{(Art. 243).}$$

Assume now that the pipe supplies an hydraulic machine at a distance of l feet from an accumulator in which h is the head. Further, suppose that n per cent. is lost by friction of the pipe, then the power transmitted in foot-lbs. per minute is

$$10Gh = 270\,h\sqrt{\frac{nh}{100l}} \cdot d^{\frac{5}{2}},$$

and the distance to which N horse power can be transmitted with a loss of n per cent. is in feet

$$l = \frac{h^3d^5n}{1{,}500{,}000N^2} \qquad \text{(nearly).}$$

With the usual pressure in accumulators of 750 lbs. per square inch, or 1,700 feet of water, this gives the simple approximate formula

$$l = 3300\,\frac{d^5n}{N^2}.$$

Thus for example, 100 horse power may be transmitted by a 5" pipe to a distance of 4 miles, or 10 horse power by a 1" pipe to a distance of 220 yards, with a loss by friction not exceeding 20 per cent. The diameter of pipe is limited by considerations of strength and cost.

The power of a motor supplied by a given pipe does not increase indefinitely as its speed increases, but is greatest when one-third of the head is lost by friction.* The maximum possible power is therefore given by the formula

$$H.P. = 220\sqrt{\frac{d^5}{l}} \qquad \text{(approximately).}$$

This is of course two-thirds the value of N in the preceding formula.

260. *Pumps.*—If the direction of motion of an hydraulic motor be reversed by the action of sufficient external force applied to drive it, while, at the same time, the direction of the issuing water is reversed so as to supply the machine at the point from which it originally proceeded, we obtain a machine which raises water instead of utilizing a head of water. Every hydraulic machine therefore may be employed to raise water as well as to do work, and most of them actually occur in this form; they are then called PUMPS, though in some cases this name would not be used in practice. Much of what has been said about motors applies equally well to pumps: the principal difference lies in the fact that the useful resistance which the pump overcomes is always reversible, whereas in the motor this is not necessarily the case. The principles of action and the classification of hydraulic machines are, in the main, the same in both cases. Some points omitted while considering motors as being of most importance in pumps, and certain differences of action between the two will now be briefly noticed. Certain machines occurring principally as pumps will be mentioned.

(1) If the direction of motion of an overshot wheel be reversed a machine is obtained which is known as a "Chinese Wheel." It picks up water in its buckets and raises it to a height somewhat less than the diameter of the wheel. This machine is little used, but a reversed breast wheel is frequently employed in drainage operations, under the name of a "scoop," or "flash" wheel. The working pair is here a turning pair, but in the chain pump we find an example in which one of its elements is a chain passing over pulleys. The chain is endless and is provided with flat plates fitting into a vertical pipe, the lower end of which is below the surface of the water, and through which the water is raised. In the common dredging machine the closed channel (p. 440) is replaced by buckets. In a third class of weight machines the water occupies a moveable chamber and forms with it a kinematic pair with only one solid element, while it forms, with the link attached to the earth, a working pair which has also but one solid element. The Archimedian screw, and certain varieties of "scoop" wheel, in which the water enters

---

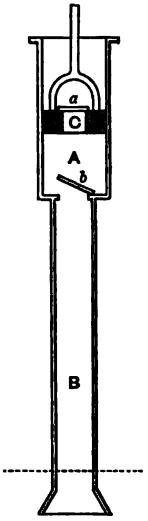
* This result was pointed out to the writer by Mr. Hearson. It appears to be little known.

the scoop at the circumference of the wheel and is delivered at the centre, are examples of this kind.

(2) The most common forms of pumps are the "lift" or "force" pumps, which consist of a chamber which expands to admit the water to be lifted and contracts in the act of lifting; they are therefore pressure machines like those considered in Art. 255-6, but reversed. The name "pump" originally applied to these machines alone.

Fig. 188 shows a common lift pump. *A* is a cylinder at a certain height h_1 above the water to be raised, *C* is a piston working in the cylinder by the action of which the water is lifted. The piston has orifices in it which permit the water to pass through. The orifices are closed by a valve, as is also the opening at the bottom of the cylinder. These valves are simple "flaps" which open on hinges to permit the water to pass upwards, but close the passage to motion in the opposite direction, thus acting as a ratchet (p. 153). Assuming the piston at the bottom of its stroke, at rest close to the bottom of the cylinder, let it be supposed to rise; the valve *b* will rise and allow air to pass if any. After several strokes the air will be nearly exhausted, and if h_1 be not too great the empty space will be filled with water raised from the tank by atmospheric pressure. Thus the water will pass into the cylinder closely following the piston. At the top of the stroke the piston commences to descend, *b* closes and *a* opens, allowing the water to pass above the piston. This water is now raised by the piston to any required height. In force pumps the process is the same, but the water passes out through an orifice in the bottom of the cylinder instead of through the piston; the raising of the water above the level of the cylinder is done in the down stroke instead of the up.

Fig. 188.



The difference between this action and that of a pressure motor lies mainly in the valves, which here open and close automatically by the action of the water, instead of by external agency. Further, the pump wholly or partly works by *suction*, a method by no means peculiar to pumps, for it also occurs in motors, but not so frequently. The height of the water barometer is 34 feet, but the height to which a pump will work by suction is not so great. When the piston is at the bottom of its stroke there must, for safety, always be a certain clearance space below. This space always contains air, the pressure of which diminishes as the piston rises, but cannot be reduced to zero. Further, a certain pressure is required to overcome the weight and friction of the valve before it opens. At least 3 feet of the lift is absorbed in this way, and generally considerably more. To obtain a high vacuum for scientific purposes, air pumps are specially designed to meet these difficulties. Also, leakage must be allowed for and the diminution on account of friction and inertia, which will be considerable if the speed be too great or the pipes too small, as will be understood on reference to Arts. 255-6, all of which applies to pumps as much as to motors. It is hardly necessary to observe that power is neither gained nor lost by the use of suction; it simply enables the working cylinder to be placed above the water to be lifted, an arrangement which is in most cases convenient. The limit in practice is about 25 feet.

Pumps are commonly, but not always, single-acting; they are worked by the direct action of a reciprocating piece, or by means of a rotating crank. In the first case, when independent, a piston acted on by steam or water pressure is attached to a prolongation of the pump plunger: a crank and fly-wheel is often added, as in Fig. 4, Plate II., p. 109, to control the motion and define the stroke. When driven by the crank three working cylinders, placed side by side with a three-throw crank, are commonly used, in order to equalize the delivery, and so to avoid the shocks due to changes of velocity. An air-chamber, forming a species of accumulator, may also be used with the same object. An arrangement of pumps, as applied by Messrs. Donkin & Co. to raise water from a well 200 feet deep and force it to a height of 143 feet above the engine house, may be mentioned as an example. A set of lift pumps at the bottom of the well worked by "spear" rods from the surface, are combined with a set of force pumps in the engine house itself. The speed of these pumps is about 80 feet per minute, and they deliver about 600 gallons per minute. Pumps almost always have a certain "slip," that is, they deliver less water than corresponds to the piston displacement and number of strokes: in this example the slip was 12 per cent. The efficiency of the pumps and mechanism of the engine was found to be 66 per cent. by careful experiments.*

In raising water from great depths in mines, force pumps at the bottom of the mine are used, worked by heavy "spear" rods from a beam engine at the surface. The weight of the rod supplies the motive force during the downward stroke of the pump; while the engine, which is single-acting, raises the rods again during the downward stroke of the steam piston.

DESCRIPTION OF PLATES IX. AND X.

In order further to illustrate the action of water-pressure machines Plates IX. and X. have been drawn.

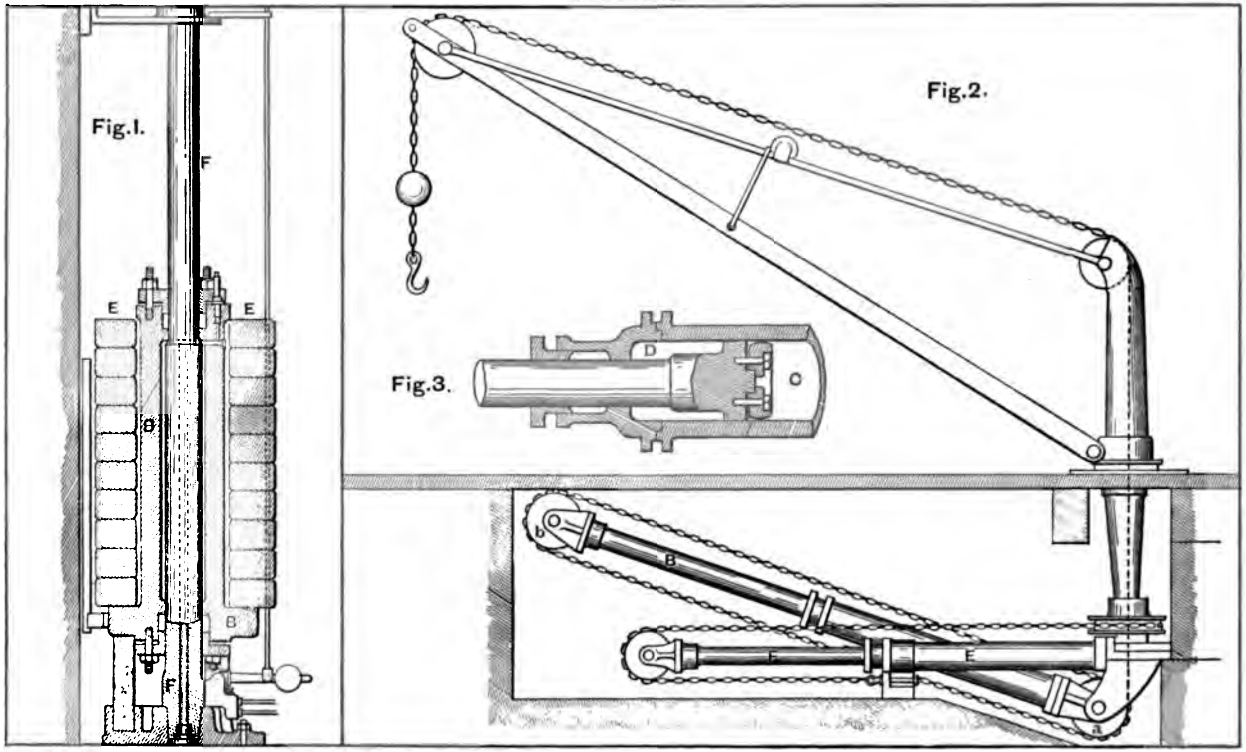
---

* *Minutes of Proceedings of the Institution of Civil Engineers*, vol. 66.

ray, and
cientif
iculties
ount of
o great
2554
nece-
actice
ater to
e limit

orked
ating
team
r: a
con-
hree
are
the
s of
ent
ed
se
d
h
e
s

PLATE IX.



To face page 453.

PLATE X.

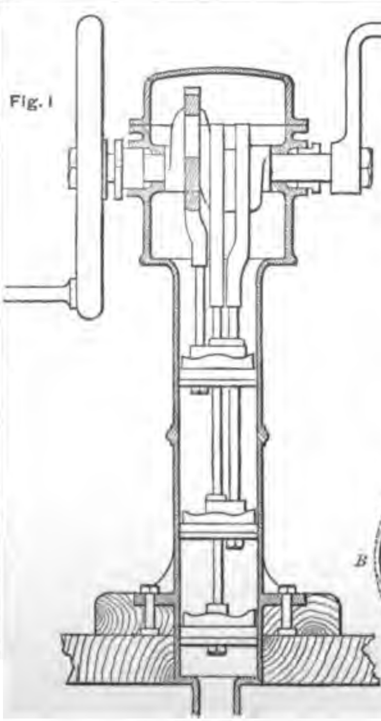


Fig. 1

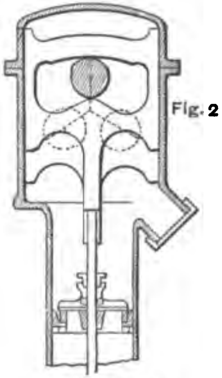


Fig. 2

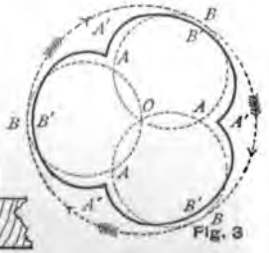


Fig. 3

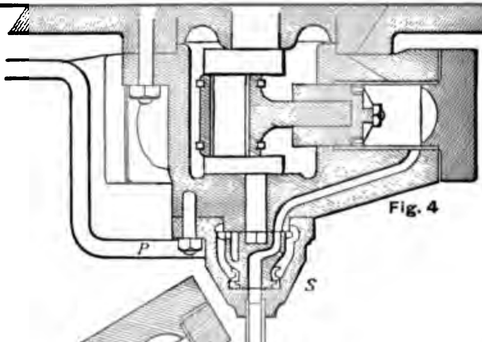


Fig. 4

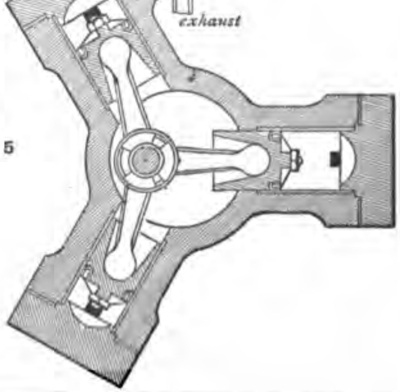


Fig. 5

To face page 453.

Fig. 1, Plate IX., shows the differential accumulator described on page 441.

In Fig. 2 is represented an hydraulic crane, designed by Sir. W. Armstrong, for lifting weights of 2 to 3 tons. In it the hydraulic power is applied to rotate the crane as well as to lift the weight.

In order to effect the lift the high-pressure water from the accumulator is admitted to the cylinder A, and forces out the plunger B. There are two pulleys at a and two at b. One end of the chain is secured to the cylinder A, it is led round b, then round a, again round b, then under the second pulley at a up through the hollow crane post on to the weight as shown. The effect of this arrangement is that any movement of the plunger B is at the hook multiplied four times.

If B is simply a plunger working in a stuffing box, then the expenditure of energy is always the same whatever weight is being lifted, and the amount must be equal to that which corresponds to lifting the maximum possible weight.

This is an objection which is common to all such machines. The surplus energy is expended in overcoming frictional resistances (p. 442). To mitigate this evil, in cranes of high power the plunger has a piston end, which fits a bored cylinder, and is provided with a cup leather, as shown in Fig. 3. The sectional area of the plunger is about one-half that of the cylinder. If a light weight is to be lifted, water is admitted to both sides of the piston, and the difference of the pressures, equal to what would be exerted on a simple plunger, is available for effecting the lift. When it is required to lift a heavy weight water is admitted to the side C only of the piston, the annular space D being put in communication with the atmosphere. Thus the full pressure due to the area of the piston is exerted with the corresponding expenditure of water.

For the purpose of rotating the crane a pair of cylinders, E, are provided, of which one only is shown in the figure. The thrusting out of the plunger F of one of them by the pressure of the water causes the other to be drawn in by means of a chain which passes around a recessed pulley secured to the crane post.

In Plate X., Figs. 1 and 2 show the construction of Downton's Pump, so much used on board ship. In the barrel work three buckets with flap valves, as shown in Fig. 2. The rods to which the upper and second buckets are attached are necessarily out of centre. The rods to the lower buckets pass through deep stuffing boxes in the buckets above, and thus the buckets are maintained from canting seriously. The movement of the buckets is effected by a three-throw crank, the crank pins, which are not round, being set at 120° apart. These pins fit and work in a curved slot in the bucket rod heads. Assuming the admission of no air but water only from below, the discharge of the pump will at each instant equal the displacement of the fastest upward moving bucket. Accordingly the rate of discharge may be represented by a curve, as in Fig. 3. If the slot in the rod head were straight and the pin round, then, the crank moving uniformly, in direction shown, the velocity of discharge would be represented by the radii from O to the dotted curve $BABABA$, which is made up of parts of three circles, the position of the radius being that of either of the three cranks. The effect of the curved slot is to diminish the maximum and increase the minimum discharge, as shown by the full curve $B'A'B'A'B'A'$.

Figs. 4 and 5 of this Plate are sections of the hydraulic engine referred to on page 445, employed to rotate a capstan. It need only be further added that a single rotating valve V suffices for admission and exhaust of all three cylinders. The high-pressure water is supplied by the pipe P to the passage S surrounding the valve and exhausted from the cylinders through the central passage.

EXAMPLES.

1. In estimating the power of a fall of water it is sometimes assumed that 12 cubic feet per second will give 1 H.P. for each foot of fall: what efficiency does this suppose in the motor? *Ans.* ·72.

2. An accumulator ram is 9 inches diameter, and 21 feet stroke: find the store of

energy in foot-lbs. when the ram is at the top of its stroke, and is loaded till the pressure is 750 lbs. per square inch? *Ans.* 1,000,000 foot-lbs.

3. In a differential accumulator the diameters of the spindle are 7 inches and 5 inches; the stroke is 10 feet: find the store of energy when full, and loaded to 2,000 lbs. per square inch. *Ans.* 377,000 foot lbs.

4. A direct-acting lift has a ram 9 inches diameter, and works under a *constant* head of 73 feet, of which 13 per cent. is required by ram friction and friction of mechanism. The supply pipe is 100 feet long and 4 inches diameter. Find the speed of steady motion when raising a load of 1,350 lbs., and also the load it would raise at double that speed?

Ans. Speed = 2 feet per second.
Load = 150 lbs.

5. In the last question, if a valve in the supply pipe is partially closed so as to increase the co-efficient of resistance by 5½, what would the speed be? *Ans.* 1·6 f.s.

6. Eight cwt. of ore is to be raised from a mine at the rate of 900 feet per minute by a water-pressure engine, which has four single-acting cylinders, 6 inches diameter, 18 inches stroke, making 60 revolutions per minutes. Find the diameter of a supply pipe 230 feet long, for a head 230 feet, not including friction of mechanism. *Ans.* Diameter = 4 inches.

7. Water is flowing through a pipe 20 feet long with a velocity of 10 feet per second. If the flow be stopped in one-tenth of a second, find the intensity of the pressure produced, assuming the retardation during stoppage uniform. *Ans.* 62 feet of water.

8. If λ be the length equivalent to the inertia of a water-pressure engine, F the co-efficient of hydraulic resistance, both reduced to the ram, v_0 the speed of steady motion; find the velocity of ram, after moving from rest through a space x against a constant useful resistance. Also find the time occupied.

Ans. $v^2 = v_0^2\left(1 - \epsilon^{-\frac{Fx}{\lambda}}\right)$; $t = \frac{\lambda}{F \cdot v_0} \log_\epsilon \frac{v_0 + v}{v_0 - v}$.

9. An hydraulic motor is driven from an accumulator, the pressure in which is 750 lbs. per square inch, by means of a supply pipe 900 feet long, 4 inches diameter; what would be the maximum power theoretically attainable, and what would be the velocity in the pipe at that power? Find approximately the efficiency of transmission at half power. *Ans.* H.P. = 240; $v = 22$; efficiency = ·96 nearly.

10. A gun recoils with a maximum velocity of 10 feet per second. The area of the orifices in the compressor, after allowing for contraction, may be taken as one-twentieth the area of the piston: find the maximum pressure in the compressor in feet of liquid. *Ans.* 560 to 594.

11. In the last question assume weight of gun 12 tons; friction of slide 3 tons; diameter of compressor 6 inches; fluid in compressor water; find the recoil. *Ans.* 4 feet 2½ inches.

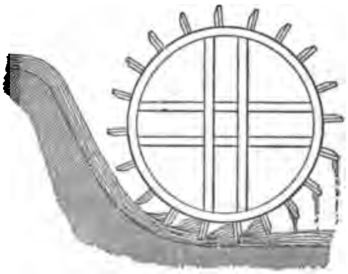
12. In the last question find the mean resistance to recoil. Compare the maximum and mean resistances each exclusive of friction of slide. *Ans.* Total mean resistance = 4·4 tons. Ratio = 2·5.

261. *Impulse and Reaction Machines in General.*—The source of energy may be a current of water or the head may be too small to obtain any considerable pressure, and it is then necessary to have some means of utilizing the energy of water in its kinetic form. A machine for this purpose operates by changing the motion of the water and utilizing the force to which the change gives rise. If the water strikes a moving piece and is reduced to rest relatively to it, the machine works by "impulse," and if it be discharged from a moving piece, by

"reaction." There is no difference in principle between these modes of working, and both may occur in the same machine. In either case, the motive force arises from the mutual action between the water and the piece which changes their relative motion. Machines of this class are also employed for high falls when the low speed of pressure machines renders their use inconvenient or impossible. The water is then allowed to attain a velocity equivalent to a considerable portion of the head immediately before entering the machine, so that its energy is, in the first instance, wholly or partially converted into the kinetic form.

The simplest machine of this kind is the common undershot wheel, consisting of a wheel (Fig. 189) provided with vanes against which the water impinges directly. Let the velocity of periphery of the wheel be V, then the water after striking the vanes is carried along with them at this velocity. If, then, the original velocity of the water be v, the diminution of velocity due to the action of the vanes will be $v - V$. Let W be the weight of water acted on per second, then the impulse on the wheel must be

Fig. 189.



$$P = \frac{W(v - V)}{g},$$

but if A be the sectional area of the stream,

$$W = Avw,$$

this being the weight of water per second which comes in contact with all the vanes taken together.

$$\therefore\ P = \frac{w}{g}Av\,(v - V).$$

The power of the wheel is PV foot-lbs. per second, and the energy of the stream is $Wv^2/2g$, therefore

$$\text{Efficiency} = \frac{2V(v - V)}{v^2}.$$

This is greatest when $V = \frac{1}{2}v$ and its value is then ·5, showing that the wheel works to best advantage when the speed of periphery is one-half that of the stream, but that the efficiency is low, never exceeding ·5.

Such wheels may be seen working a mill floating in a large river, or in other similar circumstances, but they are cumbrous and, allowing for various losses, not included in the preceding investigation; their efficiency is not more than 30 per cent. In the early days of hydraulic machines, they were often used for the sake of simplicity or, as in the example shown in the figure, from a want of comprehension of their principle.* In mountain countries, where unlimited power is available, they are still found. The water is then conducted by an artificial channel to the wheel, which sometimes revolves in a horizontal plane. When of small diameter their efficiency is still further diminished.

In overshot wheels and other machines operating chiefly by weight the head corresponding to the velocity of delivery is partly utilized by impulse, and the speed of the wheel is determined by this consideration. In all cases of direct impulse, if h is that part of the head operating by impulse, the speed of maximum efficiency is

$$V = \tfrac{1}{2}\sqrt{2gh} = 4\sqrt{h},$$

or in practice somewhat less, and at that speed at least half that head is wasted. The great waste of energy in this process is due partly to the velocity V with which the water moves onward with the wheel, and partly to breaking-up during impulse. It is in fact easy to see that one-fourth the head is wasted by each of these causes. To avoid it, the water must be received by the moving piece against which it impinges without any sudden change of direction, and must be discharged at the lowest possible velocity, effects which may be produced by a suitably-shaped vane curved so as to deflect the water gradually and guide it in a proper direction. The principle on which such a vane is designed may be explained by the annexed diagram. In Fig. 190, AB is a vane, moving vith velocity V in a given direction, against which a jet strikes. Drawing a diagram of velocities, let Oa represent v, the velocity of the jet, and let OO' represent V. Then as before (p. 434) $O'a$ represents the velocity of the jet relatively to the vane, and, in order that the water may impinge without shock, the tangent to the vane at A must be parallel to $O'a$. The vane is now curved so as gradually to deflect the water, in doing which there is a mutual action between the jet and the vane which produces the motive force which drives the wheel. If the water leave the vane at B, its velocity relatively to the vane is represented by $O'b$ drawn parallel to the vane at B, and somewhat less than $O'a$ in magnitude, to allow for friction, unless the water be enclosed in a passage, when it will bear some given proportion to $O'a$.

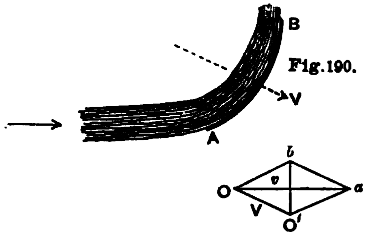


Fig. 190.

---

* See Fairbairn's *Millwork and Machinery*, from which this figure is taken, vol. I., p. 149.

The absolute velocity with which the water moves at B is now represented by Ob, and this may be arranged to deliver the water in a convenient direction with a velocity just sufficient to clear the wheel and no more. The efficiency may then theoretically be unity, and, practically, after allowing for losses, may be increased to ·65 or ·7. Vanes of this kind were applied to water wheels by Poncelet. The wheel, in this case, revolves in a vertical plane, and the water, on impinging at A, ascends to B: it then descends under the action of gravity, and is discharged at the same point A at which it entered, so that $O'b$ is approximately equal and opposite to $O'a$.

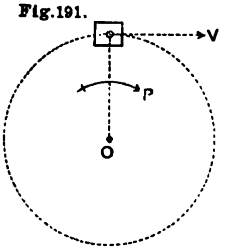
In all impulse and reaction machines there is a speed of maximum efficiency which, as in the simple case first considered, is given by the formula

$$V = k\sqrt{2gh},$$

where k is a fraction depending on the type of machine.

262. *Angular Impulse and Momentum.*—The most important of these machines are those in which the change of motion produced in the water is a motion of rotation, and it is needful to consider that form of the principle of momentum which is applicable to such cases.

Fig. 191.



In Fig. 191, W is a weight describing a circle round O with velocity V; then the product of its momentum by the radius r is called the "moment of momentum" of the weight about O. If O represent an axis to which W is attached rigidly, we may imagine it turning under the action of a force P at a radius R. The moment of P multiplied by the time during which it acts is called the "moment of impulse."

During the action of P the weight will move quicker and quicker, and the motion is governed by the principle expressed by the equation

$$\text{Moment of Impulse} = \text{Change of Moment of Momentum.}$$

If L be the moment of P, then, taking the time as one second,

$$L = \text{Change of Moment of Momentum per second.}$$

This equation is true, not only for a single weight and a single force, but also for any number of weights and any number of forces. As in other forms of the principle of momentum it is also true, notwithstanding any mutual actions or any relative movements of the weights or particles considered. Further, any radial motions which the particles possess may be left out of account, for they do not influence the moment of momentum. A particular case is when $L = O$, then the moment of momentum remains constant, a principle known as the Conservation of Moment of Momentum. The terms "moment of momentum" and "moment of impulse" are often replaced by "angular momentum," "angular impulse."

A weight rotating about an axis is capable of exerting energy in two ways. First, it may move away from the axis of rotation, overcoming by its centrifugal force a radial resistance which it just overbalances. Secondly, it may overcome a resistance to rotation in the shaft to which it is attached. In either case the work done will be represented by a diminution in the kinetic energy of the weight.

If the shaft be free, the diminution of kinetic energy must be equal to the work done by the centrifugal force, and it may be proved in this way, that if V be the velocity of rotation of the weight, r the radius,

$$Vr = \text{constant},$$

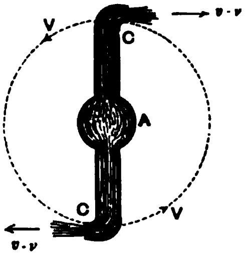
an equation equivalent to the conservation of the moment of momentum.

Conversely, energy may be applied to a rotating weight either by moving it inwards against its centrifugal force, or by a couple applied to the axis of rotation.

In turbines both modes of action occur together as we shall see presently; and the principle of momentum is the most convenient way of dealing with the question.

263. *Reaction Wheels.*—Fig. 192 shows a reaction wheel in its simplest form. CAC is a horizontal tube communicating with a vertical tubular axis to which it is fixed, and with which it rotates. Water descends through the vertical tube, and issues through orifices at the extremities of the horizontal tube so placed that the direction of motion of the water is tangential to the circle described by the orifices. The efflux is in opposite directions from the two orifices, and a reaction is produced in each arm which furnishes a motive force. There are two methods of investigating the action of this machine which are both instructive. Frictional resistances are, in the first instance, neglected.

Fig. 192.



(1) Let the orifices be closed, and let the machine revolve so that the speed of the orifices in their circular path of radius r is V. Centrifugal action produces a pressure in excess of the head h existing when the arms are at rest, the magnitude of which is $V^2/2g$, and this pressure is so much addition to the head, which now becomes

$$H = h + \frac{V^2}{2g}.$$

This quantity H may also be considered as the head "relative to the moving orifices" estimated as in Art. 239.

When the orifices are opened, the water issues with velocity v given by

$$v^2 = 2gH = V^2 + 2gh;$$

thus the water issues with a velocity greater than V, and after leaving the machine has the velocity $v - V$ relatively to the earth. The energy exerted per lb. of water is h, and this is partly employed in generating the kinetic energy corresponding to this velocity. The remainder does useful work by turning the wheel against some useful resistance, so that we have per lb. of water

$$\text{Useful Work} = h - \frac{(v - V)^2}{2g} = \frac{V(v - V)}{g},$$

and, dividing by h,

$$\text{Efficiency} = \frac{V(v - V)}{gh} = \frac{2V}{v + V}.$$

(2) A second method is to employ the principle of the equality of angular impulse and angular momentum already given in Art. 262. Originally the water descends the vertical tube without possessing any rotatory motion, but after leaving the machine it has the velocity $v - V$; its angular momentum is therefore for each lb. of water

$$\text{Angular Momentum} = \frac{(v - V)}{g} . r.$$

Now according to the principle the angular momentum generated per second is also the angular reaction on the wheel which, when multiplied by V/r, the angular velocity of the wheel, gives us the useful work done per second. Performing this operation, and dividing by the weight of water used per second, we get per lb. of water

$$\text{Useful Work} = \frac{V(v - V)}{g}.$$

This is the result already obtained, and the solution may now be completed by adding the kinetic energy on exit.

From the result it appears that the proportion which the waste work bears to the useful work is $v - V : 2V$, which diminishes indefinitely as v approaches V: but in this case the velocities become very great, since $v^2 - V^2$ is always equal to $2gh$. The frictional resistances then become very great, so that in the actual machine there is always a speed of maximum efficiency which may be investigated as follows:—

Let F be the co-efficient of hydraulic resistances referred to the orifices, then

$$(1 + F)\frac{v^2}{2g} = H = h + \frac{V^2}{2g}.$$

The useful work remains as before, and therefore

$$\text{Efficiency} = \frac{2V(v - V)}{v^2 - V^2 + F \,.\, v^2},$$

a fraction which can readily be shown to be a maximum when

$$v - V = V\sqrt{\frac{F}{1 + F}}, \text{ or } v = V\left\{ 1 + \sqrt{\frac{F}{1 + F}} \right\},$$

which value of v, when substituted in the preceding equation, will give the value of V in terms of h for maximum efficiency. The existence of a speed of maximum efficiency is well known by experience with these machines. In general it is found to be about that due to the head, so that

$$V^2 = 2gh,$$

a value which corresponds to $F = \cdot 125$, and gives an efficiency of ·67. This is about the actual efficiency of these machines under favourable circumstances; of the whole waste of energy two-thirds, that is two-ninths of the whole head, is spent in overcoming frictional resistances, and the remaining one-third, or one-ninth the whole head, in the kinetic energy of delivery.

The reaction wheel in its crudest form is a very old machine known as "Barker's Mill." It has been employed to some extent in practice as an hydraulic motor, the water being admitted below and the arms curved in the form of a spiral. These modifications do not in any way affect the principle of the machine, but the frictional resistances may probably be diminished.

264. *Turbine Motors.*—A reaction wheel is defective in principle, because the water after delivery has a rotatory velocity in consequence of which we have seen a large part of the head is wasted. To avoid this, it is necessary to employ a machine in which some rotatory velocity is given to the water before entrance in order that it may be possible to discharge it with no velocity except that which is absolutely required to pass it through the machine. Such a machine is called in general a TURBINE, and it is described as "outward flow," "inward flow," or "parallel flow," according as the water during its passage through the machine diverges from, converges to, or moves parallel to, the axis of rotation.

Fig. 193*a* shows in plan and section part of an annular casing forming a wheel revolving about an axis *XX* through which water is flowing, entering at the centre and spreading outwards. The water leaves the wheel at the outer circumference. Fig. 193*b* is similar, but the flow is inward instead of outward.

If we consider a section *aa* made by a concentric cylinder of length y and radius r, the flow will be

$$Q = u \,.\, 2\pi r y,$$

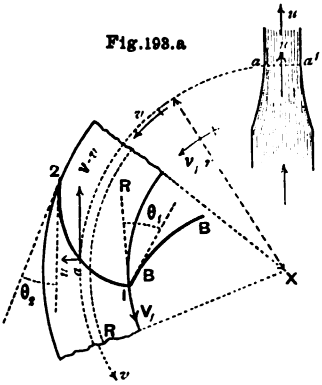


Fig.193.a

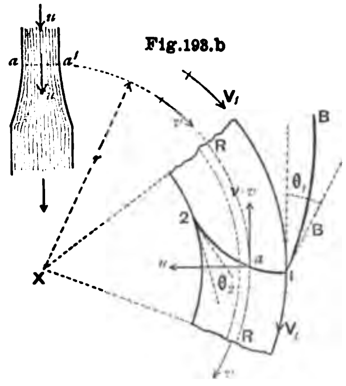


Fig.193.b

where u is the radial velocity or, as we may call it, the "velocity of flow." The area of the section ($2\pi r y$) may conveniently be called the "area of flow." The value of Q is everywhere the same, and therefore ury must be constant. It is generally desirable to make u constant or nearly so, and then the form of the casing is such that ry is constant. Whether this be so or not, the value of u can always be calculated at any radius for a given wheel with a given delivery.

The water which at any given instant is at a given distance r from the axis may be considered as forming a ring *RR*, which rotates while at the same time it expands or contracts according as the flow is outward or inward. The velocity of the periphery of this ring may be described as the "velocity of whirl," and if it be called v, the moment of momentum of a ring, the weight of which is W, is

$$M = \frac{W}{g} \,.\, vr.$$

If the wheel has no action on the water, this quantity cannot be altered, and we must then have

$$vr = \text{Constant}.$$

The water then forms what we have already called a "free vortex" (Art. 239), with the addition of a certain radical velocity u, in consequence of which the rings change their diameter. The paths of the particles of water are then spirals, the inclination of which depends on the proportion between u and v.

The case now to be considered is that in which the moment of momentum of the rotating rings is gradually reduced during their passage through the wheel by the action of suitable vanes attached to it. An impulse is thus exerted on the wheel which furnishes the motive force. The moment of this impulse is given by the equation,

$$L = \frac{wQ}{g}(v_1 r_1 - v_2 r_2),$$

where wQ is the weight of all the rings passing through the machine in a second, and the suffixes 1, 2 refer to entrance and exit respectively, as indicated in the figures for the two cases of outward flow and inward flow. The turbine works to best advantage when the water is discharged without any whirl, that is when $v_2 = 0$, and putting aside friction the only loss then is that due to the velocity of flow u, which may be made small by making the wheel of sufficient breadth at the circumference where the water is discharged.

In practice there are of course always frictional resistances, but, for given velocities, the impulse on the wheel is not altered by them, so that the moment of impulse is always given by the above equation. Suppose, now, h the *effective* head found from the actual head by deducting (1) the height due to the velocity of delivery, (2) the friction of the supply pipe and passages in the wheel, (3) the loss (if any) by shock on entering the wheel; then

$$\text{Work done per second} = wQh.$$

But, if V_1 be the speed of periphery of the wheel at the radius r_1 where the water enters, V_1/r_1 is the angular velocity of the wheel, and $L \, . \, V_1/r_1$ is the work done per second. We have then for the case where there is no whirl at exit

$$V_1 v_1 = gh.$$

The effective head h in this formula includes (1) a part equivalent to the useful work, and (2) a part equivalent to the frictional resistances to the rotation of the wheel, such as friction of bearings and friction of the water surrounding the wheel (if any) on its external surface. This last item is often described as "disc friction."

The whirl before entrance is communicated by fixed blades BB, curved, as shown in the figures, so as to guide the water in a proper direction on entrance to the wheel. It is the use of these guide blades which characterizes the turbine as distinguished from the reaction wheel.

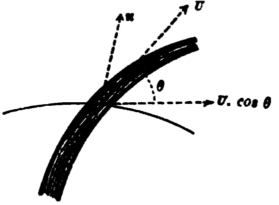
The whirl at different points, either in the wheel or outside it, depends on the angle of inclination of the vanes or guide blades to the periphery. These blades are so numerous that the water moves between them nearly as it would do in a pipe of the same form. If θ be the angle such a pipe (Fig. 194) makes with the periphery at any point at which the water is flowing through it with velocity U, the radial and tangential components of that velocity will be $U.\sin\theta$ and $U.\cos\theta$. The first of these is always the velocity of flow u, whether the pipe be fixed or whether it be attached to the revolving wheel. In the fixed pipe the second is the velocity of whirl which we may call v', and we have for motion before entrance

$$v' = u\,.\cot\theta.$$

In the moving pipe, however, it is the velocity of whirl relatively to the revolving wheel, and this is $V - v$, therefore

$$V - v = u\,.\cot\theta.$$

Fig. 194.



Suppose the vanes of the wheel are radial at the circumference where the water enters. In order that the water may have no velocity of whirl relatively to the wheel on entrance, and that the water may enter without shock, we must then have $v' = V_1$, that is, the value of θ for the fixed guide blade at entrance should be given by

$$\tan\theta_1 = \frac{u}{V_1}.$$

Further, the water should be discharged without whirl, that is, v should be zero at the circumference where the water leaves the wheel, hence

$$\tan\theta_2 = \frac{u}{V_2}.$$

The inclination of the fixed blades at entrance, and of the vanes at entrance and exit is thus determined. At intermediate points it would be desirable that it should so vary that vr should diminish uniformly from entrance to exit in order that the action of all parts of the vane upon the water may be the same. This condition would completely determine the form of the vane, but, in practice, any "fair" form would be a sufficient approximation.

Supposing the vanes thus designed $v_1 = V_1$, and the speed of periphery of the wheel at the circumference where the water enters is then given by the simple formula

$$V_1 = \sqrt{gh},$$

a value which applies to the outer periphery of an inward-flow and the inner periphery of an outward-flow turbine (see Appendix).

The formula shows that the turbine works to best advantage when the speed of periphery at entrance is that due to *half* the *effective* head: and it never can be advantageous to run it quicker. But, if the wheel be wholly immersed in water, the frictional resistance to rotation will be considerable, and as that resistance varies as the square of the speed the wheel may be run slower without much reduction of efficiency, or, it may be, even with an increase.

Many forms of outward-flow turbines exist, of which the best known was invented by Fourneyron, and is commonly known by his name. The inward flow or vortex turbine was invented by Prof. James Thomson. For descriptions and illustrations of these machines the reader is referred to the treatises cited at the end of this chapter.

265. *Turbine Pumps.*—Impulse and reaction machines are always reversible, and every motor may therefore be converted into a pump by reversing the direction of motion of the machine and of the water passing through it. If, for example, in the re-action wheel of Fig. 192 we imagine the wheel to turn in the opposite direction with velocity V, while by suitable means the water is caused to move in the opposite direction with velocity $v - V$, so as to enter the orifices with velocity v, it will flow through the arms to the centre and be delivered up the central pipe. The only difference will be that the lift of the pump will not be so great as the fall in the motor on account of frictional resistances. So, any turbine motor is at once converted into a turbine pump by reversing the direction of its motion and supplying it with water moving with a proper velocity. An inward-flow motor is thus converted into an outward-flow pump, and conversely.

No inward-flow pump appears as yet to have been constructed, though it has occasionally been proposed. The "centrifugal" pump so common in practice is, of course, always an outward-flow machine.

The earliest idea for a centrifugal pump was to employ an inverted Barker's Mill, consisting of a central pipe dipping into water connected with rotating arms placed at the level at which water is to be delivered. This machine, which must be carefully distinguished from the true reversed Barker's Mill mentioned above, operates by suction. Its efficiency, which may be investigated as in Art. 263, is very considerable (Ex. 4, p. 473), but there are obvious practical inconveniences which prevent its use in ordinary cases. The actual centrifugal pump is a reversed inward-flow turbine.

All that was said about motors in the last article applies equally well to pumps, and the same formula

$$Vv = gh$$

applies, V being the speed of rotation of the wheel, now usually called the "fan" and v that of the water, both reckoned at the outer periphery where the water issues. The quantity h is now the *gross* lift found by adding to the actual lift, the head corresponding to the velocity of delivery, the friction of the ascending main, the friction of the suction pipe and passages through the wheel into the main, and the losses by shock at entrance and exit.

A pump, however, works under different conditions from a motor, and corresponding differences are necessary in its design. The energy of a fall can by proper arrangements be readily converted, wholly or partially, into the kinetic form without any serious loss by frictional resistances, and the water can, therefore, be delivered to the wheel with a great velocity of whirl to be afterwards reduced by the action of the wheel to zero. When such a motor is reversed, the water enters without any velocity of whirl, and leaves with a velocity, the moment of momentum corresponding to which represents the couple by which the wheel is driven. To carry out the reversal exactly, this velocity ought to be reduced to as small an amount as possible in the act of lifting. Now the reduction of a velocity without loss of head is by no means easy to accomplish, and (see Appendix) always requires some special arrangement.

In Thomson's inward-flow turbine, when reversed, the water is discharged with a velocity of whirl which is equal to the speed of periphery V, and given by the formula

$$V = \sqrt{gh}.$$

The corresponding kinetic energy represents at least half the power required to drive the pump, and if it be wasted, as was the case in some of the earlier centrifugal pumps constructed with radial vanes, the efficiency is necessarily less than ·5, and in practice will be at most ·3. To avoid this loss, the wheel must revolve in a large "vortex chamber," at least double the diameter of the wheel from the outer circumference of which the ascending main proceeds. The water before entering the main forms a free vortex, and its velocity is reduced one-half as it spreads radially from the wheel; three-fourths the kinetic energy is thus converted into the pressure form. The speed of periphery in pumps of this class is that due to *half* the *gross* lift. Assuming their efficiency as ·65, the gross lift is found by an addition of 50 per cent. to the actual lift.

Many examples of vortex-chamber pumps exist, but they are comparatively rare, probably because the machine is more cumbrous; in practice a different method of reducing the velocity of discharge is generally employed. Instead of the vanes being radial at the outer periphery, they are curved back so as to cut it at an angle θ, given by the formula (p. 463)

$$V - v = u \,.\, \cot \theta,$$

the velocity of whirl is thus reduced from V to kV, where k is a fraction, and the speed is then

$$V=\sqrt{g\frac{h}{k}}$$

If the efficiency be supposed ·65, and the velocity be reduced in this way to one-half its original value, this gives about $10\sqrt{H}$ for the speed where H is the actual lift. The greater speed is a cause of increased friction as compared with the vortex-chamber arrangement, but on the other hand the friction of the vortex is by no means inconsiderable, and this is so much subtracted from the useful work done.

The centrifugal pump in this form was introduced by Mr. Appold in 1851, and is commonly known by his name.

Another important point in which the pump differs from the motor is in the guidance of the water outside the wheel. In the motor there are four or more fixed blades which guide the water to the wheel; but in the pump the outer surface of the chamber surrounding the wheel forms a single spiral guide blade. The whole of the water discharged from the wheel rotates in the same direction, and in order that the discharge may be uniform at all points of the circumference the sectionaf area of this chamber should increase uniformly from zero at one side ol the ascending main to a maximum value at the other side. In some of the earlier designs of centrifugal pumps it was supposed that some of the water would rotate one way and some the other, but in fact all the discharged water rotates with the wheel, and the passage should be so designed as to permit this, the area corresponding to the proposed velocity of whirl. There are, however, examples in which the water is discharged in all directions into an annular casing, and guided by spiral blades parallel to the axis of rotation. (See a paper by Mr. Thomson, *Min. Proc. Inst. C.E.*, vol. 32.)

Centrifugal pumps work to best advantage only at the particular lift for which they are designed. When employed for variable lifts, as is constantly the case in practice, their efficiency is much reduced and does not exceed ·5. It is often much less.

265A. *Approximate investigation of the Efficiency of a Centrifugal Pump.*—Few centrifugal pumps utilize more than a small fraction of the energy of motion possessed by the water at exit from the wheel, and an investigation of their efficiency on the supposition that this energy is wholly wasted is therefore of considerable interest.

Let h_0 be the actual lift, and let all frictional losses except that specified be neglected; then, if u be the velocity of flow, and v the velocity of whirl at exit, the loss of head is $(u^2+v^2)/2g$, and the gross lift is

$$h=h_0+\frac{u^2+v^2}{2g}.$$

Substituting this value of h in the formula for V, and replacing u by its value $(V-v)\tan\theta$, we obtain

$$Vv=gh=gh_0+\frac{v^2+(V-v)^2\tan^2\theta}{2}.$$

Adding $\frac{1}{2}V^2$ to each side, and re-arranging the terms,

$$\tfrac{1}{2}V^2=gh_0+\tfrac{1}{2}(V-v)^2 \,.\, \sec^2\theta,$$

a formula from which we find

$$\text{Efficiency}=\frac{h_0}{h}=\frac{V^2-(V-v)^2\sec^2\theta}{2Vv}$$

$$=\sec^2\theta\left(1-\tfrac{1}{2}\left(\frac{v}{V}+\frac{V}{v}\sin^2\theta\right)\right).$$

This result shows that the efficiency is greatest when

$$v=V \,.\, \sin\theta;$$

and on substitution we find

$$\text{Maximum efficiency}=\sec^2\theta(1-\sin\theta)=\frac{1}{1+\sin\theta}.$$

The speed of maximum efficiency is found from the equation

$$\tfrac{1}{2}V_0^2=gh_0+\tfrac{1}{2}V_0^2\sec^2\theta \,.\, (1-\sin\theta)^2,$$

which gives

$$V_0^2=(1+\operatorname{cosec}\theta)gh_0.$$

The proper velocity of flow is

$$u_0=V_0 \,.\, \tan\theta(1-\sin\theta),$$

and the area of flow through the periphery of the wheel should be made to give this velocity with the intended delivery.

At any other speed V the velocity of flow will be given by

$$u^2=(V^2-2gh_0)\sin^2\theta,$$

and the efficiency may be found by the preceding formula.

In the best examples of centrifugal pumps working at a suitable lift, it is probable that enough of the energy of motion on exit from the wheel is utilized to provide for the various frictional resistances neglected in this investigation. But in ordinary cases this will not be true, and the efficiency is much reduced. The theory, taking into account all the resistances, is too intricate to enter on here.

When a centrifugal pump is started the fan is filled with water which, in the first instance, rotates as a solid mass with the fan. If the radius of the inner periphery be m times that of the outer where m is a fraction, it will not commence to deliver water till the speed reaches the value

$$V^2(1-m^2)=2gh_0.$$

But when once started, the speed may be reduced below this value without stopping the delivery, provided that some of the energy of motion on exit from the wheel is utilized. This has been observed to occur in practice, and it will serve as a test of efficiency.

266. *Limitation of Diameter of Wheel.*—For a given fall in a motor or lift in a pump the diameter of wheel in a turbine is limited, because the frictional resistances increase rapidly with the diameter.

Let u be the velocity of flow, d the diameter, b the inside breadth of wheel at exit; then the delivery in cubic feet per second is

$$Q = ub\pi d.$$

Now, if the breadth b be too small as compared with the diameter, the surface friction of the passages through the wheel will be too great, as in the case of a pipe the diameter of which is too small for the intended delivery. Thus b is proportional to d: also, we have seen that u is proportional to V_1, that is to $\sqrt{h}$, and it follows therefore, by substitution for b and u, that

$$Q = Cd^2\sqrt{h},$$

where C is a co-efficient.

If the wheel be wholly immersed in the water the surface friction (Ex. 8, p. 473) is relatively increased by increasing the diameter. On investigating how great the diameter may be without too great a loss we arrive at the same formula.

Where it is of importance to have as large a diameter as possible to reduce the number of revolutions per minute, the diameter of wheel in a pump or a turbine is therefore found by the formula

$$d = \sqrt{\frac{G}{c\sqrt{h}}}.$$

If G be the delivery in gallons per minute, h the actual fall in feet, d the external diameter also in feet, the value of c for an outward-flow turbine is about 200. In a centrifugal pump the value is probably not very different, h being now the actual lift, but it varies in different types of pump.

Centrifugal pumps cannot generally be employed for high lifts, partly because it becomes increasingly difficult to utilize the energy of motion on exit from the wheel, and partly on account of disc friction. The fan rotates more than twice as fast as the wheel of a turbine, and the disc friction is consequently more than four times as great.

267. *Impulse Wheels.*—The formula

$$V_1^2 = gh,$$

which gives the speed of a turbine wheel in terms of the effective head, also gives the velocity of whirl at entrance, and therefore shows that, of the whole head employed in driving the wheel and producing the velocity of flow, one-half operates by impulse. The remainder operates by pressure, and this class of turbines is therefore not simple impulse, but impulse-pressure machines. It is necessary therefore that the wheel should revolve in a casing, and that the passages should be always completely filled with water. The diameter of wheel is then limited as explained in the last article, and for a small supply of water and a high fall the number of revolutions per minute becomes abnormally great. This consideration and the necessity of adaptation to a variable supply of water render it often advisable to resort to a machine in which the passages are actually or virtually open to the atmosphere. The whole of the energy of the fall is then converted into the kinetic form before reaching the wheel, and consequently operates wholly by impulse.

A wheel of this kind approaches closely in principle to the Poncelet water wheel mentioned in Art. 261, but is often still described as a "turbine," because the water is guided by fixed blades before reaching the wheel. A common example is the Girard turbine. The flow of the water is here parallel to the axis of the wheel, spiral guide blades are ranged round the circumference of a cylinder like the threads of a screw in order to give the necessary whirl to the water before entrance. The wheel is provided with a similar set of spiral vanes curved in the opposite direction, which reduce it to rest as it passes through. In the French *roue à poire* the wheel is conical, the water enters at the circumference, and, guided by spiral vanes, descends to the apex where it is discharged.

268. *Propellers.*—The subject of propellers is outside the limits of simple hydraulics for reasons already indicated when considering the resistance of ships (Art. 249). Nevertheless they may be regarded as hydraulic machines, and their connection with the machines already referred to forms a proper subject for consideration.

Every propeller operates by means of the mutual action between it and the water on which it acts consequent on a change of velocity which it produces in the water of the sea; it is therefore an impulse and reaction machine applied so as to produce a propelling force which drives the vessel through the water. Since the resistance of the vessel is directly astern, the change of motion produced is sternward so far as it is of any utility for the purpose. Some forms of propeller, as, for example, the screw, give lateral motions to the water, but the energy thus employed is wasted.

(1) The simplest form of propeller is that in which the water is drawn into the vessel through orifices of proper size, and projected by means of a centrifugal pump through two orifices in the side of the vessel so placed that the water issues directly astern. The reaction of the issuing jet furnishes a propelling force on the vessel. The problem here is just the same as that of the simple reaction wheel already considered in Art. 263: the fact that the orifices move in straight lines instead of in a circle making no difference in the propelling reaction. Hence, if R be the resistance of the vessel, v the velocity of discharge, V the velocity of the vessel,

$$R = \frac{wQ}{g}(v - V).$$

The engine power is employed in the first instance in creating a head h, but, supposing this known, the question is unaltered, and therefore neglecting frictional resistances,

$$\text{Efficiency} = \frac{2V}{v + V} = \frac{1}{1 + e}.$$

If we call the counter efficiency $1 + e$, and if we further suppose A to be the *joint* area of the orifices, and $K.\Delta^{\frac{2}{3}}V^2$ the resistance of the ship (Art. 249), we shall find by substitution that for all speeds

$$K.\Delta^{\frac{2}{3}} = \frac{wA}{g}(1 + 2e)\ 2e,$$

a formula which shows that the efficiency is increased by increasing the size of the orifices, and enables us to calculate the size for a given efficiency.

In every propeller, in the absence of frictional resistances and of any disturbance due to the passage of the vessel through the water, *the efficiency of a propeller is greater the greater the quantity of water on which it operates.* In the case of the jet propeller which we are now considering, this general conclusion is modified by the effect of frictional resistances. If we make the same supposition as to these as in the reaction wheel, the efficiency will be found, as before, to be greatest for a certain value of v/V, and this will correspond to a certain definite area of orifice. The circumstances being somewhat different, a different way of expressing the frictional resistances would probably more closely represent the facts, but the general conclusion must be the same. It is of little use to consider this more closely, as the disturbance of the water passing near the vessel, produced by drawing a large quantity of water through the orifices of entry, must modify in an unknown way the resistance of the ship. Hence the best magnitude and position of the orifices of entry and efflux can only be found by careful experiments. Such experiments have not hitherto been carried out, but it may be considered probable that the jet propeller cannot compete with other forms of propeller so far as efficiency is concerned. If it is ever introduced, it will be for the sake of facility in manœuvring or some other similar reason.

(2) If the action of paddles be observed, two streams of water are seen proceeding from the floats which play the part of the jets in a jet propeller. The most efficient kind are those in which the floats turn about axes on which they are mounted in such a way as to enter and leave the water without any shock. The streams are then simple jets of area not very different from that of the floats, and are driven back with a velocity which is about the same as that of the floats themselves. If, then, v be the speed of the paddles as calculated from their diameter and revolutions, V the speed of the ship, the propelling reaction is given by the same formula as for jets, namely,

$$R = \frac{w}{g} Q \, . \, (v - V),$$

the velocities being in feet per second, and Q the quantity acted on in cubic feet per second. The energy exerted per second is, however, now Rv, and therefore

$$\text{Efficiency} = \frac{V}{v}.$$

For given values of v and V the efficiency is less than that of the simple jet when the frictional resistances are left out of account. The reason of this is that the value just found includes the loss due to breaking up as the paddles strike the water and drive it upwards in a mass of foam before it settles down to the nearly undisturbed motion of the streams. In smooth water, when the paddles are not too deeply immersed, the efficiency however of paddles is far greater than that of the jet, because the area of orifice which can practically be employed is so limited in the jet and the frictional losses in the pump and passages so great.

(3) In rough water the efficiency of paddles is much reduced, and this is also the case when the immersion varies in consequence of alterations in the displacement of the vessel due to consumption of fuel on a voyage or other causes. In sea-going vessels, therefore, the paddles are replaced by a screw propeller.

In this case the action of the propeller is much more complex: the water has a rotatory velocity communicated to it as well as a sternward velocity, and these velocities are different for each portion of the screw blade. Further, the water in which the screw works has very complex motions due to the action of the sides of the ship upon it, a circumstance which affects the resistance of the ship as well as the action of the propeller. For these reasons the screw is not so efficient, other things being the same, as well constructed paddles. On the other hand the quantity of water acted on is large, and the action is not greatly influenced by the circumstances just mentioned as reducing the efficiency of paddles.

On comparing a screw propeller with the machines already considered, it will be perceived that it is a parallel-flow impulse wheel reversed, with two important modifications. First, the fixed guide blades are omitted. It is true that it has been proposed to employ such blades in order to avoid the loss occasioned by the transverse velocity given to the water which is useless for propelling purposes, but the gain of efficiency, if any, is very small. Secondly, the number of blades in the wheel is large; whereas in a screw propeller there are often only two, and in all cases they occupy much less than half the whole circumference.

Both these modifications are due to the same cause, and arise from the fact that in the propeller the changes of velocity produced by the action of the blades are small compared with the whole velocity of the water through the propeller, whereas in the wheel they are large. Hence the frictional resistances in the propeller are disproportionately great and render it necessary to reduce the surface exposed to the water as much as possible.

This reasoning also shows that the frictional losses in the centrifugal pump and passages of a jet propeller must be great. Such a pump should receive the water at the velocity of the ship, and gradually increase its velocity to that attained on efflux from the orifices. The frictional resistances will depend on the whole velocity, but the propelling reaction on the difference of velocities. Or if the alternative be adopted of checking the motion of the water on entry, and afterwards giving it the whole velocity of efflux, the reduction of velocity will be difficult to accomplish without considerable loss, and the propelling reaction will not be greater than before. All unnecessary changes of velocity are a cause of loss.

EXAMPLES.

1. In a reaction wheel the speed of maximum efficiency is that due to the head. In what ratio must the resistance be diminished to work at four-thirds this speed and what will then be the efficiency? Obtain similar results when the speed is diminished to three-fourths its original amount.

Ans. Efficiency = ·63 or ·64.
Ratio = ·84 or 1·14.

2. Water is delivered to an outward-flow turbine, at a radius of 2 feet, with a velocity of whirl of 20 feet per second, and issues from it in the reverse direction at a radius of 4 feet, with a velocity of 10 feet per second. The speed of periphery at entrance is 20

feet per second, find the head equivalent to the work done in driving the wheel. *Ans.* 24·22 feet.

3. In a Fourneyron turbine the internal diameter of the wheel is 9½ inches, and the outside diameter 14 inches. The effective head (p. 462) is estimated at 270 feet: find the number of revolutions per minute. *Ans.* 2,200.

NOTE.—These data are about the same as those of a turbine erected at St. Blasien in the Black Forest.

4. An inverted Barker's Mill (p. 464) is used as a centrifugal pump. If the co-efficient of hydraulic resistances referred to the orifices be ·125, show that the speed of maximum efficiency is that due to twice the lift, and find the maximum efficiency. *Ans.* Maximum efficiency = ·75.

5. A centrifugal pump delivers 1,500 gallons per minute. Fan 16 inches diameter. Lift 25 feet. Inclination of vanes at outer periphery to the tangent 30°. Find the breadth at the outer periphery that the velocity of whirl may be reduced one-half, and also the revolutions per minute, assuming the *gross* lift 1½ times the actual lift. *Ans.* Breadth = ¾ inch. Revolutions = 700.

6. In the last question find the proper sectional area of the chamber surrounding the fan (p. 466) for the proposed delivery and lift. Also examine the working of the pump at a lift of 15 feet. *Ans.* 24 sq. inches.

7. A jet of water moving with a given velocity, strikes a plane perpendicularly. Find how much of the energy of the jet is utilised in driving the plane with given speed. Determine the speed of the plane for maximum efficiency, and the value of the maximum efficiency. *Ans.* Speed of maximum efficiency = one-third that of jet. Maximum efficiency $= \frac{8}{27}$.

8. Assuming the ordinary laws of friction between a fluid and a surface, and supposing that any motion of the fluid due to friction does not affect the question: find the moment of friction (L), and the loss of work per second (U), when a disc of radius a rotates with speed of periphery V.

$$\textit{Ans.}\quad L = f \cdot \frac{2\pi}{5} \cdot a^3 V^2 ; \quad U = f \cdot \frac{2\pi}{5} \cdot a^2 \cdot V^3 .$$

9. If the rotating disc in question 8 be surrounded by a free vortex of double its diameter, show that the loss by friction of the vortex on the flat sides of the vortex chamber is 2½ times the loss by friction of the disc.

10. The resistance of the *Waterwitch* at 8 knots is 5,500 lbs., the orifices of her jet propeller are each 18 inches by 24 inches, what must be the delivery of her centrifugal pump in gallons per minute to propel her at this speed, and what will be the efficiency, neglecting frictional resistances.

Ans. Velocity of efflux = 29·3 per second.
Delivery = 66,000 gallons per minute.
Efficiency = ·63.

11. In the last question find the H.P. required for propulsion, assuming the efficiency of the pump and engines ·4. *Ans.* 525.

12. If the jet propeller of the *Waterwitch* be replaced by feathering paddles, what will be the area of the stream driven back for a slip of 25 per cent. Find the efficiency and the water acted on in gallons per minute.

Ans. Joint area of streams = 34 square feet.
Efficiency = 75.
Delivery = 236,000 gallons per minute.

REFERENCES.

The subject of hydraulic machines is very extensive, and it is impossible within the limits of a single chapter to do more than give a general idea of their working. For descriptive details and illustrations the reader is referred, amongst other works, to

GLYNN. *Power of Water.* Weales' Series.
FAIRBAIRN. *Millwork and Machinery.* Longman.
COLYER. *Water-Pressure Machinery.* Spon.
BARROW. *Hydraulic Manual.* Printed by authority of the Lords Commissioners of the Admiralty.

The theory of hydraulic machines is further developed in Professor Unwin's work on Hydraulics, already cited on page 438, and the various special treatises therein mentioned, also in Rankine's treatise on the *Steam Engine and other Prime Movers.*

CHAPTER XXI.

PNEUMATICS AND THERMODYNAMICS.

269. *Preliminary Remarks.*—An elastic fluid under pressure is a source of energy which, like a head of water in hydraulics (p. 439), may be employed in doing work of various kinds by a machine, or simply in transferring the fluid from one place to another. In hydraulics we commence with the case of simple transfer, but the density of gases is so low that, unless the differences of pressure considered are very small, the inertia and frictional resistances of the fluid employed in a pneumatic machine have little influence: it is the elastic force, which is the principal thing to be considered. In studying pneumatics, therefore, we commence with machines working under considerable differences of pressure and then pass on to consider the flow of gases through pipes and orifices together with those machines in which the inertia and frictional resistances of the fluid cannot be neglected.

270. *Expansive Energy.*—The special characteristic of an elastic fluid is its power of indefinite expansion as the external pressure is diminished. While expanding, it exerts energy of which the fluid itself is, in the first instance, the source, whereas the energy exerted by an incompressible fluid is transmitted from some other source. Expansive energy is utilized by enclosing the fluid in a chamber which alternately expands and contracts; the common case being that of a cylinder and piston.

Fig. 195 represents in skeleton a cylinder and piston enclosing a mass of expanding fluid. Taking a base line *aa'* to represent the stroke, set up ordinates to represent the total pressure S on the piston in each position; a curve $1Q2$ drawn through the extremities of these ordinates is the Expansion Curve. Reasoning as in Art. 90, p. 178, the area of this curve represents the energy exerted as the piston moves from the position 1, where the expansion commences, to the position 2, where it terminates. One common case was considered in the article cited, namely, that in which the expansion curve is a common hyperbola. This is included in the more general supposition,

$$Sy^n = S_1 y_1^n = S_2 y_2^n,$$

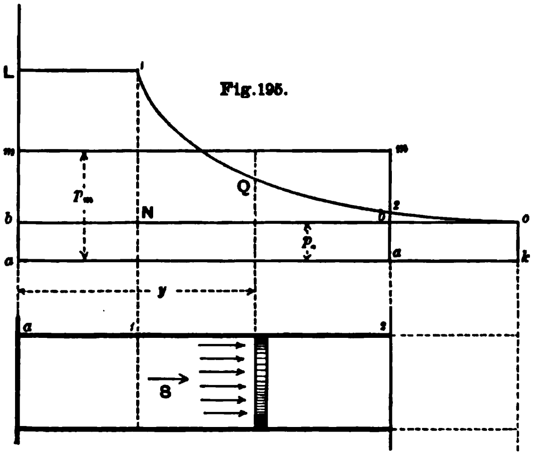


Fig. 195.

where y is the distance of the piston from the end a of its stroke, and n is an index which, for the particular case of the hyperbola, is unity. Most cases common in practice may be dealt with by ascribing a proper value to n; for air it ranges between 1 and 1·4, and for steam it is roughly approximately unity. The suffixes indicate the points at which the expansion commences and terminates.

If, now, E be the energy exerted during expansion,

$$E = \int_{y_2}^{y_1} S dy = S_1 y_1^n \int_{y_1}^{y_2} y^{-n} dy = S_1 y_1^n . \frac{y_2^{1-n} - y_1^{1-n}}{1-n}.$$

This formula may be written in the simpler form

$$E = \frac{S_1 y_1 - S_2 y_2}{n-1},$$

in applying which, the terminal pressure S_2 is supposed to have been previously found from the equation

$$S_2 = S_1 . \left(\frac{y_1}{y_2}\right)^n.$$

It is, for brevity, convenient to write

$$y_2 = r y_1 ; \quad S_2 = x . S_1,$$

where r is a number known as the "ratio of expansion," and x is a fraction derived from r by the formula

$$x = \left(\frac{1}{r}\right)^n.$$

The formula for E then takes the simple form

$$E = S_1 y_1 \cdot \frac{1 - rx}{n - 1} = P_1 V_1 \cdot \frac{1 - rx}{n - 1}.$$

If $n = 1$ the formula fails and is replaced by

$$E = S_1 y_1 \log_e r = P_1 V_1 \log_e r.$$

The value of E is here, in the first instance, expressed in terms of the total pressure on the piston, but as in Art. 255, p. 441, we may replace S by PA, and Ay by V, so that $S_1 y_1$ is replaced by $P_1 V_1$. In "rotatory" engines and pumps the expanding chamber is not a simple cylinder and piston, but is formed from a turning pair. Or, more generally, the chamber pair may be formed from any two links of a kinematic chain which it may be convenient to select for the purpose. In its last form the formula is applicable in every case. If the expansion curve be not given in the form supposed, the value of E is determined graphically by measuring the area of the curve, in doing which, when the chamber is not a simple cylinder, the base of the diagram must represent the volume swept out by the chamber pair, and the ordinates the pressures per unit of area.

271. *Transmitted Energy.*—The energy exerted by an elastic fluid consists not merely of that derived from the expansive power of the fluid pressing against the piston, but also of that which is transmitted in the same way as would be the case if it were incompressible. The fluid is supplied from a reservoir, which may either be an accumulator in which it is stored by the action of pumps, or a vessel in which, by the action of heat, it is generated or its elasticity increased. In any case, so long as the cylinder remains in communication with the reservoir the fluid enters at nearly constant pressure, and energy is exerted on the piston just as in the water pressure engine. During this period of admission the energy exerted is

$$L = S_1 y_1 = P_1 V_1 = 144 p_1 V_1,$$

the notation being as in the last article. It is usually convenient to express volumes in cubic feet and pressure in lbs. per square inch. We must thus replace P by $144p$.

The whole energy exerted on the piston is now

$$U = L + E = L \cdot \frac{n - rx}{n - 1},$$

which for the case of the hyperbola becomes

$$U = L(1 + \log_e r.)$$

The mean pressure on the piston is conveniently denoted by p_m, and is represented in the figure by the ordinate of the line mm so drawn that the area of the rectangle *ma* is equal to the area of the diagram. Its value is given by the formulæ

$$p_m = \frac{p_1}{r} \cdot \frac{n - rx}{n - 1}; \; p_m = p_1 \cdot \frac{1 + \log_e r}{r}.$$

A reservoir filled with an elastic fluid at high pressure is an accumulator, the absolute amount of energy stored in which is the expansive energy or the total energy according as the pressure is not, or is, maintained by the addition of fresh fluid in place of that discharged, the expansion being supposed indefinite in either case. With the law of expansion already supposed, when n is greater than unity, rx vanishes when the expansion curve is prolonged indefinitely. The total absolute energy is then

$$U_1 = P_1V_1 \cdot \frac{n}{n - 1},$$

where V_1 is the whole volume of fluid considered.

When n is not greater than unity, U_1 is infinite. In practice, however, there is always a "back" pressure P_0 on the working piston, or, more generally, on the sides of the chamber in which the fluid is enclosed. In overcoming this, the work P_0rV_1 is done, and nothing is gained by prolonging the expansion beyond the point 0 at which the terminal pressure P_2 has fallen to P_0. The corresponding ratio r_0 is given by the formula

$$\log r_0 = \frac{1}{n} \log\left(\frac{1}{x_0}\right) = \frac{\log P_1 - \log P_0}{n}.$$

The available energy is found by writing $r = r_0$, $x = x_0$ in the value of U, and subtracting $P_0r_0V_1 = P_1V_1r_0x_0$. The result is

$$\frac{n}{n - 1} \cdot P_1V_1 (1 - r_0x_0) = U_1 - U_0,$$

being the difference of the values of U when the expansion commences at 1 and at 0. It is always finite, and is graphically represented by the area $L10bb$.

In the transmission and storage of energy by elastic fluids this quantity plays the same part as the "pressure-head" in hydraulics, to which indeed it reduces if n be supposed very great, r_0 unity, and V_1 the volume of a lb. of water. It is the energy of a given quantity of fluid due to a given difference of pressure. The term "head" may be used for this when the quantity considered is a mass of 1 lb. When $n = 1$

$$U_1 - U_0 = P_1V_1 \log_e r_0.$$

When the reservoir is not kept full the only available energy is the expansive energy, less the work done in overcoming P_0 through the volume $V_0 - V_1$. This is graphically represented by the curvilinear triangle $N01$, and is most conveniently given by the formula

$$A \, . \, E = U_1 - U_0 - (P_1 - P_0)V_1.$$

272. *Cycle of Mechanical Operations in a Pneumatic Motor—Mechanical Efficiency.*—Motors operating by the pressure of an elastic fluid may be described generally as Pneumatic Motors. They are either supplied from an accumulator as in hydraulic motors of the same class, or they may be heat-engines serving as the means by which heat energy is utilized. In either case the mechanism of the motor is the same, and consists of a chamber which expands to admit the fluid and contracts to discharge it, with a proper kinematic chain for utilizing the motion of the chamber pair.

In water-pressure engines the contraction to expel the water from the chamber is not considered, because all pressures are reckoned above the atmosphere, and the pressure in the accumulator is so great that small differences of pressure may be disregarded. With elastic fluids it is commonly different: the "exhaust" of the chamber must be taken into account.

Returning to Fig. 195, suppose that the piston has reached the end of its stroke, the cylinder is then filled with fluid of a certain pressure p_2 which may be supposed known. Let now a valve be opened allowing the cylinder to communicate with the atmosphere, or with a reservoir containing fluid at a lower pressure p_0. The fluid in the cylinder then rushes out into the reservoir, and the pressure in the cylinder speedily subsides to p_0; the fluid expands in this process, but its expansive energy is wasted in producing useless motions in the air which afterwards subside by friction. After subsidence let the piston be moved back by an external force applied to it which supplies the energy necessary to overcome the "back" pressure p_0. The fluid is discharged from the chamber, and so long as the communication with the exhaust reservoir is open the pressure remains constantly p_0. We represent this on the diagram by drawing a horizontal line *bb*, the ordinate of which is p_0. The work done in overcoming back pressure is $144p_0V_2$ and is represented on the diagram by the rectangle *ba*; this is so much subtracted from the energy exerted by the motor.

Thus the volume of the chamber goes through a cycle of changes alternately expanding and contracting. During expansion energy is exerted, the corresponding mean pressure p_m is the "mean forward pressure." During contraction work is done, and the corresponding mean pressure is the "mean back pressure." The difference between the two is the "mean effective pressure" which measures the useful work done, as shown by the equation

$$\text{Useful work} = (p_m - p_0)144\,V_2,$$

and is graphically represented by the area of the closed figure $L12bb$.

In most cases the moveable element of the chamber pair divides the chamber into two parts, one of which expands while the other contracts, and conversely: the motor is then described as "double acting." The force acting on the moving piece is then the difference between the forward pressure in one chamber and the back pressure in the other, and when the stress on the parts of the machine is to be considered this is the effective pressure upon which the stress depends (p. 216). For all other purposes, however, the back pressure is to be taken as just explained.

If the pressures p_1, p_0 in the supply and exhaust reservoirs be given, and also the form of the expansion curve, the only waste of energy in this process arises from incomplete expansion. Imagine the expansion curve prolonged to the point o where it meets the back pressure line, and suppose the stroke lengthened so as to reach this point, then additional work would be done by the fluid which would be represented graphically by the area of the curvilinear triangle $2ob$. This area represents energy lost by unbalanced expansion, and to avoid it the expansion must be "complete," that is, the fluid must be allowed to expand till its pressure has fallen to p_0, the pressure in the exhaust reservoir, a condition seldom fulfilled in practice, because the loss by friction and other causes becomes disproportionately great. Leaving this out of account, a pneumatic motor is capable of exerting only a certain maximum amount of energy, quite irrespectively of the nature of its mechanism, but dependent only on the pressures between which it works and the nature and treatment of the fluid. A motor which reaches this maximum power may be described as *mechanically* perfect, and the ratio of the actual useful work done to the theoretical maximum may be described as the MECHANICAL EFFICIENCY of the motor.

In practice the mechanical efficiency is diminished not only by incomplete expansion, but also by a portion of the fluid being retained in the "clearance" space of the chamber after the exhaust is completed. This, however, is a detail which cannot be considered here. The theoretical maximum is clearly the same as the store of energy in the fluid used, already found in the last Article, and denoted by $U_1 - U_0$. The consumption of fluid (neglecting clearance) is one cylinder full, at the terminal pressure, in each stroke.

273. *Pneumatic Pumps.*--A pneumatic like an hydraulic motor may

be reversed by applying power to drive it in the reverse direction, and the machine thus obtained is a Pump which takes fluid at a low pressure and compresses it into a reservoir at high pressure.

The cycle in the pump is the same as the cycle in a motor, but the operations take place in reverse order. As the chamber expands fluid is drawn in from the low-pressure reservoir and energy is exerted on the piston by the original "back" pressure: as the chamber contracts the fluid is compressed till it reaches the pressure p, when a valve opens and admits it to the high-pressure reservoir. There is, however, this important difference, namely, that the process of unbalanced expansion in the motor cannot be reversed; and therefore, if the pump is to operate on the same weight of fluid, the volume of the working cylinder must be enlarged so that the expansion curve may start from o. If this be supposed, the compression curve will, for the same fluid treated in the same way, be identical with the expansion curve of the motor. If there were no unbalanced expansion the motor would be exactly reversible, and the condition of a motor being mechanically perfect may therefore be described by saying that it must be mechanically reversible. The difference of working of the valves in pumps and motors has already been referred to in Art. 260.

Air pumps are employed either to compress atmospheric air to a high pressure or to exhaust a chamber. In the second case the atmosphere is the high pressure reservoir into which the exhaust air is compressed. In the old "atmospheric" engine the steam was employed merely to produce a vacuum, the motive force being the pressure of the atmosphere: this machine is therefore a reversed air pump.

In all pneumatic motors a pump is required to replace the fluid in the supply reservoir. Unless the motor be a heat engine this pump must be driven by external agency, and the whole process is one of distribution, transmission, and storage of energy, as in water-pressure engines. In Whitehead torpedoes indeed, and in some other similar cases, the reservoir is not kept full: but the motor then works with constantly diminishing power till the store of energy is exhausted. In condensing steam engines an air pump is employed to exhaust the condenser.

The work done in pumping is found by the formulæ of the last article. Examples will be found at the end of this chapter.

274. *Indicator Diagrams.*—The pressure existing in the chamber of a pneumatic machine may be graphically exhibited by means of an instrument called an Indicator. In steam engines especially its use is indispensable to enable the engineer to study the action of the steam.

Figs. 1 and 2, Plate XI., show an indicator in elevation and section.

S is a drum revolving on a vertical axis, A is a cylinder communicating with the steam cylinder, the pressure in which is to be measured. P is a pencil connected by linkwork with a small piston H so as to move with it up or down in a vertical line. The piston is pressed down by a spring which measures the pressure, while the drum, by means of a cord passing over pulleys and connected with the steam piston, revolves through arcs exactly proportional to the spaces traversed by it. A card is folded round the drum, and as the engine moves a curve is traced by the pencil upon it which shows the pressure at each point of the stroke. In practice many precautions are necessary to secure accuracy in the diagram; the more so the higher the speed, because the friction and inertia of the parts of the indicator, together with unequal stretching of the cord and inaccuracy in the reducing motion connecting the drum with the steam piston, may give rise to serious errors. To diminish the effect of inertia the stroke of the indicator piston is made short and multiplied by linkwork.

In the example shown (Crosby's patent) the spring applied to the drum to keep the cord tight has a tension which increases as the drum rotates from rest. This increase compensates for the inertia of the drum, and is said to give a more nearly uniform tension of the cord.

Fig. 196 shows an indicator diagram taken in this way from the high-pressure cylinder of a compound engine.

BB is the atmospheric line drawn on the card by the indicator pencil when the cylinder communicates with the atmosphere. AA is the vacuum line laid down on the diagram at a distance below BB, which represents the pressure of the atmosphere, as found by the barometer, reckoned on the scale of pressures. Then on the same scale any pressure shown by the indicator is the absolute pressure when measured from AA.

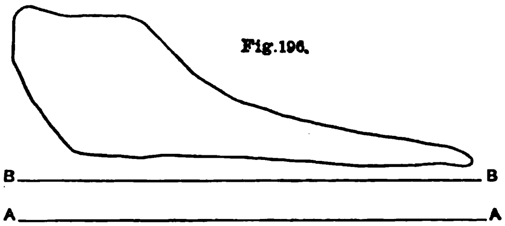
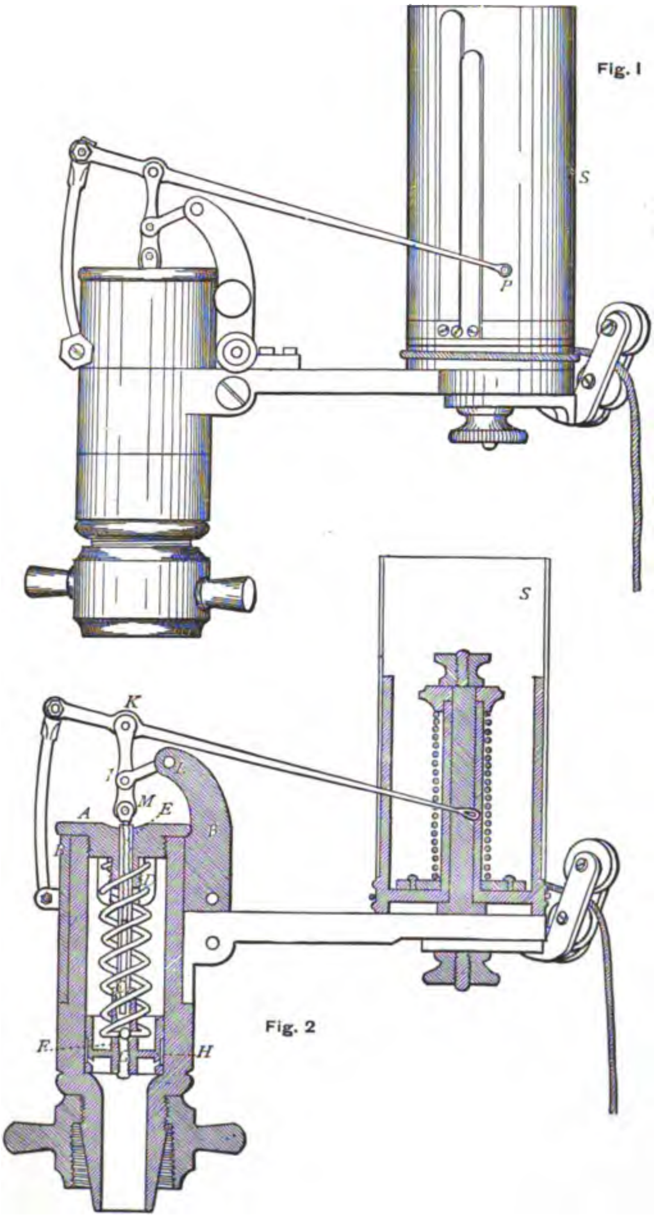


Fig. 196.

The figure drawn is a closed curve bearing a general resemblance to the diagram (Fig. 195), which was drawn to represent the cycle of operations of a motor. The principal difference is that the corners of the theoretical diagram are rounded off in the actual diagram, an effect principally due to the valves closing gradually instead of instantaneously.

PLATE XI.



To face page 482

Also at the end of the return stroke a certain amount of steam is retained in the cylinder and compressed behind the piston, causing the very considerable rounding off observable at the left-hand corner.

In every case the mean effective pressure may be determined graphically by measuring the area of the diagram and dividing by the length of the stroke. This, with the number of revolutions per 1′ determines the horse-power for an engine of given dimensions, and the consumption of steam in cubic feet per 1′ for each horse-power thus "indicated" can be found. The *weight* of steam used, however, cannot be found without measurement of the feed water used, because the steam always contains an unknown amount of water mixed with it.

275. *Cycle of Thermal Operations in a Heat Engine.*—So far all that has been said applies equally well to all pneumatic motors, though its most important application may be to the case where the fluid serves as the means whereby mechanical energy is obtained through the agency of Heat. We now go on to consider very briefly the principles which apply especially to heat engines.

In heat engines the pump necessary to replace the fluid in the supply reservoir, or discharge it from the exhaust reservoir, is worked by energy derived from the working cylinder, so that the engine is self-acting. Now, if the condition of the fluid were the same in the pump as it is in the working cylinder, as much energy would be required to drive the pump as is supplied by the motor, or, in practice, more; a necessary condition therefore that any useful work should be done is that, by the agency of heat, the condition of the fluid should be changed so that its mean density, while being forced into the supply reservoir, shall be greater than when doing its work in the working cylinder. Hence the fluid must be heated in the supply reservoir, and cooled in the exhaust reservoir, and therefore, in every heat engine, in addition to the cycle of mechanical operations there is a cycle of thermal operations consisting of an alternate addition and subtraction of heat; the heat in question being supplied by a body of high temperature and abstracted by a body of low temperature.

In non-condensing steam engines the pump is the feed pump which supplies the boiler with the fluid in the state of water; in the boiler heat is supplied which converts it into steam of density many hundred times less than that of water. The pump is, in this case, very minute, and requires a trifling amount of energy to work it. In condensing engines we have, in addition, the air pump.

In air engines the compressing pump is generally a conspicuous part of the apparatus and requires a large fraction of the power of the motor to drive it; because the changes of density due to the alternate heating and cooling are comparatively small.

276. *Mechanical Equivalent of Heat.*—Heat and mechanical energy are mutually convertible; a unit of heat corresponding to a certain definite amount of mechanical energy which is called the "MECHANICAL EQUIVALENT" of heat. In British units the thermal unit is equivalent to 772 ft.-lbs.

The statement here made is the First Law of the Science of Thermodynamics, and it shows that quantities of heat may be expressed in units of work, and, conversely, quantities of work in units of heat. In dealing with questions relating to heat and work, a common unit of measurement must be selected. In most cases the thermal unit is adopted, and quantities of work reduced to such units by division by 772.

In heat engines the cycle of thermal operations consists of an alternate addition of heat (Q), and subtraction of heat (R), so that, if kU be the useful work,

$$kU = Q - R,$$

that is, the work is done at the expense of an equivalent amount of heat which disappears during the action of the engine. In steam engines this has been tested experimentally by measuring the heat supplied in the boiler and the heat discharged from the condenser. The difference should be, and in fact is found to be, the thermal equivalent of the work done by the engine. The ratio kU/Q is called the "absolute," or sometimes, for reasons we shall see presently, the "apparent" efficiency of the engine. It is always a small fraction: in the best steam engines, for example, it does not exceed ·15 losses connected with the furnace and boiler not being included. (Comp. Art. 278.) The quantity U is here the theoretical maximum (p. 478) for a pneumatic motor working between the given limits of pressure, and k is the "mechanical" efficiency.

277. *Mechanical Value of Heat. Thermal Efficiency.*—In stating the first law of thermodynamics nothing is said about the temperature at which the heat is used. In other words, the mechanical equivalent of heat is just the same whether the temperature be low or high. Yet common experience tells us that the value of heat for mechanical purposes depends very much on this circumstance. The heat discharged from the condenser of a condensing steam engine, or with the exhaust steam of a non-condensing engine, is of little value for the purposes of the engine. So obvious is this fact that the first attempts at connect-

ing the work done by a heat engine with the heat supplied to it may be partly described as attempts to show that temperature, not quantity, was equivalent to energy, heat being supposed as indestructible as matter.

It is now known, however, that difference of temperature is not in itself energy, but merely an indispensable condition that heat may be capable of being converted into work. The power of a heat engine depends on difference of temperature, being greater, the greater that difference is; but in all cases only a fraction of the heat supplied is converted into mechanical energy.

In the converse operation of converting mechanical energy into heat it is possible, by employing it in overcoming frictional resistances, to obtain an amount of heat equal to the energy employed, but such processes are always irreversible. The only way of converting heat into work is by means of a heat engine in which the rejection of heat at low temperature is as essential as the supply of heat at high temperature.

Difference of temperature is wasted if heat be allowed to pass from a hot body to a cold one without the agency of steam, air, or some other body, the density of which is changed by its action. When once wasted it cannot be recovered, a fact of common experience which is expressed by stating the Second Law of Thermodynamics.

Heat cannot pass from a cold body to a hot one by a purely self-acting process.

By a "self-acting" process in this statement is meant any process of the nature of a perpetual motion which is independent of any external agency. By the employment of mechanical energy drawn from external bodies, heat may be made to pass from a cold body to a hot one, the amount of energy required being greater the greater the difference of temperature. And the method recently introduced of raising steam, without the use of a furnace, by means of heat derived from the exhaust steam condensed in a solution of caustic soda, shows that energy derived from chemical affinity may serve the purpose. But, if no energy is employed, no heat will pass.

Difference of temperature must therefore be carefully utilized, and since the smallest difference of temperature is sufficient to cause heat to pass from a source into the air or steam which exerts energy, it at once follows that the process of conversion of heat into work will be most efficient if all the heat be supplied while the fluid has the temperature of the source of heat, and all the heat rejected while it has the temperature of the body which subtracts heat. These are the conditions of maximum efficiency, and if they are satisfied it is possible to show that a mechanically perfect motor (p. 480) supplied with heat Q will exert the energy.

$$U = Q \cdot \frac{t_1 - t_2}{t_1 + 461},$$

t_1, t_2 being the temperatures Fahrenheit of addition and subtraction of heat. This is true whatever be the nature of the heat engine employed for the purpose, and no more heat can be converted into work under any circumstances. A heat engine which satisfies these conditions may be described as "thermally perfect."

If two bodies be at the same temperature heat may be made to flow in either direction from one to another, the actual direction being determined by a difference which may be made as small as we please: that is, the process is *reversible.* Hence the conditions of maximum thermal efficiency may also be described by saying that the cycle of thermal operations must be "thermally reversible." And the condition that an engine may be both mechanically and thermally perfect may be completely described by stating that the engine is reversible.

Whichever way we adopt of stating the result it follows at once that a unit of heat has a certain definite MECHANICAL VALUE given by the equation

$$M = 772 \cdot \frac{t_1 - t_2}{t_1 + 461},$$

where t_1, t_2 are the temperatures between which it can be used.

278. *Thermal Efficiency.*—If an engine be mechanically perfect the work done per unit of heat will be simply the mechanical value, if the conditions of maximum efficiency are satisfied. In general, however, some of the heat will be supplied at a lower temperature than the source of heat, and some will be abstracted at a higher temperature than that of the refrigerator. When this is the case difference of temperature is wasted and there is a corresponding loss of thermal efficiency. If the temperature is known at which the air or steam is, while it is being supplied with a certain quantity of heat, or while a certain quantity of heat is being abstracted, the mechanical value of that heat can be found corresponding to that temperature. This quantity represents the work actually done since the engine is supposed mechanically perfect, and the same calculation being made for all the heat supplied or abstracted, the total actual work will be known. Dividing this by the total quantity of heat the actual work (U) per unit of heat will be known. The ratio

$$e = \frac{U}{M}$$

may be described as the "THERMAL EFFICIENCY" of the engine.

In practice the engine will not be either mechanically or thermally perfect: its efficiency will then be the product (*ek*) of the mechanical efficiency and the thermal efficiency. The efficiency thus calculated is estimated relatively to an engine which is mechanically and thermally perfect, and may be described as the "relative" or "true" efficiency, as distinguished from the "absolute" or "apparent" efficiency defined in a former article.

To estimate the efficiency of a heat engine without any reference to the temperatures between which the heat can be used is very misleading. The true efficiency of the best condensing steam engines is about 55 per cent., instead of 15 per cent. as it appears to be merely from the quantity of heat used. In comparing engines of different kinds, however, the same limits of temperature should be employed. (See Appendix.)

279. *Compound Engines.*—The working fluid may be discharged from one contracting chamber into a second which simultaneously expands. In many cases an intermediate reservoir is employed, which receives the fluid from the first chamber and supplies it to the second; the two chambers are then virtually separate, and form two distinct motors, the power of which can be separately calculated. The sum of the two is the power of the compound motor; it is necessarily the same as if the fluid had been used with the same expansion curve between the same extreme pressures in a single chamber; except that the frictional resistance of the passages between the chambers and the intermediate reservoir represents a certain loss of energy in the compound motor which does not occur in the simple one. When there is no intermediate reservoir there is no distinct period of admission or expansion in the low-pressure chamber, but the power may still be determined graphically for each chamber, and the results added.

In every case the energy of the fluid is the same, and cannot be affected by the mechanism employed to utilize it, unless its density or elasticity be altered by contact with the sides of the chamber in which it is enclosed. In steam engines, however, the action of the sides of the cylinder has great influence by condensing steam as it enters the cylinder. The liquefied steam is re-evaporated towards the end of the stroke as the temperature of the steam falls, but the process is nevertheless a very wasteful one. The action is greater the greater the degree of expansion employed, because the range of temperature is greater, and the gain by expansion is thus in great measure neutralized or even converted into a loss. By employing two cylinders instead of one the expansion is divided into two parts each of moderate amount, and liquefaction may be in great measure avoided. Compound engines are therefore being used more and more wherever economy of fuel is a consideration, and in marine practice have almost superseded the simple engine.

The principal losses in steam engines are (1) a mechanical loss due to incomplete expansion, and (2) a thermal loss due to liquefaction. One of these cannot be diminished without increasing the other; but considerable economy may be effected by the use of a "steam jacket," by the employment of superheated steam, and by compounding.

280. *Internal Energy. Internal Work.*—The distinction between internal work and external work was pointed out in Art. 92, p. 181, and the corresponding distinction between internal and external energy of motion in Art. 134, p. 250. These distinctions are principally important in fluids, because the extreme mobility of their parts renders internal motions, of great magnitude, of common occurrence. We have already seen in Chap. XIX. how energy is dissipated by the internal action of liquids; in gases the same dissipation occurs, and is even more important.

In liquids the absorption of energy is almost completely irreversible, but in gases it is not so. We may have internal energy as well as internal work: the greater part of the expansive energy of a gas being due to internal actions.

The state of an elastic fluid is completely known when its pressure and volume are known, but these quantities are capable of any variation we please within wide limits, provided only that we have unlimited power of adding or subtracting heat. If, however, a third quantity, the temperature, be considered, it will be found that the three are always connected together by a certain equation depending on the nature of the fluid, so that when any two are given the third is known. For example, in the so-called "permanent" gases such as dry air, the equation is very approximately

$$PV = c \,.\, T,$$

where T is the temperature reckoned from the "absolute" zero, a point 461° F. below the ordinary zero of Fahrenheit's scale, and c is a constant which for pressures (P) in lbs. per square foot and volumes (V) in cubic feet per lb. has, for dry air, the value 53·2. The "state" of the fluid is completely known if any two of these three quantities are known, but not otherwise.

To produce a given change of *state* a certain definite amount of work must be done in overcoming molecular resistances; this is the internal work, and is the same under all circumstances. But in gaseous fluids, the molecular forces being reversible, may tend to give rise to the change of state, and then we have internal energy instead of internal work. Taking the first case: if the change be at constant volume this internal work will be the total work done; but in general the volume changes, and in consequence external work is done, the amount of which depends not merely on the change of state, but also on the way in which that change is carried out. The total amount of work is the sum of the internal and external work: it is done by the agency of heat energy supplied from without, so that we write

Heat Expended = Internal Work + External Work,

the three quantities being expressed in common units.

For the application of this equation to questions relating to the formation of steam the reader is referred to a treatise on the Steam Engine by the present writer. We have now to consider the flow of gases through pipes and orifices, for which purpose the equation is written

Expansive Energy = Internal Energy + Heat Supply,

or, in other words, of the whole expansive energy of the fluid, a part is derived from internal molecular forces, and a part from heat supplied from without.

If no heat is supplied from without the expansive energy is equal to the internal energy: this case is called "adiabatic" expansion, obtained by writing $n = 1{\cdot}4$ in the formulæ of Art. 270. More generally, it is shown in treatises on thermodynamics that the internal work done in changing the temperature of a lb. of air from T_1 to T_2 is $I_2 - I_1$, where

$$I = K_v . T = 2{\cdot}5\ PV,$$

K_v being the specific heat at constant volume, which is $2{\cdot}5\ c$. Hence when the temperature falls from T_1 to T_2 the internal energy supplied by the fluid is $K_v\ (T_1 - T_2)$ and the equation becomes for a heat supply Q

$$E = 2{\cdot}5\ c\,(P_1 V_1 - P_2 V_2) + Q.$$

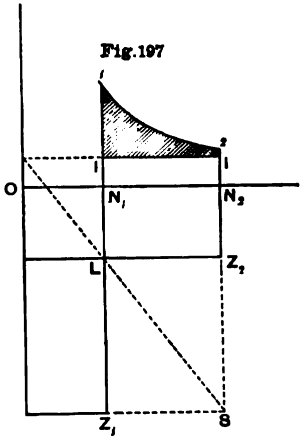
This is the fundamental equation from which all cases may be derived.

If heat be supplied to a permanent gas at a uniform rate as the temperature falls, it may be shown that the law of expansion is $PV^n =$ constant, as supposed in Art. 270, and this is generally permissible with sufficient approximation. The expansion index n then depends upon the proportion which Q bears to E.* If $Q = E$ the expansion is hyperbolic, and the whole of the expansive energy is derived from heat supplied from without. The manner in which the expansive energy (E) depends on the heat supply (Q) is well seen by the annexed diagram (Fig. 197). Let, as before, the ordinates of the point 1 represent the pressure and volume before expansion and those of the point 2 after expansion, 1, 2 being the expansion curve. Set downwards N_1Z_1, N_2Z_2 each equal to 2½ the corresponding pressure ordinates, and complete the rectangles OZ_1, OZ_2. Then complete the rectangle Z_1Z_2, and draw the diagonal SL to meet the vertical through O. Finally through the intersection draw II horizontally: then the rectangle IN_2 will be found to be the difference of the rectangles OZ_1, OZ_2, and therefore represents the internal energy exerted during expansion. Thus the area $12II$ (shaded in the figure) represents the heat supply: which will depend not only on the points 1, 2, that is, on the change of state of the air, but also on the form of the expansion curve, that is, on the way in which the change takes place.

---

* For further explanation of the statements here made the reader is referred to Cotterill's *Steam Engine*, chapter IV.

Fig. 197



When the pressure of air is changed the changes of temperature are generally so great that, unless the process be very rapid, they are accompanied by a flow of heat to or from external bodies. The amount of heat thus abstracted from, or supplied to the air, is unknown, and the index n cannot, therefore, be found accurately. An approximate value is assumed according to circumstances in each special case.

When air is compressed into a reservoir for the purpose of storing energy, its temperature cannot remain permanently above that of surrounding bodies, and the process will be most economical when the temperature is kept as low as possible by surrounding the pump with cold water. We then assume $n = 1$, but the result of the calculation will generally be too small. The heat to be abstracted is the thermal equivalent of the absolute amount of work done in compression alone. The energy exerted in pumping is *greater* than this by the amount of energy transmitted by the air (if any) which leaves the reservoir, and *less* by the energy exerted by the back pressure which here serves as a source of energy. (See Art. 271, p. 478.)

When air expands from a reservoir in which it is stored at a moderate pressure it receives heat from without, but the amount is uncertain and cannot be relied on. We therefore assume $n = 1{\cdot}4$, though the result of the calculation will be too small. The ratio of results for $n = 1{\cdot}4$ and $n = 1$ will be a measure, though a crude and imperfect one, of the efficiency of the process of storage and transmission of energy by compressed air. The efficiency is less the higher the pressure. If, as in the Whitehead torpedo, the pressure employed is very high, it will be necessary to reduce it by wire-drawing. As will be seen farther on, the mechanical energy dissipated by internal resistances then re-appears as heat which maintains the temperature. The effect of this is so great that the process may even (practically) be economical.

The various points here mentioned are illustrated by Ex. 1-4, p. 502. It must be understood, however, that the air is supposed dry, a condition seldom satisfied in practice. Indeed, in many machines of this class water is injected into the working cylinder. Some remarks on this point will be found in the Appendix.

281. *Steady Flow through a Pipe. Conservation of Energy.*—Referring to Fig. 167, p. 413, suppose that the reservoir is closed, and that it contains an elastic fluid at high pressure which is flowing through the pipe. Unless the change of pressure be very small, difference of level may be disregarded as relatively unimportant (p. 475), and we have only to consider differences of pressure, while, on the other hand, we must now remember that, when the pressure changes, energy is exerted by expansion as well as by transmission. The energy transmitted from the reservoir *to* any point where the pressure is P and volume V is P_0V_0, where the suffix indicates the state of the fluid in the reservoir. Of this the amount PV is transmitted through the point, and the difference $P_0V_0 - PV$ together with the expansive energy E is employed in generating the kinetic energy which the gas possesses in consequence of the velocity u with which it is rushing through the pipe at the point considered. Thus, if the motion be steady,

$$\frac{u^2}{2g} = 3{\cdot}5\,(P_0V_0 - PV) + Q,$$

where Q is the heat (if any) supplied during the passage from the reservoir to the point. If no heat be supplied,

$$\frac{u^2}{2g} + 3{\cdot}5\ PV = \text{Constant},$$

an equation which may also be written

$$\frac{u^2}{2g} + K_p\,.\,T = \text{Constant},$$

where K_p is the specific heat at constant pressure. If we have to do with any elastic fluid other than a permanent gas, $3{\cdot}5\ PV$ must be replaced by $I + PV$, where I is the internal energy, and if the question be such that the elevation of the point considered has any sensible influence, the term z must be added as in the corresponding case of an incompressible fluid.

The equation for a compressible fluid, however, is much more general than that for an incompressible fluid, because the internal energy is taken into account, and consequently any energy exerted in overcoming frictional resistances is replaced by an equivalent amount of heat generated. It follows that the equation is true whether there be frictional resistances or whether there be none, provided that the internal motions have time to subside and be converted into heat by friction, and provided that none of the heat thus generated is transmitted to external bodies.

It sometimes happens that we have to consider cases where a quantity of heat Q is supplied to a permanent gas during its passage from a point 1 to a point 2, we shall then have the equation

$$Q = K_p\,(T_2 - T_1) + \frac{u_2^2 - u_1^2}{2g},$$

an equation which is true, however great the variations of pressure or temperature are, and whether or not there are frictional resistances.

282. *Velocity of Efflux of a Gas from an Orifice.*—The most important applications of the equation for the steady flow of a gas are to the discharge of air or steam from an orifice and to the flow of air through long pipes.

In the first case the frictional resistances are small and are consequently neglected. It will be desirable to give a method of treating the question which is independent of the general equation.

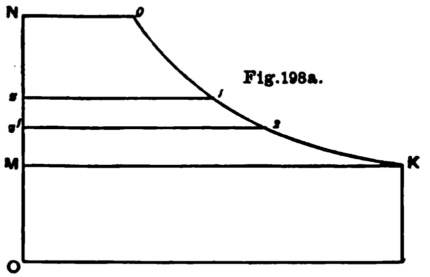


Fig.198a.

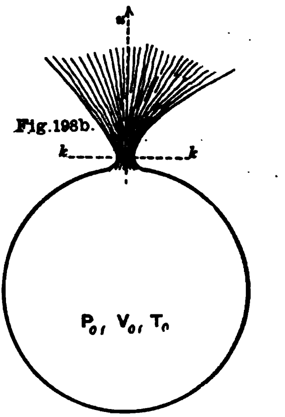


Fig.198b.

In Fig. 198a $o12k$ represents the expansion curve for a small portion of the gas as it rushes out of the reservoir A (Fig. 198b) in which it is confined through a small orifice into the atmosphere. The jet contracts at issue to a contracted section *kk*, nearly as in the case where the fluid is incompressible, and then, in general, expands again in some such way as is shown in the figure. The velocity through the contracted section may be denoted by u, and the pressure there by P. The area of the contracted section is connected with the area of the orifice by the equation

$$A = kA_0,$$

as on page 409, k being a co-efficient.

Each small portion of the fluid expands from the state represented by the point o on the diagram to that represented by K: in some intermediate state it will be represented by a point 1 on the expansion curve, and immediately after by 2, a point near to 1. Let u_1, u_2 be the corresponding velocities, then

$$\frac{u_2^2 - u_1^2}{2g} = \frac{P_1 - P_2}{w} = V.\delta P,$$

where w is the mean density and V the mean specific volume represented graphically by the mean of $z1$, $z'2$. Hence $V.\delta P$ is represented by the area of the strip cut off by these ordinates. Dividing the whole area into strips, the area of each strip represents the corresponding change in $u^2/2g$, so that the total area represents the final value of this quantity. We have then

$$\frac{u^2}{2g} = \text{Area } NoKM = \int_P^{P_0} VdP = h$$

The quantity h thus found and graphically represented is the "head" due to difference of pressure, as fully explained in Art. 271.

Assuming the expansion curve $PV^n =$ Constant, as before,

$$\frac{u^2}{2g} = \frac{n}{n-1}(P_0V_0 - PV) = \frac{nc}{n-1}(T_0 - T).$$

Now, if the expansion be adiabatic $n = 1{\cdot}4$, and $nc/(n-1)$ is equal to K_p, so that the result might have been written down at once from the general equation of the preceding article.

Employing the notation of Art. 270, but replacing the suffix 1 by the suffix 0, the velocity of efflux is given by the formula

$$\frac{u^2}{2g} = \frac{n}{n-1}.P_0V_0(1 - rx).$$

283. *Discharge from an Orifice.*—The weight of gas discharged per second from an orifice of contracted area A is now found from the formula

$$W = \frac{Au}{V},$$

where V is the specific volume of the gas at the instant of passing through the contracted section, and therefore supposing A unity the weight per unit of area is given by

$$W^2 = 2g \cdot \frac{n}{n-1} \cdot \frac{P_0 V_0}{V}(1 - rx).$$

For V we now write rV_0 and finally obtain

$$W^2 = 2g \cdot \frac{n}{n-1} \cdot \frac{P_0}{V_0} \cdot \frac{1 - rx}{r^2}.$$

In applying this formula x must be supposed known and r calculated from it by the equation on p. 476.

It will be found on examination that as x diminishes from unity W increases to a maximum value and then diminishes again to zero. That is, if the pressure in the throat of the jet at the contracted section be diminished the discharge does not increase indefinitely, but reaches a maximum and then decreases. On substitution for r in terms of x it will be seen that for a given pressure (P_0) in the reservoir W is greatest when $x^{\frac{2}{n}} - x^{\frac{n+1}{n}}$ is greatest.

This will be found to be the case when

$$x = \left(\frac{2}{n+1}\right)^{\frac{n}{n-1}}.$$

The expansion is adiabatic, and the values of n with the resulting values of x for maximum discharge are shown in the annexed table.

NATURE OF GAS.	VALUE OF n.	VALUE OF x.
Dry Air,	1·408	·527
Superheated Steam,	1·3	·546
Dry Saturated Steam,	1·135	·577
Moist Steam,	1·1	·582
	1	$\frac{1}{\sqrt{e}} = \cdot 6$

The discharge is therefore a maximum when the external pressure is from ·5 to ·6 the pressure in the reservoir. For dry air it will be found on substitution that the maximum discharge per second per unit of contracted area is

$$W_m = \frac{3 \cdot 9\, P_0}{\sqrt{P_0 V_0}} = \frac{P_0}{1 \cdot 87 \sqrt{T_0}},$$

and for dry steam

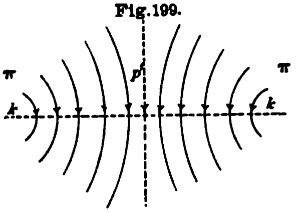
$$W_m = \frac{3 \cdot 6\, P_0}{\sqrt{P_0 V_0}}.$$

The pressure P_0 was originally supposed expressed in lbs. per square foot, but it may now be taken as lbs. per square inch in the numerator of these fractions, in which case W_m will be the discharge per square inch.

The diminution of the discharge on diminution of the external pressure below the limit just now given, is an anomaly which had always been considered as requiring explanation, and M. St. Venant had already suggested that it could not actually occur. In 1866 Mr. R. D. Napier showed by experiment that the weight of steam of given pressure discharged from an orifice is really independent of the pressure of the medium into which the efflux takes place; and in 1872 Mr. Wilson confirmed this result by experiments on the reaction of steam issuing from an orifice.*

The explanation lies in the fact that the pressure in the centre of the contracted jet is not the same as that of the surrounding medium. The jet after passing the contracted section suddenly expands, and the sudden change of direction of the fluid particles gives rise to centrifugal forces which cause the pressure to increase on passing from the surface of the jet to the interior on the principle explained on page 415. This will be better understood by reference to the annexed figure (Fig. 199) which shows a longitudinal section of the jet at the point where the contraction of transverse section is greatest. The particles describe curves the radius of curvature of which increases from a small minimum value at the surface k to an infinite value at the centre. The pressure p increases from that of the medium (π) at k to a maximum p' at the centre, the increase being very rapid at first and afterwards more gradual. The problem is therefore far more complicated than we have supposed, each small portion of the jet having its own pressure and (consequently) its own velocity and density.

Fig. 199.



The results of experiment however suggest that an approximate solution may be obtained by the assumption of a mean pressure in the throat of the jet, with a corresponding mean velocity; this mean pressure being that which gives maximum discharge in every case in which that quantity is greater than π. At lower pressures it is to be assumed equal to π.

Adopting this hypothesis we see that whenever steam is discharged into the atmosphere from a boiler the pressure in which is greater than, about, 25 lbs. per square inch absolute, or 10 lbs. above the atmosphere, the formula given above for maximum discharge is to be used. If we assume the mean value 252 for $\sqrt{P_0 V_0}$ this gives $p_1/70$ for the weight discharged from an orifice per square inch of effective area per second, the pressure p_1 being the absolute pressure in the boiler expressed in lbs. per square inch. Contraction and friction must be allowed for by use of a co-efficient of discharge, the value of which however is more variable than that of the corresponding co-efficient for an incompressible fluid. Little is certainly known on this point.

---

* *Discharge of Fluids*, by R. D. Napier. Spon, 1866. *Annual of the Royal School of Naval Architecture for* 1874.

283A. *Flow of Gases through Pipes.*—Returning to the general equation, we have now to examine the case where air or steam flows through a pipe of considerable length. As in the case of water, the frictional resistances are then so great that most of the head is taken up in overcoming them. The velocities of the fluid are therefore not excessive, and the value of $u^2/2g$ varies comparatively little.

Now in the equation

$$\frac{u^2}{2g} + K_p \,.\, T = \text{Constant}$$

the numerical value of K_p is about 184, and therefore a variation of temperature of a single degree will correspond to a great change in $u^2/2g$; it may therefore be assumed that the temperature remains very approximately constant provided only that the difference of pressure at the two ends of the pipe is not too great compared with its length.

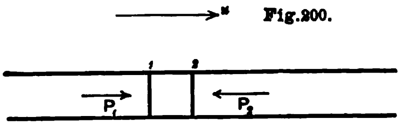


Fig. 200.

In Fig. 200 suppose 1, 2 to be two sections of the pipe at a distance Δx so small that, in estimating the friction, the velocity may be taken at its mean value u; then the force required to overcome friction is

$$R = f \,.\, s \,.\, \Delta x \,.\, u^2,$$

where s is the perimeter and f the co-efficient of friction, defined as on page 419. Replacing this by a new co-efficient f' as in the passage cited,

$$R = f' \frac{w}{2g} \,.\, s\, \Delta x \,.\, u^2,$$

in which equation w means the weight of unit-volume of the gas. Now, it was pointed out on page 431 that surface friction was a kind of eddy resistance, and that in the case of water it was proportional to the density. This leads us to suppose that in fluids of varying density, not f but f' is a constant quantity. Replacing w by its equivalent $1/V$, we obtain, suppressing the accent of f,

$$R = f \,.\, \frac{s \Delta x}{V} \,.\, \frac{u^2}{2g}.$$

We now apply the principle of momentum which will be expressed by the equation

$$(P_1 - P_2)A = W \,.\, A\frac{u_2 - u_1}{g} + R,$$

where W is the weight of gas flowing through the pipe per unit of area per second, and the suffixes refer to the two sections in question, the area of which is A. Now, the motion through the pipe being steady, W is the same throughout, so that

$$\frac{u}{V} = W = \text{Constant}.$$

By substitution for W and writing H for $u^2/2g$, an equation is obtained which when written in the differential form becomes

$$- V \,.\, \delta P = \delta H + f \,.\, \frac{\delta x}{m} \,.\, H,$$

m being the hydraulic mean depth.

Next, if T be the temperature, which, as remarked above, is sensibly constant,

$$P = \frac{cT}{V} = W \,.\, \frac{cT}{u} ; \therefore \delta P = - W \,.\, \frac{cT}{u^2} \,.\, \delta u.$$

Substitute again for W and u, we then find

$$- V \,.\, \delta P = \frac{1}{2}\, cT \,.\, \frac{\delta H}{H}.$$

On substitution for $V \,.\, \delta P$ the value of which has just been given, the differential equation becomes integrable by dividing by H, and we obtain on performing the integration

$$\frac{1}{2}\, T \left\{ \frac{1}{H_0} - \frac{1}{H} \right\} = \log_e \frac{H}{H_0} + f \,.\, \frac{l}{m},$$

where l is the length of the pipe, and H_0, H the values of $u^2/2g$ at entrance and exit respectively. In application of this equation the term containing the logarithm is small as compared with the rest, and may generally be omitted: also

$$\frac{H}{H_0} = \frac{p_0{}^2}{p^2},$$

a ratio which is known if the pressures are the data of the question.

The value of the co-efficient is found by experience to be nearly the same as for water, that is, about ·007. In the case of steam cT is to be replaced by the nearly constant product PV, which is to be taken from a table for this quantity so as to obtain a mean value according to the pressure considered.

The equation just found must not be applied to cases in which the difference of pressure is too great compared with the length of the pipe. The friction is then not great enough to prevent the velocity from becoming excessive; the temperature then sensibly falls, instead of remaining constant as supposed in the calculation. An equation can be found which takes account of the fall of temperature when necessary, but in such cases as commonly occur in practice, the supposition of constant temperature is sufficiently approximate. When the difference of pressure is small the equation will be found to reduce to the hydraulic formula for flow in a pipe. This case will be considered presently.

The head is given by the formula

$$h = PV.\log_e\frac{p_0}{p} = PV\log_e r,$$

and the power expended in forcing the air through is Wh or $PAu.\log_e r$ ft. lbs. per 29 ft. of sectional area per 1″.

284. *Flow of Gases under Small Differences of Pressure.*—When the differences of pressure are small and no heat is added or subtracted, a gas flows in the same way as a liquid of the same mean density. In the case of air the mean specific volume is found from the equation

$$V = \frac{cT}{P} = \frac{T}{40},$$

the units being feet and pounds, the mean pressure that of the atmosphere, and the temperature measured on Fahrenheit's scale from the absolute zero. At 59° this gives $V = 13$ cubic feet, but the actual volume will vary slightly from variations in the mean pressure.

The small differences of pressure with which we have now to do are commonly measured by a syphon gauge in inches of water. One inch of water is equivalent to a pressure of 5·2 lbs. per sq. ft.

If now ΔP be the difference of pressure in lbs. per sq. ft., i the corresponding number of inches of water, the head due to it will be, as in Art. 283,

$$h = V.\Delta P - \frac{T}{7{\cdot}7}.i.$$

The velocity due to this head, or, what is the same thing, the volume discharged per sq. ft. of *effective* area per second in the absence of frictional resistances, is in cubic feet

$$u = \sqrt{2gh} = 2{\cdot}89\sqrt{Ti},$$

and the weight-discharge in pounds per second

$$W = \frac{u}{V} = 155{\cdot}6\sqrt{\frac{i}{T}}.$$

At 59° one inch of water gives a head of 67·5 feet and a discharge of 65·9 cubic feet, or 5·07 lbs. per second; but at 539° the head is 130 feet and the discharge 91·3 cubic feet, or 3·67 lbs., results which show that the effect of a given difference of pressure is entirely different according to the temperature of the flowing air. This is a point which must always be borne in mind in applying hydraulic formulæ to the flow of gases. Frictional resistances are taken into account by the employment of a co-efficient as in hydraulics, and as elsewhere explained, there is reason to believe that the values of these co-efficients are the same, except so far as they may be dependent directly or indirectly on the co-efficient of contraction (p. 430). Co-efficients of contraction are more variable in air than in water, but their average value does not differ widely in the two cases, and may provisionally be assumed the same.

In particular, it is well established that the formula for the discharge of a pipe in cubic feet per second (p. 420),

$$Q = k\sqrt{\frac{h'}{l}}\,.d^{\frac{5}{2}},$$

applies to air as well as to water with the same value of the co-efficient k, that is (d in feet) about 40. The head h' is calculated, as just explained, according to the temperature of the air, for a given difference of pressure.

It is sometimes necessary to consider the flow of some gas other than atmospheric air. In approximately permanent gases this is easily done if we know the density of the gas. For example, the density of common coal gas is about ·43, air being unity. The value of c in the formula $PV = cT$ is then proportionately increased, but in other respects the formulæ are unaltered, the index of the adiabatic curve and the constants 2·5, 3·5 which depend on it remaining unaltered. The formula for small differences of pressure may also be employed for the non-permanent gases, such as steam, with a corresponding modification.

Pneumatic machines in which the variation of pressure is small are analogous to hydraulic machines, and most of what was said in the last chapter is applicable to them. The common fan, for example, is a centrifugal pump, the lift of which is the difference of pressure reckoned in feet of air, that is, at ordinary temperatures, about 67 feet for each inch of water. The speed of periphery is $\sqrt{gh}$ in feet per second, where h is the lift increased, as explained in the case of the pump, on account of frictional resistances and the curving-back of the vanes.

Fans are employed to produce a current of air for the purpose of ventilating a mine, ship, or structure of any kind. In mines they are often 30 feet diameter or more. The pressure required is here small and the speed moderate. They are also used to produce a forced draught in torpedo boats, or the blast of a smithy fire. The pressure is then 5 inches of water and more, corresponding to a lift of 300 feet and upwards. The speed of periphery is consequently excessive, and for the comparatively great pressures required for a foundry cupola or a blast furnace, it is necessary to resort to some other form of blowing machine. The speed of periphery might perhaps be reduced by the use of fixed guide blades to give the air a rotatory motion before entering the fan.

285. *Varying Temperature. Chimney Draught.*—If the temperature of the flowing air is varied by the addition or subtraction of heat, its density will be altered during the flow, and it is then necessary to know the mean density, in order that we may be able to calculate the "head" due to a given difference of pressure as measured by the water gauge.

In Fig. 201 OX, OY are axes of reference parallel to which ordinates are drawn as usual to represent volumes and pressures. A given difference of pressure ΔP is represented by the difference ab of a pair of ordinates. The original volume of the air is represented by $a1$. Suppose now that in flowing through a passage of any description the air is heated, as for example in passing through a furnace, the volume increases greatly while the pressure falls slightly; this will be represented by the curve 12, terminating at a point 2. The form of the curve will depend on the law of heating, and will be very different according to the state of the fire; if the bars of the grate be blocked by clinker and the surface of the fire be free from special obstruction, most of the frictional resistance and corresponding fall of pressure will occur before the air is heated, and the curve will slope rapidly near 1 and slowly afterwards; while, conversely, if the fire be covered with fresh fuel and the great bars clear, the reverse may be true. After being heated let the air pass through a boiler tube, by which heat is abstracted, till it reaches the chimney: the volume then diminishes greatly while the pressure falls slightly, as shown by the curve 23, terminating at a point 3, such that $b3$ represents the volume of the air in the chimney. The form of the curve 23 will depend on the law according to which the tube abstracts heat. The area of the whole figure $a123b$ represents the "head" due to the whole difference of pressure ΔP, and it will now be obvious that this head will vary according to circumstances which cannot be precisely known. Thus the mean density cannot be found except

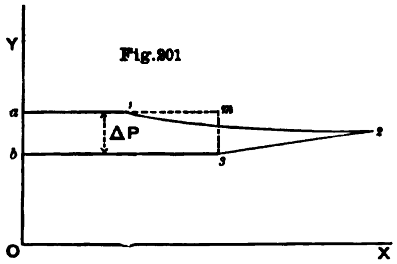


Fig. 201

by empirical formulæ derived by direct experience, and consequently applicable only to the special cases for which they have been determined. It has hitherto been most usually assumed in the case of a furnace and boiler that the mean density was that of the air in the chimney, which amounts to supposing that the forms of the curves 12 23 are such that the area of the rectangle $a3$ is equal to the area of the whole figure. This is the supposition employed by Rankine, and in many cases it appears to lead to correct results.

In every case of the flow of heated air it must be carefully considered what the mean density will probably be. Its value can often be foreseen without difficulty. It is only in the case of long passages, where the air suffers great frictional resistance while being heated or cooled, that it is uncertain what value to adopt.

The draught which draws air through a fire may be produced artificially or by the action of a chimney. In the latter case there is a difference of pressure within and without the chimney at its base due to the difference of weight of a column of air of the height of the chimney at the temperature of the chimney and at that of the atmosphere. Radiation causes the temperature of the air to be less in the upper part of the chimney and so diminishes the draught, and frictional resistances have the same effect. If these be disregarded the draught in inches of water will be

$$i = 7\cdot7\left\{\frac{1}{T_0} - \frac{1}{T}\right\}l,$$

where T_0 is the temperature of the atmosphere, T that of the chimney, while l is the height of the chimney in feet. The temperatures are here reckoned from the absolute zero.

If, for example, the temperature of the chimney be 539° F., and that of the atmosphere 59° F., the height of chimney required for a draught of 1 inch of water will be about 141 feet, or in practice more on account of friction and radiation.

The effect of this draught in drawing air through a furnace or through passages of any kind will vary according to the circumstances which have just been explained.

EXAMPLES.

1. Find the store of energy in the reservoir of a Whitehead torpedo. Capacity 5 cubic feet. Pressure 70 atmospheres.

Ans. If $n = 1$ 2,420,000 ft.-lbs., or 3,130 thermal units.
$n = 1\cdot4$ 1,092,000 „ or 1,414 „
Ratio of results = ·45.

2. In the last question the air is supplied to the torpedo engines by a reducing valve

so that the pressure in the supply chamber remains constantly at 13 atmospheres: find the available energy.

Ans. If $n = 1$ 1,900,000 ft.-lbs.
$n = 1{\cdot}4$ 1,346,000 „

NOTE.—The difference between these results and the preceding is the effect of wire-drawing (resistance of valve). The efficiency of the process may be taken as 1346/2420 or ·56. The supply chamber is supposed small.

3. Air is stored in a reservoir the pressure in which is maintained always nearly at 10 atmospheres: find the store of energy per cubic foot of air supplied from the reservoir.

Ans. If $n = 1$ 48,700 ft.-lbs.
$n = 1{\cdot}4$ 35,700 „
Ratio = ·733.

4. A chamber of 100 cubic feet capacity is exhausted to one-tenth of an atmosphere: find the work done, assuming $n = 1$.

Here if the chamber be imagined to contract, compressing the air still remaining in it, the energy exerted will be due to the pressure of the atmosphere, and the difference between this and the work done in compression will be available for other purposes. In exhausting this is reversed. *Ans.* 142,000 ft.-lbs.

5. Find the mechanical efficiency of an engine so far as due to incomplete expansion (ratio r): assuming the expansion hyperbolic.

Ans. If R be the ratio of complete expansion,

$$\text{Efficiency} = \frac{1 + \log_e r - \dfrac{r}{R}}{\log_e R}.$$

6. In the last question obtain numerical results for a condensing engine, taking the back pressure at 2 lbs. and boiler pressure 60 lbs.

Ans. Ratio of expansion,	1	2	5	10,
Efficiency, - -	·284	·48	·72	·87.

7. Find the comparative mechanical efficiencies in a condensing and a non-condensing engine. Back pressure in condensing engine 2, in non-condensing 16. Boiler pressure 60 and 100. Ratio of expansion 5 in both cases.

The engines must here be supposed to have the same lower limit of pressure of 2 lbs.; and the result for the non-condensing engine includes the loss by the actual back pressure being 16 lbs. *Ans.* ·72, ·46.

8. Find the loss by wire-drawing between two cylinders from one constant pressure of 60 lbs. to another constant pressure of 40 lbs. Expansion hyperbolic. *Ans.* ·405 PV.

9. One vessel contains A lbs. of fluid at a given pressure P_A, and a second B lbs. of the same fluid at a lower pressure P_B. A communication is opened between the vessels, and fluid rushes from A to B: find the loss of energy.

The loss here is the difference between the energy exerted by A lbs. expanding from V_A to V, and the work done in compressing B lbs. from V_B to V: where V_A, V_B are the specific volumes of the fluid in A and B, and V that of the fluid after equilibrium has been attained, found from the formula

$$V = \frac{AV_A + BV_B}{A + B}.$$

Hence the loss is very approximately

$$\text{Loss} = \frac{AB}{A + B} \cdot \frac{(V_B - V_A)(P_A - P_B)}{2}.$$

10. In a compound engine the receiver is half the volume of the high-pressure cylinder, and at release the pressure in the cylinder is 25 lbs. per square inch, while that in the receiver is 15 lbs. per square inch: find the loss of work per lb. of steam. Obtain the results also when the receiver is double instead of one half the cylinder.

Ans. Case I., 1638 ft.-lbs.
Case II., 3873 „

11. In a condensing engine find the mean effective pressure and the consumption of steam in cubic feet per I.H.P. per minute at the boiler pressure: being given, back pressure 3, boiler pressure 60 lbs. per square inch (absolute), ratio of expansion 5.

Ans. Mean effective pressure = 28·33 lbs. per square inch.
Consumption of steam = 1·62 cubic feet per minute.

12. If the volume of 1 lb. of dry steam at the boiler pressure be taken in the preceding question as 7 cubic feet and the liquefaction in admission 20 per cent.: find the weight of steam consumed in lbs. per I.H.P. per hour. *Ans.* 17·5.

13. Find the mechanical value of a unit of heat, the limits of temperature being 600° and 60°; 300° and 100°; 400° and 212°.

Ans. 393, 203, 169 ft.-lbs.

14. The limits of temperature in a heat engine are 350° and 60°; find the thermal efficiency when two-thirds of the whole heat supplied is used between 300° and 100°, one-sixth between 200° and 100°, and one-sixth between 250° and 100°. *Ans.* ·705.

15. In question 6, on account of a gradual increase in the liquefaction the thermal efficiency at the several ratios of expansion mentioned is assumed as ·9, ·85, ·7, ·5: find the true efficiency. *Ans.* ·256, ·408, ·504, ·435.

16. In a compound engine the pressure of admission is 100 lbs. per square inch, the steam is cut off at one-third in the high-pressure cylinder, the ratio of cylinders is 2½: the back pressure is 3 lbs. per square inch, the large cylinder 40 inches diameter, and the speed of piston 400 feet per second. Find the H.P., neglecting wire-drawing and sudden expansion. *Ans.* 567.

17. In the last question suppose that the engine has a very large intermediate reservoir, and that the cut-off in the low-pressure cylinder is ·5; find the pressure in the reservoir, neglecting wire-drawing, also the loss per cent. by sudden expansion at exhaust from the high-pressure cylinder, and the percentage of power developed in the two cylinders.

Obtain the results also for a cut-off of one-third in the low-pressure cylinder.

Ans.

	Cut-off ½.	Cut-off ⅓.
Pressure in reservoir, - - - - - - - -	26·7	40
Loss by sudden expansion per cent., - - - -	·8	·7
Percentage of power in high-pressure cylinder, - -	46·5	32·4
,, ,, low-pressure ,, - -	52·6	67·6

18. Compare the efficiencies of the simple and compound engine, assuming the liquefaction the same at the best ratio of expansion, which is 5 in the simple engine and 7 in the compound engine, while in the latter 5 per cent. of the work is lost by wire-drawing between the cylinders. Back pressure and boiler pressure in both cases 3 lbs. and 84 lbs. respectively.

Ans. Gain by compounding 2½ per cent.

19. In question 16, instead of supposing the whole expansion represented by a single hyperbolic curve, assume that at the end of the stroke in the high-pressure cylinder the steam is dry, while at the end of the stroke in the low-pressure cylinder the steam contains 10 per cent. water. Obtain the required result for the cut-off ·5 and find the weight of steam used (exclusive of jacket steam) in lbs. per I.H.P. per hour. Also obtain the results when the steam at the end of the stroke in the high-pressure cylinder contains 30 per cent. water, all other data remaining the same.

Ans.

	Case I.	Case II.
Pressure in reservoir, - - - - - - -	22·5	14·9
Percentage of power in high-pressure cylinder, -	55	37·5
,, ,, low-pressure ,, -	55	62·5
Lbs. of steam per I.H.P. per hour, - - - -	13	16·5

NOTE.—The results of this question may be readily obtained by use of tables of the properties of steam. They show clearly the great influence on the working of a compound engine of the relative liquefaction in the cylinders. If liquefaction be permitted in the high-pressure cylinder the compound engine loses its advantage.

20. Air at a pressure of 1,000 lbs. per sq. inch and a temperature of 539° expands to 6 times its volume without gain or loss of heat; find the pressure and temperature at the end of the expansion. *Ans.* $p = 81$, $t = 27°$.

21. In the last question suppose the air at the end of the expansion to have a pressure equal to $1\frac{1}{3}$ times that given by the adiabatic law, and heat to be supplied at a uniform ratio as the temperature falls; find the index of the expansion curve and the work done during expansion. Compare the heat supplied with the work done and find the specific heat.

Ans. $n = 1·174$. Specific heat = ·228.
Work done = 82,000 ft.-lbs. Ratio = ·575.

22. Air is contained in a vessel at a pressure of 25 lbs. per sq. inch and temperature 70°. What will be the velocity with which the air issues into the atmosphere (pressure 15 lbs. per sq. inch)? Also find the discharge and the head.

Ans. $h = 13,420$: $u = 930$ ft. per second.
$W = 34·26$ lbs. per sq. inch of orifice per minute.

23. In the last question find the initial pressure corresponding to maximum discharge for all external pressures less than that of the atmosphere. Find this discharge.

Ans. Pressure = 28·5 lbs. per sq. inch.
Discharge = $39\frac{1}{3}$ lbs. per sq. inch per minute.

24. What weight of steam will be discharged per minute from an orifice 2 inches diameter, the absolute boiler pressure being 120 lbs. per sq. inch. Co-efficient of discharge ·7. *Ans.* 227 lbs.

25. Air flows through a pipe 6 inches diameter and 4,000 feet long; the initial pressure is 20 and the final pressure 15 lbs. per square inch; temperature 70°; find the velocities and the discharge. $4f = ·03$.

Ans. Velocity at entrance = 39 feet per second.
" exit = 52 feet "
Discharge = 4 lbs. per sq. ft. = ·78.

26. In the last question find the loss of head and the H.P. required to keep up the flow. *Ans.* $h' = 8,124$ feet. H.P. = $11\frac{1}{3}$.

27. Steam at 50 lbs. rushes through a pipe 3 inches diameter and 100 feet long with velocity at entrance of 100 feet per second; find the loss of pressure. $4f = ·03$. *Ans.* 1·6.

REFERENCES.

For descriptive details and illustrations of the mechanism of steam engines the reader is referred amongst other works to

THURSTON. *History of the Growth of the Steam Engine.* International Scientific Series. Kegan Paul.
SENNETT. *The Marine Steam Engine.* Longman.
RIGG. *Practical Treatise on the Steam Engine.* Spon.

APPENDICES.

APPENDIX.

A. NOTES AND ADDENDA.

I.—STATICS OF STRUCTURES.

RANKINE's treatise on *Applied Mechanics* appeared in 1858. The sixth edition is quoted in the following notes by the letters A.M.

PAGE 2. "The word STRESS has been adopted as a general term to comprehend various forces which are exerted between contiguous bodies or parts of bodies, and which are distributed over the surface of contact" (A.M., p. 68). It appears from this that RANKINE's use of the word is confined to internal forces, but by some writers it is employed for all forces whether external or internal. Ties and struts are, however, defined as in the text (A.M., p. 132).

PAGE 3. The total load on the platform of a timber bridge carrying an ordinary roadway may be assumed as 250 lbs. per sq. ft., of which 120 represents the weight of a closely packed crowd, and the remainder is the weight of the roadway and platform. The weight of a timber roof (slate or tile) is from 12 to 24 lbs. per sq. ft. The travelling load on railway bridges is commonly estimated at 1 ton per foot-run.

PAGE 14. The diagram of forces for a funicular polygon under a vertical load was (probably) first given by ROBISON in his treatise on *Mechanical Philosophy*, Vol. I. Dr. Robison died in 1805, and this work is a collection of his papers published in 1822.

PAGE 20. In the Saltash bridge the compression member of each girder is a tube of elliptical section 15 feet in breadth, 8 feet in depth. A pair of chains, one on each side, carry the platform.

PAGE 20. A. Of the various methods of constructing a parabola the most convenient is that in which a curve is drawn through the intersections of a set of lines radiating from a point, with a set of equidistant lines drawn parallel to a fixed line: the radiating lines being drawn so as to cut off equal intercepts on another fixed line. It can easily be proved that this curve is the funicular polygon proper to a uniform load without introducing any properties of the parabola.

Page 21. Let P be the vertical tension of the chain at the point P, then, since $dy/dx = P/H$, where H is constant,

$$\frac{d^2y}{dx^2} = \frac{1}{H} \cdot \frac{dP}{dx} = \frac{w}{H}.$$

This equation is equally true if w vary according to any law, and is therefore the general differential equation of a cord or linear arch under any vertical load. Particular cases are:—

(1) *The Common Catenary.* Here if m be the weight of a unit of length of the cord, ds an element of arc,

$$w = m \cdot \frac{ds}{dx} = m\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

the equation then becomes, if $H = m \cdot c$,

$$\frac{d^2y}{dx^2} = \frac{1}{c}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Divide by the right-hand member, multiply by dy/dx, and integrate, then

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{y}{c} \text{ or } \frac{d^2y}{dx^2} = \frac{y}{c^2},$$

an equation which, by integration and a proper determination of the constants, gives for the form of the curve

$$y = \frac{c}{2}\left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}\right).$$

(2) *The Catenary of Uniform Strength.* Here, if T be the tension of the chain at P,

$$T = m\lambda = w\lambda\,\frac{dx}{ds},$$

where λ is the length equivalent to the stress (p. 80),

$$\therefore \frac{d^2y}{dx^2} = \frac{T}{H\lambda} \cdot \frac{ds}{dx} = \frac{1}{\lambda} \cdot \left(\frac{ds}{dx}\right)^2.$$

Integrating by the same process as before we find

$$y = \lambda \log_\epsilon \sec\frac{x}{\lambda}$$

as the equation to the curve.

In ordinary cases there is very little difference between the catenary and the parabola, and these curves therefore are not of much interest.

If the form of an arch be not such as corresponds to the distribution of the load on it, a horizontal force will be necessary for equilibrium, and the investigation of the magnitude of this force is a problem of wider application. Let p be the intensity of this force per unit of length of a vertical ordinate, then H is no longer constant, but is given by

$$\frac{dH}{dy} = p, \text{ also } \frac{dP}{dx} = w \text{ and } \frac{dy}{dx} = \frac{P}{H},$$

three equations from which p can be found for any distribution of load and form of arch. This is the general problem of the linear arch. For examples see A.M., p. 199. If p = const., w = const., we obtain the ellipse as the proper form of arch to sustain the pressure of a great depth of earth.

Page 30. On reciprocal diagrams of forces in general the reader is referred to a memoir by **Clerk Maxwell** in the transactions of the Royal Society of Edinburgh for 1870.

The notation used in the text was suggested by **Henrici** in the course of a discussion on a paper by **Crofton** read before the Mathematical Society in 1871. The figure in the text was drawn at the time by the writer to illustrate the method. The notation was afterwards given by Bow in the treatise referred to.

Page 33. It is convenient to have a general term for the tendency to separate into parts due to the action of external forces on a structure or part of a structure. The term "straining action" used in the text is taken from Ch. II., Part III., of a treatise on *Shipbuilding* (London, 1866), edited and in great part written by **Rankine**. By some writers this tendency to separate would be called "stress," and for a simple thrust or pull there is no objection to doing so (A.M., p. 132). In more complex cases a separate word is preferable, as the conception is very different. (Comp. p. 271.)

Page 44. In some of his engines, before the introduction of cast iron, **Watt** employed a timber beam trussed with iron rods, forming a Warren girder in two divisions with diagonals inclined at about 30° to the horizontal. This is perhaps the earliest example of such a construction. (**Robison**, Vol. II., p. 14.)

Page 52. See Plate VIII., p. 399.

Page 56. The method here detailed is given by **Rankine** in his work on Civil Engineering (p. 242), who ascribes it to **Latham**. If M be the bending moment, F the shearing force, w the load per foot-run, we have the equations

$$\frac{d^2M}{dx^2}=\frac{dF}{dx}=w,$$

which are the symbolical expression of the method. They may be used to find by integration the bending moment and shearing force at any section due to a given load, the constants of integration being found by considering that the bending moment is zero at two points, which must be known if the problem is determinate. (See Art. 38, p. 76.)

Page 64. See Ch. II., Part III. of the work on *Shipbuilding*, cited above.

Page 70. The properties of funicular polygons were first thoroughly investigated by **Culmann**, who based upon them a complete system of graphical calculation. In the semi-graphical methods employed in this treatise the integral calculus, trigonometry, and even, to a great extent, algebra, are replaced by geometrical constructions, but arithmetic is still used, and certain steps of the various processes are conducted by numerical calculations. For example, in Ch. II., the supporting forces of a loaded beam are found by the ordinary process of taking moments. In the modern purely graphical methods every step is taken graphically, whatever the calculation be. For example, the displacement of a vessel at a given draught, or her stability at a given angle at heel, would be found without the use of arithmetic.

The pressure of other matter, and the amount of illustration required, have prevented the writer from giving any account of these methods in this treatise. At present they can hardly be considered suitable for an elementary work, though, if graphical calculations were introduced into our schools, the case might be different. A full account of them will be found in **Chalmers**'s treatise referred to in the text (p. 73). Readers of German or Italian will find the principles of "graphical arithmetic" clearly explained in a small compass in a treatise the German translation of which is entitled *Elemente des Graphischen Calculs*, von Dr. L. CREMONA. Leipzig. 1875.

PAGE 64. The property of the funicular polygon expressed by the equation $Hy = M$ follows immediately by comparing the equations

$$\frac{d^2y}{dx} = \frac{w}{H}; \quad \frac{d^2M}{dx^2} = w,$$

of which one gives the form of the polygon for a given load, and the other the bending moment due to the same load.

Another fundamental property is that any two sides of the polygon must meet on the line of action of the load on that part of the polygon which lies between the two sides. When the load is vertical and represented by a curve, as in Fig. 36*a*, p. 61, this is equivalent to saying, that any two tangents to the curve of moments must intersect on the vertical through the centre of gravity of the area of the curve of loads between the corresponding ordinates. (See p. 303.)

The funicular polygon, considered as a line of transmission of stress, will be again referred to in the notes to Ch. XVII.

PAGE 78. A full account of the Forth Bridge will be found in a paper by BAKER, published in the Report of the British Association for 1882.

PAGE 79. The theory of *linear* arches is merely an introduction to the theory of arches in general. Arches are of two kinds—(1) the stone or brick arch (2) the metallic arch. In either case the theorem of the text is of equal importance. In a blockwork arch the linear arch corresponding to the load shows the direction and position of the resultant of the mutual action between the blocks, and must therefore (p. 310) fall within the middle third of the arch ring. (A.M., p. 258.)

PAGE 81. See CLERK MAXWELL'S memoir referred to above (p. 508).

PAGE 85. For the effects of changes of temperature, see p. 514.

PAGE 87. One of the most remarkable suspension bridges which have been constructed is the East River Bridge at New York, opened in May, 1883. The principal opening of this bridge is 1,600 feet span, the platform 85 feet wide, and 135 feet above the water. Cables, four in number, each of 145 square inches *net* area, constructed of 19 steel wire ropes, each containing 278 wires. Estimated strength of wire, 170,000 lbs. per square inch.

II.—KINEMATICS OF MACHINES.

PAGE 93. Referring to Figs. 1, 2, Plate II., p. 111, it seems clear that the sector pair *CD*, Fig. 1, differs kinematically much more from the turning pair *BA* than it does from the sliding pair *CD* of Fig. 2. The writer, therefore, would have been disposed to classify the three lower pairs as the "oscillating pair," the "turning pair," and the "screw pair." This, however, would have probably involved more considerable alterations in REULEAUX'S nomenclature than would have been justified in a general elementary treatise.

PAGE 95. The three incomplete lower pairs are considered by **REULEAUX** as higher pairs. The writer here follows **GRASHOF** (Theoretische Maschinen-Lehre, Band II.)

PAGE 100. Diagrams of velocity are considered generally by **CLERK MAXWELL** (*Matter and Motion*, p. 28). The application to mechanism is, so far as the writer is aware, new.

PAGE 100. The construction of curves of position and velocity of a piston has, for many years past, formed a regular part of the course of instruction at Greenwich, and formerly at South Kensington.

PAGE 104. In Owen's air compressor two such mechanisms (Fig. 50) are placed face to face with the guide A and block D common, a steam piston is connected with d and the air-pump piston with the corresponding point d of the other mechanism. The result is that a uniform pressure of steam compresses the air.

PAGE 108. Stannah's pump has been introduced since the publication of **REULEAUX**'s work. The example there given is a mechanism used in the polishing of specula.

PAGE 111. The double-slider mechanism, with sliding pairs and turning pairs alternating, is common in collections of mechanisms, but is not often found in practice. It is omitted in **REULEAUX**'s enumeration. The example given (Rapson's slide) and Stannah's pump, were pointed out to the writer by Mr. Hearson.

PAGE 157. The propositions relating to centrodes have long been known and are, perhaps, stated as clearly by **BELANGER** in his excellent treatise on kinematics (*Traité de Kinématique*, Paris, 1864) as by **REULEAUX** himself. In the author's opinion it is the conception of a kinematic chain which constitutes **REULEAUX**'s great contribution to the theory of mechanism. It is virtually a complete reconstruction of the whole theory of machines, while the centrodes are only a method of stating results which was already known. The kinematic formulæ employed by **REULEAUX** to indicate the component elements of a mechanism, in the same manner as a chemical formula shows the composition of a substance, may be regarded as indispensable, if it be attempted to proceed with the study of descriptive mechanism.

PAGE 166. The author has ventured on the introduction of the terms "driving pair," "working pair." They are simply the natural adaptation of the well-known phrases "driving point" and "working point" to **REULEAUX**'s theory.

PAGE 169. The term "multiple chains" has also been introduced by the author.

III.—DYNAMICS OF MACHINES.

The impossibility of a perpetual motion and the practical application of the principle of work were well understood by **SMEATON** and others of our great engineers of the last century. Smeaton's papers, read before the Royal Society in 1759–82, were long regarded as an engineering text-book by his successors. The language in which their ideas are expressed, however, are not regarded as consistent with NEWTON's teaching, and this circumstance perhaps concealed the real importance of the ideas themselves. At any rate, although the term "energy" was proposed by YOUNG, no considerable use was made of them by students of mechanical science until the publication by PONCELET, in 1829, of the *Introduction à la Mécanique Industrielle*, a work which has had a great influence on the study of mechanics. The third edition of this work (Paris, 1870) published after PONCELET's death, will be quoted by the abbreviation *Méc. Ind.* Poncelet's methods were explained, and considerable additions made to the theory of machines, by MOSELEY in his *Mechanical Principles of Engineering* (London, 1843).

PAGE 178. This method was probably employed for the first time by WATT in his expansion diagram. See ROBISON's *Mechanical Philosophy*, Vol. II. It is given by PONCELET (*Méc. Ind.*, p. 66).

PAGE 180. The terms "statical" stability, "dynamical" stability, in relation to vessels were introduced by MOSELEY (*Phil. Trans.*, 1850). They have been criticized by REYNOLDS, perhaps not without justice, but are too firmly rooted to be displaced.

PAGE 181. "Force is an action between two bodies, either causing or tending to cause, change in their relative rest or motion." (A.M., p. 15.) The distinction between internal work and external work is due to PONCELET (*Méc. Ind.*, p. 30).

PAGE 183. The language in which writers on mechanics have expressed the distinctive character of frictional resistances has been severely criticized by REULEAUX in the notes to his work on the Kinematics of Machines (*Kennedy's translation*, p. 595). The author by no means supposes that he can escape this universal censure, for the difficulty of expressing abstract principles in a form to which no objection can be made is almost insuperable. As, however, CLERK MAXWELL remarks in reference to a different question, the language in which a truth may be expressed is less important than the truth itself. Friction always causes energy to disappear, and is never a source of mechanical energy except indirectly through the agency of thermal energy. In mechanics this is a distinction of fundamental importance and justifies, in the author's opinion, the use of such phrases as "loss of energy."

The extension of the term "reversible" from a machine to the resistances which are overcome by the machine has been ventured on, though with some hesitation. The old term "active" can hardly be considered suitable.

PAGE 183. "Envisagé sous ce point de vue, le principe de la transmission du travail comprend implicitement toutes les lois de l'action réciproque des forces, sous un énoncé qui en facilite infinitement les applications à la Mécanique industrielle, qu'on pourrait nommer la *Science du travail* des forces. Dès les premier pas des jeunes éléves dans l'étude, cet énonce, en effet, se présente à eux comme une sorte d'axiome évident par lui-même, et donc la demonstration leur semble superflue aussitôt qu'ils ont bien saisi ce qu'on entend par *travail mécanique*, et qu'il leur est clairement démontre que ce travail, reduit en unités d'une certaine espèce est dans les arts, l'expression vraie de l'activité des forces" (*Méc. Ind.*, p. 3). This passage from PONCELET is quoted to show how clearly it was seen, even before the discovery of the conservation of energy in its complete form, that the principle of work ought to be regarded as fundamental, and not merely as a deduction from certain equations.

PAGE 184. The modifications made here in the old statement of the principle of work, as applied to machines, are necessary consequences of REULEAUX's conception of a kinematic chain.

PAGE 188. The value 33,000 ft.-lbs. per minute was derived by BOULTON and WATT from experiments on the work done by the powerful dray horses employed in London breweries.

PAGE 190. To avoid misapprehension, it may be here stated that in this, as much as in the preceding section, the object is to explain and to verify, from results given in elementary treatises on dynamics, the principle of work : not in any sense to demonstrate it.

PAGE 192. Except in the use of the word "kinetic" instead of "actual," the statement here is in the form given by RANKINE (A. M., p. 500). The author is entirely of (the late) Mr. W. R. Browne's opinion that this is the best form, and has always used it himself. The idea of energy being stored in a body in motion perhaps first appears clearly in MOSELEY's treatise.

PAGE 202. The construction by means of which curves of crank effort are drawn was given by PONCELET, but it does not appear that any such curves were actually drawn until they were given by ARMENGAUD in his treatise on the Steam Engine. In Fig. 97, to save room, the curves are placed half above and half below the base, but otherwise the figure is that of ARMENGAUD ; it is far the most convenient form for applications.

PAGE 209. The stress due to centrifugal action on the rim of a wheel is given by a formula (p. 264) which may be written in the simple form $V^2=g\lambda$, where λ is the length due to the stress (p. 80). A velocity of 80 ft. per second gives a length of only 200 feet, or about one-fifth of the stress cast iron would safely bear in tension. The inequality of distribution produced by inextensible arms tying together opposite points on the rim of the wheel probably increases the maximum stress about 50 per cent. ; but the principal reason for the low limit required for safety is the alternate bending backwards and forwards of the arms as energy is alternately stored and restored by the wheel. The speed is occasionally increased to 100 feet per second. The author is indebted to Prof. Unwin for the information that when the wheel is in segments the speed should be limited to 40 feet per second.

PAGE 211. The method here given occurred to the author many years back ; but it has (we believe) been published in *Engineering*.

PAGE 212. On the effect of inertia of reciprocating parts in high speed engines the reader is referred to *Ueber Dampfmaschinen mit hoher Kolbenge-schwindigkeit* von J. F. RADINGER. Wien, 1872.

PAGE 220. In the original Brotherhood engine (*Vienna Exhibition*, 1873) the steam was admitted to the central chamber and exhausted at the outer ends of the cylinders : but in recent examples the central chamber communicates with the exhaust.

PAGE 224. The Friction Circle was defined and its use explained by RANKINE in his treatise on *Millwork and Machinery*, p. 428.

PAGE 233. Since the text was written, a *Report of the Committee on Friction* of the Institution of Mechanical Engineers has been published. (See *Engineering*, Nov. 16th, 1883.) From these experiments it appears that the mode of lubrication has a great influence on the results, and that when the lubrication is perfect the moment of friction is nearly constant whatever the pressure, as in the surface friction of a fluid. The friction here, no doubt, is chiefly due to the viscosity of the lubricant. The co-efficient was sometimes as low as 1/1000th.

In the discussion of this report (*Engineering*, Feb. 1st, 1884) the reporter (Mr. Tower) stated that in locomotive axles 300 lbs. per sq. inch was the limiting pressure, whereas in the crank pins the limit was as high as 1,000 lbs. per sq. inch, the difference being that the load on the pins is alternating whereas that on the axle is constant.

PAGE 250. The distinction between internal and external kinetic energy is pointed out by RANKINE (A.M., p. 508).

PAGE 256. On Governors in general the reader is referred to a paper by CLERK MAXWELL in the *Proceedings of the Royal Society*, No. 100, 1868. A full account of the principles of construction of centrifugal regulators will be found in *Theoretische Maschinenlehre*, Band III. Leipzig. 1879, von Dr. F. GRASHOF.

PAGE 263. The utility of balance weights, sufficiently heavy to neutralize completely the horizontal forces, is by no means universally admitted. The vertical forces introduced are very great (Ex. 17, p. 267), and, should they synchronize (p. 348) with the period of vertical oscillation of the engine on its springs, most dangerous results might follow.

PAGE 264. The stress due to rotation is best expressed by the formula $V^2 = g\lambda$—given above.

IV.—STIFFNESS AND STRENGTH.

PAGE 273. If the temperature of a bar be raised t°, the corresponding change of linear dimensions (strain) is given by the formula

$$e = \frac{t}{K},$$

where K is roughly approximately constant for the same material.

If a change of length be forcibly prevented during the change of temperature, the stress

$$p = \frac{E}{K}t$$

will be produced. The change of temperature corresponding to 1 ton per square inch is K/E. This quantity in degrees Fahrenheit and the value of K are given for various materials in the annexed table:—

Material.	Value of K.	Value of K/E.
Wrought Iron,	147,500	$11^\circ{\cdot}3$
Cast Iron, - -	162,000	20°
Copper, - - -	104,500	16°
Brass, - - -	95,400	25°

PAGE 307. If an elastic solid or, more generally, a set of connected pieces of perfectly elastic material, be under the action of any number of forces P_1, P_2, ..., and any number of couples M_1, M_2, ..., in equilibrium, the value of U must be

$$U=\tfrac{1}{2}\Sigma Px+\tfrac{1}{2}\Sigma Mi,$$

where x_1, x_2, ..., are the displacements of the points of application of the forces and i_1, i_2, ..., the angular displacements of the arms of the couples. For if the forces gradually increase from zero, always remaining distributed in the same way, each part of the load (P) will exert the energy $\frac{1}{2}Px$, since the space moved through (x) must clearly be proportional to P. The same argument applies *mutatis mutandis* to couples. Hence the whole energy exerted must be given by the above formula, and this is always represented by the energy stored up in the system when the parts are perfectly elastic.

Now, imagine the solid immoveably fixed at three or more points, and let one of the forces P_1 be increased by a small quantity δP_1, all the other forces retaining their original magnitudes. The effect of this is that the points of application of all the forces move through certain small spaces (δx), and the arms of all the couples through certain small angles (δi). The total additional work done will be

$$\delta U=\Sigma P\delta x+\Sigma M\delta i.$$

But, on differentiating the value of U on the supposition that P_1 alone varies, we find

$$2\,.\,\delta U=\Sigma P\delta x+\Sigma M\delta i+x_1\,.\,\delta P_1,$$

and therefore by substitution

$$\delta U=x_1\,.\,\delta P_1.$$

A similar equation is derived by supposing one of the couples to vary, and we obtain the general equations

$$\frac{dU}{dP}=x;\ \frac{dU}{dM}=i,$$

that is, the displacements are the partial differential co-efficients of U with respect to the forces.

The forces to be considered are partly weights or other loads of known magnitude, and partly arise from the stress between the bounding surfaces (real or ideal) of the solid and external bodies. The boundary forces must be consistent with statical equilibrium, but subject to this condition are determined by equations found by differentiating the function U. In particular, when the bounding surface is fixed, the partial differential co-efficients of U with respect to the corresponding forces must be zero. The value of U is then in most cases (perhaps always) a minimum, as stated in the text.

It appears then that whenever the elastic potential can be found and expressed in terms of the external and boundary forces acting on the system, the necessary equations for determining the boundary forces and the deflection produced by the external forces can all be found by differentiation of U and by the conditions of statical equilibrium. As an example, take the case of a beam loaded in any way and fixed at the ends. Let the beam be AB (Fig. 28), and let the notation be as on pages 40, 41, then (p. 306)

$$U=\int\frac{M^2}{2EI}dx.$$

Substitute for M by the formula on page 41, and integrate between the limits l and o, we find

$$2EI\,.\,U=\tfrac{1}{3}(M_A{}^2+M_AM_B+M_B{}^2)l+\int_0^l m^2dx+\frac{2M_A}{l}\,.\int_0^l m(l-x)dx$$
$$+\frac{2M_B}{l}\,.\int_0^l mxdx.$$

The integrals are most conveniently expressed in terms of, S the area of the curve of moments (m), z the distance of its centre of gravity from A, and, $\bar{y}$ the distance of its centre of gravity from AB. The formula then becomes, dividing by 2,

$$EI\,.\,U=\tfrac{1}{6}(M_A{}^2+M_AM_B+M_B{}^2)l+S\frac{M_A(l-z)+M_Bz}{l^2}+S\bar{y}.$$

The potential is thus expressed in terms of the load on the beam and the bending moments at its ends. The latter may have any values we please consistently with statical equilibrium, and the partial differential co-efficients of U with respect to $M_A\,M_B$ will be the slopes at the ends. In particular, if the ends are fixed horizontally.

$$2M_A+M_B+6\frac{l-z}{l^2}\,.\,S=0,$$

$$2M_B+M_A+6\frac{z}{l^2}\,.\,S=0,$$

equations which determine $M_A\,M_B$ and express that the function U is then a minimum. In the particular case of a symmetrical load

$$M_A=M_B=-\frac{S}{l}$$

The value given on page 301 for the particular case of a uniform load will be found to agree with this result.

The potential for a continuous beam may be immediately deduced, by addition of the potentials for each span taken separately, in terms of the bending moments at the points of support. The theorem of three moments (p. 306) for the case of supports on the same level, then follows at once by differentiating with respect to the moment at the middle point of support.

In all cases, differentiation of U with respect to any portion of the external load will give the deflection at the point where that load is applied.

In applying this method care must be taken that the supporting forces, in terms of which the potential is expressed, are independent: if they are not, then the equations of statical equilibrium will be conditions subject to which U will be a minimum. To take a simple example, suppose a perfectly rigid four-legged table standing on four similar elastic supports and loaded in any way, then

$$U=n(P_1{}^2+P_2{}^2+P_3{}^2+P_4{}^2),$$

where P_1, P_2, P_3, P_4, are the part of the whole load resting on each leg, and n is some multiplier. Here the forces P are partly determined by three statical equations for equilibrium of the table, and only one additional equation is found by making U a minimum.

This method was explained and applied to a number of examples in some papers by the author which appeared in the *Philosophical Magazine* for 1865; the demonstrations there given, however, were insufficient. The author at that time supposed it to be new, but it had already been given in a memoir by M. E. F. MÉNABRÉA, *Comptes Rendus*, vol. xlvi. (1858), page 1056.

PAGE 313. The lateral disturbance is here supposed small. With a larger disturbance the pillar would return even if the value of W were equal to $2EI/l^2$, and with a greater value would bend over into a position of equilibrium given by the formula

$$W=\left(\frac{\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\right)^2 \cdot \frac{2EI}{l^2},$$

where θ is the angle subtended by the circular arc into which the pillar is bent.

PAGE 314. Referring to Fig. 133, p. 310, let O be the origin of rectangular coordinates, ON axis of x, and y the horizontal ordinate of any point in the curve OA. Then the bending moment at that point is, very approximately,

$$M= W(a+\delta-y).$$

The differential equation of bending is therefore (p. 298)

$$\frac{d^2y}{dx^2}=\frac{W\,(a+\delta-y)}{EI}.$$

For brevity it is convenient to write

$$W=m^2 \,.\, EI,$$

where m for a uniform transverse section is constant. The equation then becomes

$$\frac{d^2y}{dx^2}+m^2 \,.\, y=m^2(a+\delta).$$

The integral of this equation is well known to be

$$y=a+\delta+A \,.\, \cos mx+B \,.\, \sin mx,$$

where A and B are the constants of integration. For the case shown in the figure $y=0$, $dy/dx=0$, when $x=0$ and the equation becomes

$$y=(a+\delta)\;(1-\cos mx).$$

If we now put $x=l$ we have $y=\delta$, and therefore

$$a+\delta=\frac{a}{\cos ml},$$

an equation which determines a definite value of δ, the lateral deviation of the summit of the pillar, unless $a=0$, when we must have $\cos ml=0$, that is

$$W=\frac{\pi^2}{4} \cdot \frac{EI}{l^2}.$$

This is the equation given in the text; and for the pillar flat or rounded at both ends it applies with the modification there explained.

If, however, the pillar be fixed at one end while the other is constrained to lie in the same vertical, as will be the case when one end is flat, and the other rounded, the equation requires modification. We must now suppose a horizontal force P at the top of the pillar to prevent lateral deviation, and the bending moment is then, since $a=0$, $\delta=0$,

$$+M= +P(l-x)-Wy.$$

Making the same substitution as before, we find

$$\frac{d^2y}{dx^2}+m^2 . y=m^2\frac{P}{W}(l-x),$$

the integral of which equation is

$$y=\frac{P}{W}(l-x)+A \,.\, \cos mx+B \,.\, \sin mx.$$

To determine the constants we have $y=0$ when $x=0$ and when $x=l$; also $dy/dx=0$ when $x=0$, therefore

$$A=-\frac{P}{W}l;\ mB=\frac{P}{W};\ A\cos ml+B.\sin ml=0,$$

from which we find that P is zero unless

$$\tan ml=+ml,$$

a transcendental equation which, being solved by trial, gives $ml=4{\cdot}493$, that is

$$W=2{\cdot}047\ \pi^2\frac{EI}{l^2}.$$

It is sufficiently approximate to replace 2·047 by 2, as is done in the text, and this value is much more approximate than the value 16/9 employed by RANKINE (*Useful Rules and Tables*, p. 220), which was obtained by supposing the summit of the pillar to deviate laterally until it is in the same vertical as the point of contrary flexure.

PAGES 315 and 316. When the pillar is absolutely straight and homogeneous and of uniform transverse section, the lateral deflection due to an actual deviation a is given by the formula

$$a+\delta=\frac{a}{\cos ml}=a.\sec\frac{\pi}{2}\sqrt{\frac{p}{p_0}},$$

and the formula on p. 316 for the effect of deviation becomes

$$\left(\frac{f}{p}-1\right)\cos\frac{\pi}{2}\sqrt{\frac{p}{p_0}}=\frac{qa}{nh}.$$

In any actual example, however, this formula would not be exact any more than that given in the text. Each particular example will have its own formula. The result of all such formulas, however, must be nearly the same for a small deviation. Further, a great proportional change in the deviation, always supposing it small, produces little change in the crushing load, and this probably explains why experiment gives tolerably definite values of the crushing load although its precise amount must depend on accidental circumstances.

PAGE 318. It is worth noticing that Gordon's formula with the theoretical values of the constants may be written

$$\frac{1}{p}=\frac{1}{f}+\frac{1}{p_0}.$$

On substituting this value of p in the deviation formula we get

$$\frac{qa}{nh}=\frac{f}{f+p_0}.$$

PAGE 328. The formulæ given in different books for the moment of resistance of a shaft of rectangular section exhibit considerable discrepancies. COULOMB, to whom the formula for a circular section is due, supposed that in every case

$$T=f.\frac{I}{r_1},$$

where I is the polar moment of inertia and r_1 is the outside radius. In a rectangular section of sides a and b this gives

$$T=\tfrac{1}{6}\,fab\sqrt{a^2+b^2},$$

which for a square section of side h becomes

$$T={\cdot}2357\,f.\,h^3.$$

If these results were correct it would appear that a shaft of given sectional area was stronger the more unequal the sides were, a result quite contrary to experience. In a memoir on torsion published in the *Mémoires de l'Institut* for 1856, Barré de Saint Venant investigated the question thoroughly, and showed that the exact formula for a square section is

$$T = 1{\cdot}6653 f . \left(\frac{h}{2}\right)^3 = {\cdot}2082 f h^3,$$

giving about 88 per cent. of the preceding result. On comparing this with the moment of resistance of a circular section of equal area we find the ratio to be ·738, as stated in the text. The general result for a rectangular section with sides in any proportion can be expressed only as an infinite series involving a separate calculation for each case. The calculation, however, may be replaced approximately by an empirical formula: that given in the text is not exactly the same as that used by Saint Venant, but leads to nearly the same result. The result given in the text for an elliptic section is exact. (See also *Additional Notes*, page 531.)

Rankine (A.M., p. 358) gives $\cdot281 fh^3$ as the result of Saint Venant's calculations without further explanation. This value is greater than that given by Coulomb's hypothesis, and is certainly too large.

Page 356. A line of stress may be regarded as the geometrical axis of a curved rod which is in tension or compression, as the case may be, under the action of a load perpendicular to itself. The whole solid, therefore, may be conceived as made up of a set of rods, each of which is a rope or linear arch in equilibrium under a transverse load. Each rod transmits stress in the direction of its length. If there be no lateral stress the rods are straight, but otherwise they are curved. In a framework structure loaded at the joints, the bars of the frame may be regarded as lines of stress except at the joints where those lines assume complex forms. The tendency of modern science is to regard all force as being due to the transmission of stress through a medium of some kind, even in such cases as that of gravity, where no medium perceptible to our senses exists. All forces on this conception are represented by a system of lines of stress.

Page 362. The theory of elastic solids has been much more fully treated with reference to practical application by Grashof, Saint Venant, and other continental writers than in any English treatise. The author is chiefly indebted to Grashof's work, *Die Festigkeits Lehre* (Berlin, 1866), a new edition of which has (we believe) recently appeared. An attempt has been made to distinguish clearly between those parts of the subject which are necessarily true either exactly or to a degree of approximation which is capable of being exactly calculated, and those parts which depend on hypotheses more or less probable. The first are placed in the present chapter; the second in the chapter which follows.

Page 363. Attempts have been made to prove by theoretical reasoning that, in a perfectly elastic isotropic material, the value of m is necessarily 4, and the demonstration is still considered valid by some authorities. It is, however, more probable that such reasoning simply shows that matter is not constituted in the way supposed in the demonstration. It is difficult to obtain material which is really perfectly isotropic, but all the experimental evidence at present goes to show that m may have various values from 2 to infinity.

PAGE 364. Some other points in the theory of bending may here be noticed :—

(1) The effect of curvature is that a lateral stress p' must exist on the longitudinal layers given by the same equation as is used for thick hollow cylinders under internal fluid pressure (p. 368), viz.,

$$\frac{d}{dr}(p'r) = p.$$

Replacing r by $R+y$, and p by the value given in the text, we find

$$\frac{d}{dy}(p'r) = \frac{Ey}{R},$$

and therefore, by integration,

$$p'r = \frac{Ey^2}{2R} + \text{constant}.$$

Since p' is zero at the outer surface where y is $\pm\frac{1}{2}h$,

$$p' = \frac{E}{8R^2}(4y^2 - h^2) = p_1 \cdot \frac{4y^2 - h^2}{4Rh},$$

where p_1 is the stress due to the bending at the outer surface, and r is replaced by its mean value R. At the neutral surface p^1 is greatest, but even there has only the very small value

$$-p' = p_1 \cdot \frac{h}{4R}.$$

This lateral stress is therefore never great enough to have any perceptible influence on the elasticity of the layers.

(2) It has been stated on page 279 for the case of tension, page 286 for the case of bending, and page 328 for the case of torsion, that the stress on any transverse section is the same, however the straining forces are applied to a bar, provided only that their resultant be given in magnitude and position. This may be regarded as a general principle applicable in all cases. Any other distribution of stress produced on a transverse section by friction or other external forces applied directly to it will change with great rapidity on passing to transverse sections not directly exposed to such forces. It is, however, generally necessary to provide additional strength at these exceptional sections.

(3) If the beam be fixed at one end and loaded at the other, there will be shearing as well as bending on all sections. Transverse sections are then no longer plane, but are distorted by the action of the shearing force and the distribution of stress is not given exactly by the formulæ on pages 285, 333. Further, the maximum stress on the parts of the beam does not depend on the bending alone, but also on the shearing, as shown on page 357. It is, however, certain that the approximation is very close, even in short pieces, where the shearing is relatively great. The only exception is in beams of I section, in which the web is very thin, as explained in the passage cited. In practice, however, additional strength is always required in the web on account of the necessary stiffening (p. 288).

The various approximations here referred to have been all examined very thoroughly, especially by SAINT VENANT, and there is no doubt that the formulae are substantially exact within the limit of elasticity.

PAGE 366. The lines of stress for a thick hollow cylinder under internal fluid pressure, and also under the action of tangential stress applied as in Ex. 6, p. 358, will be found to be equiangular spirals, the angle of the spiral depending on the portion between the fluid stress and the tangential stress.

The verification given in the text is necessary because, otherwise, we could not be sure that the assumptions on page 367 were consistent with one another. This is very well shown by supposing the cylinder to rotate and obtaining a solution of the problem when thus modified, assuming the cylinder to remain cylindrical and employing the equation of verification. It will be found that the solution thus obtained can only be true if the stress on the transverse section varies according to a certain law. If the cylinder is long it appears that this must really be the case except very near the ends. The problem of a swiftly rotating circular saw appears not as yet to have been attempted; it is found by experience that a saw to run at high speed must be hammered so as to be "tight" at the periphery. The same difficulty occurs if the material of the cylinder be not isotropic.

Page 369. Article 210 belongs more properly to the next chapter and is referred to below.

It may be useful to add the general formula for the elastic potential in the case of stress in two dimensions. Let a cube (side unity) be subject to the normal stresses p, q, o, on its faces, then, if the strains be e_1, e_2,

$$U=\tfrac{1}{2}pe_1+\tfrac{1}{2}qe_2.$$

Substituting from the formula on page 362

$$U=\frac{1}{2E}\left\{p^2+q^2-\frac{2pq}{m}\right\},$$

this gives the elastic potential per unit of volume. When p and q vary as in the cylinder under internal pressure the volume of each element must be multiplied by this expression and an integration performed. This calculation is simple in the case of the cylinder.

Page 373. Tresca's experiments are described in detail, with a great variety of interesting illustrations in a series of memoirs which have been separately published (*Mémoires sur l'Ecoulement des Corps Solides*). The example in the text is taken from the second memoir (Paris, 1869). It is to be remarked that the influence of time was not taken into account. It is, however, believed that in the softer metals a much greater amount of work must be done in producing a permanent deformation, when that deformation takes place quickly, than when the slow action of hydraulic pressure is employed. Such bodies probably behave like fluids of very great viscosity.

Page 374. The more refined methods employed in the mechanical laboratories recently fitted up show that, when a bar is stretched a second time, the elasticity is sensibly perfect up to a certain limit, and then becomes sensibly imperfect up to a second limit, where drawing out commences. Further, if the testing machine be constructed in such a way as to permit the stretching force to diminish after a maximum value has been reached, drawing out will continue much longer before rupture takes place, as in the case of a stick of hot sealing wax. (See *Additional Notes*, page 533.)

Page 379. The modulus of elasticity in compression is found to be less than that in tension in cast iron as well as wrought iron in about the same ratio. This circumstance, together with the equality of the moduli for bending and tension, leads us to conjecture that the effect is due to lateral bending which cannot be wholly prevented by the trough. By some experimentalists no such difference has been found.

Page 380. The argument of Art. 217 applies equally to any case where stress is not uniformly distributed. In the hydraulic press cylinder the stress is never reversed, and the increase of strength is probably reliable.

Page 386. To test experimentally the truth of the various theories of compound strength the tensile strength of a bar should be determined, the ends of which project from a cylinder under hydraulic pressure of a given amount.

Page 389. It is much to be regretted that the resistance of cast iron to alternate bending has not been determined. Such an experiment would settle a point of the greatest importance in strength of materials, namely, whether the tenacity of a material can be increased by lateral connection. In wrought iron and steel, it would appear that it is not, the increase of apparent strength being accounted for by the raising of the elastic limit (p. 377).

Page 391. No formula of this kind is anything more than a formula of interpolation supplying the place of missing experiments. The author is led to make this remark by the elaborate manner in which such formulæ are discussed by some writers. The study of Wohler's original memoir cannot be too strongly recommended to those interested in the subject.

V.—HYDRAULICS AND PNEUMATICS.

Page 405. The value of w for pure water is greatest at 39°, and is then 62·425, while at 100° it diminishes to 62. At temperatures above 75°, 62 is more approximate than 62½, but on the other hand water is seldom entirely free from solid matter, which increases its density.

Page 409. The standard experiments on the co-efficients of velocity and contraction in the case of orifices are those made by Weisbach, and described by him in his treatise *Die Experimental Hydraulik* (Freiberg, 1855), to which the reader is referred for details. A short pipe projecting inwards is known as Borda's mouthpiece. The theoretical minimum value of the co-efficient of contraction (·5, see p. 433) is closely approached when the pipe is very thin and sharp-edged; otherwise the value is somewhat larger, say about ·55. The only case in which a co-efficient of contraction has been found theoretically is that of a long narrow slit, for which Rayleigh has obtained the value ·611. (See *Additional Notes*, page 536.)

Page 413. The use of the term "head" for the energy per unit of weight of a fluid is not free from inconvenience—the two things not being identical unless the datum level be at the surface of the fluid.

Page 415. In the flow of rivers it is well known that it is the outer side of a bend, not the inner, which suffers erosion, so that the windings of the river have a constant tendency to increase in extent. The reason of this has been explained by Thomson to be that the layers of water in contact with the bottom are greatly retarded, and hence have less centrifugal force than the upper layers. The excess pressure at the outer side of the bend is therefore partially unbalanced below, and an inward flow takes place, carrying material with it from the outer side to the inner. This was verified by experiment. (*B.A. Report* for 1876, p. 31.)

PAGE 415. The velocity of the water in any one of the ideal pipes is inversely proportional to the sectional area of the pipe. Now the form of the pipes depends solely on the form of the bounding surfaces, and it follows, therefore, that the velocities of all parts of the stream bear a fixed proportion to each other, depending only on the nature of the bounding surfaces. In the language of the theory of mechanism, the fluid forms a closed kinematic chain. The chain is closed by the pressure of the bounding surfaces, and when the velocity exceeds a certain limit the chain opens. Energy can then no longer be transmitted uniformly to all parts of the fluid, and is no longer uniformly distributed. When energy is unequally distributed, eddies are formed.

PAGE 417. That hydraulic resistances of all kinds are independent of the pressure is one of the best established laws of experimental mechanics, but how far this may be true at very high pressures is, of course, uncertain. In some books it is stated as confidently as if it were an observed fact that the friction of the skin of a vessel near the keel is greater than that near the surface on account of increased pressure, but there is no foundation for this assertion.

To the three normal laws given in the text may, in all probability, be added a fourth :—

(4) Friction is proportional to the density of the fluid.

The grounds for this statement will be seen on reference to pages 430, 499. It amounts to saying that friction is a kind of eddy resistance. If we assume this, the laws of friction are expressed by an equation similar to that employed for eddy resistance,

$$R=f'w\,.\,S\,.\,\frac{V^2}{2g}.$$

The co-efficient of friction, f', is then distinguished from the friction per sq. ft., given in the table on p. 418, and is the same as the co-efficient for pipes.

The explanation in the text of the diminution of friction in long surfaces is that given by FROUDE in his reports on surface friction, and also by BOURGOIS in his treatise referred to farther on.

PAGE 423. The formation of eddies by the meeting of different streams and the passage of water past solid bodies is familiar to all observers of the motions of fluids, and is described in the earliest treatises on hydraulics. The way in which they absorb energy has long been understood: thus PONCELET says, "En général, la production des tourbillons est l'un des moyens dont la nature se sert pour éteindre ou, *plutôt dissimuler*, la force vive dans les changements brusques de mouvement des fluides" (*Méc. Ind.*, p. 571). The italics are the author's. The passage is too long to quote at length, but is worth studying throughout. The extent to which eddy motion may prevail throughout the mass of a fluid, often without any clear indication at the surface, was not understood till long afterwards.

The theory of simple systems of eddies has of late attracted much attention, but the extreme intricacy of the internal motions of fluids will probably long defy calculation in such cases as commonly occur in practice.

The particular case mentioned in the text (Fig. 172) is one observed by the author, in which conspicuous eddies were formed, one or two at a time, with great regularity.

PAGE 428. If the motion of water in a pipe or channel be supposed of the undisturbed kind (p. 414) and viscosity be taken into account (p. 416), it is possible to find the discharge due to a given head. In the case of tubes of very small diameter it was shown by Poiseuille that the flow actually does take place according to this law, and the co-efficient of viscosity was found. The loss of head is then proportional not to the (vel.)^2, but to the simple velocity.

In pipes of ordinary diameters through which water is flowing with ordinary velocities, the loss of head is, however, certainly, approximately as the (vel.)^2, and, moreover, Boussinesq has shown that it is enormously greater than it would be according to the law for undisturbed flow with the co-efficient deduced by Poiseuille. The inevitable conclusion is, that the loss is mainly due to the formation of eddies. In the case of large rivers it is found by experiment that the velocity diminishes as the bottom is approached according to a law represented by the ordinates of a parabola, a result which is consistent with the law of undisturbed flow. Nevertheless, in this case also, the facts cannot be explained except by supposing that the resistance is due to eddies. With fluids, the viscosity of which is so small, as is the case in water, undisturbed flow only occurs at very low velocities in very small channels.

Although these facts were tolerably well established, it is only very recently that any attempt has been made to discover the connection which must exist between the viscosity of the fluid, its velocity, and the dimensions of the channel in which it flows, in order that the flow may or may not be undisturbed. This has at length been done by Reynolds, who has succeeded in connecting by a common law Poiseuille's experiment on capillary tubes and Darcy's experiments on full-sized pipes. For particulars the reader is referred to his paper published in the *Philosophical Transactions* (1883, Part III.). It need only here be mentioned that it is shown that the loss of head in a pipe may be expressed by a formula which, when stated in a simplified form sufficient for our present purpose, becomes

$$h' = 4f \cdot \frac{l}{d^{3-n}} \cdot \frac{v^n}{2g},$$

where n is an index depending on the nature and condition of the surface. When the surface is rough $n=2$, and we get the formula already given on page 420; this is the case for an incrusted pipe, but for a clean cast-iron pipe it falls off to 1·9, and in a lead pipe is 1·723. This falling off in the index in smooth surfaces is quite analogous to that already found by Froude in his direct experiments on surface friction (p. 418). (See *Additional Notes*, p. 536.)

Page 430. The formula for eddy resistance is given in this form by Poncelet, and the reasoning in the text is essentially that employed by him (*Méc. Ind.*, p. 585). It is well suited to show the real nature of the law of hydraulic resistance (p. 408). All that is supposed in this law is, that the velocities of the particles of fluid bear a fixed proportion to each other depending solely on the form of the bounding surfaces, as is actually the case in undisturbed motion. If the bounding surfaces are of invariable form the law should be accurately verified for a fluid absolutely devoid of viscosity. The causes of irregularity are explained on page 427. A variation of 20 per cent. in the course of the same experiment was actually observed by Froude.

Page 431. Until the establishment of the Institution of Naval Architects and the labours of Froude had given so great an impulse to naval science, the resistance of ships, like other branches of the subject, had been much more thoroughly studied in France than in England. The only treatise specially devoted to it, even now, is that entitled *Mémoire sur la Resistance de l'Eau* par M. Bourgois, Paris (no date).

The earlier experiments by Beaufoy and others are all very thoroughly discussed in this work, and the reader is referred to it for the values of k in the formula for eddy resistance. These experiments were chiefly on bodies of unfair form, and this is the principal reason why so little progress was made for a long period in discovering the true principles of the resistance of ships. Experiments on the resistance of a full-sized ship made in France are described by Bourgois, and it was known that it was much less than would have been supposed in the absence of such experiments, but it was not till the discovery by Froude of the true method of comparing the resistance of a ship with that of her model that any great advance was made in the subject.

The formula employed in the text is given by Rankine, and, if a single constant only is admitted, this form is perhaps the best. The value of K, however, stated by him (*Useful Rules and Tables*, p. 274) is now known to be 50 per cent. too large, being deduced not from direct experiment but from the power required for propulsion. The limit of speed to which the formula applies is also much less than that stated in the passage cited.

If three constants be admitted the formula employed by Bourgois,

$$R=aV+bV^2+cV^4,$$

will give a much closer approximation and apply up to a much higher limit of speed. No formula, however, is true at all speeds, and the only complete way of representing the resistance of a vessel is by means of a curve derived from experiments on a model. This question is far too large to enter on here.

Page 443. In finding λ in cranes and other hoisting machines the weight raised, multiplied by the velocity ratio between it and the ram, must be included in the weight of the ram.

Page 463. The vanes of a turbine-wheel are often not radial at the circumference where the water enters. The simpler case, however, is sufficient to illustrate the principle on which such machines are designed.

Page 465. The undisturbed motion of a perfect liquid within fixed boundaries is always *reversible*, that is, if every particle of liquid were imagined to be set in motion with the same velocity in the reverse direction, the motion would continue undisturbed. But if water be set in motion from rest this will generally not be the case. If, for example, we imagine a pipe connected with a tank by a mouthpiece in the form of the *vena contracta*, then, when water flows out of the tank, it will issue in a continuous stream with small loss of head; but if the motion be reversed most of the energy of motion of the water in the pipe will be wasted in the internal motions soon after entering the tank. The loss is not unavoidable as will be seen on reference to the case of a trumpet-shaped pipe (Fig. 168, p. 414), but may be rendered small by enlarging the pipe very gradually.

Page 466. The spiral passage should not be too large and should be connected with the ascending main by a carefully formed gradual enlargement.

Page 467. This calculation is partly the same as that given by Unwin, (*Min. Proc. Inst. C. E.*, vol. 32).

Page 486 a. The author has for some time past been in the habit of distinguishing between the mechanical *equivalent* and the mechanical *value* of heat.

PAGE 486 B. In the author's treatise on the steam engine (chap. viii.) attention has been called to the distinction between the mechanical losses and the thermal losses in heat engines, and the terms "mechanical" efficiency and "thermal" efficiency are here introduced to render the distinction more clear.

PAGE 490. The value of n in the adiabatic expansion of the permanent gases is probably somewhat greater than 1·4. The difference, however, is not important, and the simpler value is therefore adopted in the text.

PAGE 491. If the fluid be supposed at rest, and elevation be taken into account, we obtain

$$K_pT+z=\text{Const.}$$

or, as it may also be written,

$$3{\cdot}5\ PV+z=\text{Const.}$$

This gives the distribution of pressure and temperature of the atmosphere for "convective equilibrium" (CLERK MAXWELL'S *Theory of Heat*, 1st edition, p. 301). Energy is then uniformly distributed.

PAGE 495. The explanation here given (Fig. 199) on the whole seems the most natural, but it very probably may not be complete.

PAGE 497. This formula for the flow of air in a long pipe is believed to have been first given by UNWIN (*Min. Proc. Inst C. E.*, vol. xliii.). It is a question of considerable practical interest. By comparison with experiment it has been shown that the co-efficient is given by a formula of the same form (DARCY'S), as in the flow of water through pipes, an important verification of theoretical principles. The equation for the case where the temperature varies can be obtained without difficulty, but has not as yet been practically applied.

APPENDIX.

B. ORGANIZATION OF THE CLASSES IN ENGINEERING AND NAVAL ARCHITECTURE IN THE ROYAL NAVAL COLLEGE.

A SCHOOL of naval architecture was founded in Portsmouth dockyard so long ago as 1810, but, after existing for more than twenty years, was abolished in 1833. In 1844 it was re-established, but only to be once more abolished in 1853. In 1860 the Institution of Naval Architects was founded, and by its influence a third school was commenced under the direction of the Science and Art Department at South Kensington. For particulars respecting the two earlier of these schools the reader is referred to a paper by SCOTT RUSSELL in the *Transactions I.N.A.* for 1863. The third was afterwards incorporated with the Royal Naval College, of which it now forms a department.

This department is divided into two classes, of which the junior serves as the final stage in the training of the engineer officers of the navy, all of whom spend nine months at Greenwich immediately on entering the service, after several years spent in the dockyard. The senior is an advanced class, consisting partly of a small number of engineer officers selected by competition from the preceding, and partly of students in naval architecture originally selected by competition from the dockyard apprentices to join the junior class. The full course in the advanced class lasts three years, of which one is spent in the junior, and two in the senior class. There are also private students who generally go through the full course. The programme of these classes differs in some important respects from that of most other technical colleges, and it may be useful to describe it briefly here.

The three principal branches of study are :—

I. Pure and Applied Mathematics;
II. Applied Mechanics;
III. { Naval Architecture,
      Marine Engineering;

to each of which the time allotted is about the same. In addition, there is a course in Physics and Chemistry. The following remarks will be confined to the second and third of the principal subjects.

In APPLIED MECHANICS the subjects are

A. Elementary Subjects.
B. { Stability and Oscillation of Ships.
     Theory of the Steam Engine.
C. Wave motion. Resistance and Propulsion of Ships.

Subjects A. are pretty closely represented by the present treatise, but there are a few omissions and some additions, especially in graphical statics and elementary theory of the steam engine. The course lasts two years, each subject being commenced in the junior class and completed in the senior.

Subjects B. are commenced in the second year and completed in the third. The first is studied by students in naval architecture only, and the lectures on it are at present given by the Instructor in Naval Architecture. The second is studied by students in engineering only.

Subjects C. occupy the greater part of the third year.

In NAVAL ARCHITECTURE the course followed is very fully explained in a paper by Mr. W. H. White in the *Transactions I.N.A.* for 1877 (vol. xviii., p. 361), and it need not therefore be further considered here.

In MARINE ENGINEERING the course for the junior class occupies nine hours a week. Each of the principal parts of the marine engine, including the boiler and propeller, are taken in detail, the dimensions proper for that part determined, and the other practical questions considered which are involved in its design. An example is set, and the student is expected to work out a design from the data proposed, and to produce working drawings. About 30 of these drawings are prepared in the session, the subjects being:—

Fastenings—
Bolts and nuts, Cotters, Pin-joints, Rivetted-joints.

Details of principal parts of Engine—
Piston, Piston rod, Connecting rod, Cross-head and guides, Propeller shafting and couplings, Thrust-block, Crank-shaft, Cylinders.

Slide Valves—
Zeuner's diagram, Valve ellipse, Construction and setting of slide valves, Link motion, Expansion valves.

Boilers—
Dimensions and Structural details, Fittings.

Condensers and Air Pumps.

Indicator Diagrams--
Expanding actual diagrams of compound engines. Construction of probable diagrams.

The foregoing course is gone through by all students in engineering. Those who are selected to enter the advanced class devote six hours a week to the subject in two following sessions, under the direction of an instructor appointed specially for the purpose by the Engineer-in-Chief to the Admiralty. In the second year detailed drawings are made of the parts, and three views of the general arrangement, of a set of marine engines of large power suitable to propel a given ship at a given speed. The drawings of the details and propeller are completed, and the general drawings pencilled. In the third year the boilers are designed, and drawings made showing the disposition of the pipes and auxiliary engines. The general drawings are completed, the whole design being represented by a set of about 20 drawings.

The practical training of students both in naval architecture and in engineering takes place in the dockyards before entering the College and during the three summer months in which the College is closed. This is a point of great importance, for, quite irrespectively of the absolute necessity of such training for its own sake, no theoretical course can be thoroughly understood without some preliminary knowledge of a practical kind. A college workshop is a very imperfect substitute, and occupies time which is better spent elsewhere. The author, however, must not be understood to depreciate the importance of a "mechanical laboratory," provided with testing machines, hydraulic apparatus, steam engines, and the like, for the purpose of studying mechanics experimentally. Such a laboratory, when properly organized, is capable of rendering great service, but it in no way replaces training in a large workshop carried on for commercial purposes. Nor are these remarks intended to apply to the lower grades of technical education, in which the workshop to a great extent plays the part of a laboratory.

In the author's opinion, much the same may be said as to the use of models in teaching mechanics. An engineer does not use models; he employs drawings almost exclusively; and so, in the instruction of professional students, models are not necessary for descriptive purposes. Nor should they be used to demonstrate the laws of motion. But in explaining a mechanical principle, a model is sometimes of service; it plays the same part as the figure in a proposition of Euclid in aiding the conception of the learner. And, as before, in the lower grades of technical education, models may properly be used for demonstrative purposes, while, in the case of non-professional students, they are often indispensable for descriptive purposes. In the "steam" department of the Royal Naval College, organized for the purpose of imparting to the executive officers of the navy a knowledge of the mechanism and working of a marine engine, models are freely used in this way. On the subject of technical education in naval architecture the reader is referred to two valuable papers by Mr. John and (the late) Mr. W. Denny in the *Transactions of the Institution of Naval Architects*, the first in vol. xix., page 120; the second in vol. xxii., page 144.

NOTE.—*Some changes have recently* (1889) *been introduced in this course, but not being as yet permanently established, it has not been thought necessary to notice them.*

APPENDIX.

C.—ADDITIONAL NOTES.

(*Added on reprinting this work in December*, 1889.)

I.—CRUSHING BY BENDING.

(Ch. XIV., Pages 309-321.)

Euler's Formulæ (Page 314).—The simplest way of proving these formulæ is to assume the curve in which the pillar bends to be a curve of sines, and then to show that the sectional area is constant, a process which is the converse of that given in the Appendix (page 517). It is worth remarking that the co-efficient of elasticity of flexion employed in them is not, strictly speaking, Young's modulus (E) but $E-p$ where p is the intensity of the stress on the cross section. This, though theoretically interesting, is of no importance in practice, because of the extreme smallness of the ratio p/E in all practical cases.

In a paper published in the *Proceedings of the Cambridge Philosophical Society*, vol. iv., part ii., Greenhill has determined the greatest height of a vertical pole which is consistent with stability, that is, the greatest length of pole of given diameter which will stand upright without bending over laterally at the summit. Let $\lambda = E/w$ be the length due to a stress E for a given material, E being as usual Young's modulus. This method of measuring a stress is explained on page 80. Then for a pole of uniform transverse section of radius a the greatest height is given by the simple formula

$$h = 1{\cdot}26\sqrt[3]{\lambda a^2},$$

in which all the quantities are given in the same units. For a pole of radius a at the base diminishing in section uniformly to zero at the summit so that the longitudinal section is triangular the same formula serves, but the co-efficient is $\sqrt[3]{7{\cdot}63}$, or 1·97 instead of 1·26. For a pole of pinewood 6" diameter at the base the greatest height is about 90 feet in the first case and 140 in the second. The stress (wh) on the transverse section at the base must, as appears from what is said in the text, be much less than the crushing stress (f) of the material, a condition which would generally be satisfied. These formulæ are of considerable theoretical interest and are applied in the paper cited to questions relating to the growth of trees: it must be remembered, however, that the effect of wind pressure is neglected, a circumstance which limits considerably their practical application.

Collapse of Flues.—A formula, corresponding to Euler's formula for pillars, has been obtained for the collapse of a flue of unlimited length, by **Lévy** and **Halphen**. This formula as quoted by **Greenhill** in a letter which will be found in the *Engineer* for Feb. 1888 is

$$p=\frac{E}{4}\left(\frac{t}{a}\right)^3,$$

where t is the thickness, a the radius, both reckoned in inches, while E as usual is Young's modulus, and p the collapsing pressure. The corresponding form of Gordon's formula deduced as on page 318 will be

$$p=\frac{f.\frac{t}{a},}{1+\frac{a^2}{ct^2}}$$

where the "theoretical" value of the constant c is

$$c=\frac{E}{4f}.$$

This formula may be expected, with suitable values of the constants, to give the collapsing pressure of a flue, the length of which is so great as to have no sensible influence on its strength. In short lengths the strength is greater, as described on page 321.

II. SHEARING AND TORSION.

Ch. XV., Pages 360-370. Appendix, Page 518.

Torsion of Shafts.—It may be worth while to add to the remarks made in the Appendix that the moment of resistance of a shaft of rectangular section was often found, prior to **St. Venant's** researches, by a formula due to **Cauchy** which has been quoted in some of the older standard treatises on strength of materials. The formula in question is

$$T=\frac{1}{3}f.\frac{b^2d^2}{\sqrt{b^2+d^2}},$$

where b and d are the sides of the section.

As determined by this formula the ratio of strengths of a rectangular section and a circular section of the same area is $.8353\sqrt{\frac{2}{n+\frac{1}{n}}}$ where n is the ratio of the sides, a result which differs from the truth chiefly in the constant factor which, as determined by St. **Venant**, is ·738, the value given on page 361. This formula therefore gives results which are about 11 per cent. too large.

Centrifugal Whirling of Shafts.—On referring to page 299 it will be seen that the deflection produced by a load R placed at the centre A (Fig. 128) of a beam of uniform section is

$$\delta=\frac{Rl^3}{48EI}.$$

Suppose now a weight W fixed at the centre of a rotating shaft, there will be a centrifugal force due to this weight whenever a deflection is produced by it in such a way that the weight rotates in a circle. This centrifugal force on reference to pages 253, 254 will be found to be given by

$$R=W.\frac{\delta n^2}{35,232}$$

where n is the number of revolutions per minute. On substitution we obtain an equation for n, all dimensions being given in inches.

$$n = 1300\sqrt{\frac{EI}{Wl^3}}.$$

If n be greater than this limit the shaft will be in an unstable condition like a loaded pillar, being liable to "centrifugal whirling." When the weight W is distributed like the weight of the shaft itself, instead of being concentrated in the centre the limiting number of revolutions is greater and is found by the solution of a differential equation. The general theory is given by **Rankine** in his *Millwork and Machinery*. Centrifugal whirling is one of many things which have to be considered in machines running at excessive speed. In ordinary cases the limiting speed is not approached.

Thrust and Torsion.—When it is a question of strength only this case is dealt with on the principles explained in Chapter XVII. **Greenhill** has, however, pointed out that, if the unsupported length of the shaft be too great, it is necessary to consider its stability. Let P be the end thrust on the shaft T the twisting moment, then the greatest unsupported length consistent with stability is given by the formula

$$\frac{\pi^2}{l^2} = \frac{P}{EI} + \frac{T^2}{4E^2I^2}.$$

Let P_0 be the greatest load which by Euler's formula this length of shaft would carry considered as a pillar, and let T_0 be the greatest twisting moment consistent with strength, f being the co-efficient of resistance to twisting, then the formula may be written, supposing d the diameter

$$P_0 = P + \frac{T^2}{T_0^2} \cdot \frac{f}{E} \cdot \frac{\pi}{16} f \cdot d^2,$$

or using p_0, p to represent the stress per square inch of section,

$$p_0 = p + \frac{1}{4} \cdot f \cdot \frac{T^2}{T_0^2} \cdot \frac{f}{E}$$

As f/E is necessarily a very small fraction and T/T_0 is fractional this shows that there can be very little difference between p and p_0, so that unless these quantities themselves be small, that is the unsupported length of the shaft very great, the twisting makes no sensible difference in the stability of the shaft. The formula, therefore, in ordinary cases, though theoretically interesting, is not of practical value. It will be found with numerical applications in a paper read by Professor Greenhill before the *Institution of Mechanical Engineers* in 1883.

Maximum Shearing Stress on the Transverse Section of a Beam (Art. 188, page 334).—In the first edition of this work it was inadvertently stated that the shearing stress in this case is always greatest at the neutral surface of the beam. A more general investigation is given by **Rankine** (A. M., page 340), and also by **Alexander** and **Thompson** in their treatise on Mechanics.

III.—STRENGTH OF MATERIALS.

(Ch. XVIII., Pages 371-399. Appendix, Pages 521-522.)

Elastic Strength.—The following results obtained in the Laboratory of University College, by experiments on the stretching of wrought iron and soft steel under the direction of Professor Kennedy, may probably be considered as representative, agreeing, as they appear to do, with various experiments made by others.

(1) Irregularities (page 374) may be eliminated by a preliminary load much below the elastic strength.

(2) Increments of extension are constant up to about ·85 × Stress at the breaking down point.

(3) A sharply defined "breaking down point" exists, at which a considerable extension takes place suddenly without increased stress. The corresponding stress appears to range from ·5 to ·7 in iron, and to average ·6 in soft steel, of the ultimate tenacity.

(4) After the breaking down point the oad increases to a maximum value up to which the contraction is general throughout the bar.

(5) After a certain maximum load has been applied local contraction begins at one or more points of local weakness and continues till rupture occurs.

These results suggest a few remarks on elastic strength and on some other points of importance in strength of materials.

The sudden drawing out which occurs at a certain point is characteristic of wrought iron and soft steel. The point described above as the "breaking down point" has received various other names, such as the "yield point," the "limit of stability," of which we shall adopt the last. It may be raised as described in the text (page 378), and Unwin has suggested with great probability (*Testing of Materials*, page 66) that it is a purely artificial limit created by the process of manufacture. When a bar of iron is stretched to breaking in the workshop without recourse to delicate measuring apparatus it is the apparent limit of elasticity, being well marked by the falling off of scale and by an obvious extension accompanied by lateral contraction. In bar iron of good quality this "limit of stability" is generally from 15 to 17 tons per sq. inch.

At a stress of about ·85 per cent. of the stress corresponding to the limit of stability, that is in bar iron at from 12½ to 14½ tons per square inch, we reach a point which has been described as the "limit of proportionality" at which the elongations cease to be proportional to the load. This is the limit of elasticity as determined by accurate measurement in a bar which has already been exposed to a moderate load to eliminate irregularities. The limit usually given by the older experiments, such as the one given in the text, was much less, being about 10 tons per sq. inch, a result perhaps due to testing the bar without a preliminary elimination of irregularities, and this circumstance suggests what is no doubt the fact that this limit is artificial as well as the limit of stability depending to a great extent on the treatment undergone by the bar during manufacture and afterwards.

It is thus clear that so long as we confine ourselves to simple stretching, no point can be defined as the true limit of elasticity of the material, and this difficulty of definition has been encountered by all experimentalists. Prior to Wöhler's experiments, **Rankine** (A. M., page 274) defined Proof Strength as the maximum load which does not produce an *increased* set by repeated application. A natural extension of this is to consider the resistance to unlimited alternate stress as furnishing the true limit of elasticity, a view adopted in the text (page 389). This view is strongly confirmed by the remarkable experiments of **Bauschinger**, an account of which will be found in Professor Unwin's work cited above. **Bauschinger** shows that a stress beyond the elastic limit in tension lowers the limit in compression, and conversely. Alternate experiments thus lower the limit to a point near that obtained by **Wöhler**'s experiments. Hence experiments on unlimited alternate stress may be regarded as giving the true elastic strength. Some of these results are given in the text; they are here repeated with additions taken from Professor Unwin's work. They are applicable to all cases of tension, compression and bending. For shearing and torsion see p. 390.

TRUE ELASTIC STRENGTH.

MATERIAL.	TONS PER SQ. IN.	MATERIAL.	TONS PER SQ. IN.
Plate Iron, - -	7	Soft Steel, - -	9½
Bar Iron, - -	8	Cast Steel, - -	13½
		Copper, - - -	4¾

Much greater values of the elastic strength may be obtained as stated above (see also p. 381), but they are only reliable for some special mode of loading or some special treatment of the material. Unfortunately experiments on unlimited alternate stress are tedious to make: **Wöhler**'s results have been fully confirmed by others, but until such experiments have been made on a much greater variety of materials in many different ways many important questions on strength of materials will remain unsettled.

Maximum Load.—Returning to the results given at the commencement of this article, it will be seen that the load increases to a maximum and afterwards diminishes, a fact first observed by **Adamson** in 1878. This marks the point where local contraction begins. **Unwin** has suggested (*Testing of Materials*, p. 67) that this gives the stress at which the metal ceases to acquire increased hardness (page 378) while suffering deformation, and behaves nearly as a perfectly plastic body (page 373). He proposes therefore to call this point the "limit of plasticity." On this supposition the drawing out of the bar under a diminishing load, after passing this point, takes place under a nearly constant mean stress, rupture occurring from irregularity of distribution, or from some accidental circumstance.

This is the principal reason why the ratio of the breaking load to the contracted area is not the "true tenacity" of the metal.

Work done in Stretching.—If r be the ratio of the stress at the "limit of stability" to the ultimate tenacity, W the total breaking load, x the ultimate extension, the work done in stretching till rupture occurs is found to be approximately

$$U = Wx\frac{r+2}{3}.$$

This formula is given by **Kennedy** (*Proc. I. C. E.*, vol. lxix., page 30), the factor

$(r+2)/3$ being derived by supposing the curve of stress and strain after passing the "limit of stability" to be parabolic. Taking $r = \cdot 6$ for soft steel, he finds the work done in tearing asunder a specimen of length = 10 diameters to be 5·5 to 6·3 inch tons per cubic inch. On account of local contraction, the deformation of the metal is of course very unequally distributed; in compression of the short block described in the text (p. 379) the work done is 31·7 inch tons per cubic inch. It is this experiment which gives the best absolute test of endurance of a material.

Compression.—The bulging which occurs when a short block of ductile material is compressed is due to the instability of a cylindrical flow of the metal, and would probably occur even if there were no friction between the block and the compressing surfaces. Fracture occurs by lateral tearing asunder along longitudinal cracks when the height is small. When of greater height, the block crushes by lateral bending. In wrought iron the ratio of length to diameter at which lateral bending commences is about 3 and the corresponding crushing stress is 36,000 lbs. per sq. inch, remaining independent of the length until the ratio reaches about one-third of the limiting values given in the text (page 317), after which the length begins to influence the crushing load as described in the chapter cited (Ch. XIV.).

In rigid materials (p. 383) crushing takes place either by oblique shearing or by longitudinal cracks. Unwin (*Testing of Materials*, page 419) finds that the mode of crushing and the resistance to crushing are much influenced by the material on which the specimen rests. When bedded on a soft material, the lateral floor of this material supplies by friction a transverse force on the base of the specimen, in consequence of which it crushes by longitudinal cracks at a smaller load than if the bed were hard, in which case oblique shearing occurs. This is a highly interesting observation, but it would be premature to say that all cases of crushing by longitudinal cracks can be explained in this way.

Principle of Similitude.—The idea that in similar pieces of material under similar forces similarly applied the load necessary to produce similar deformations must be proportional to the sectional area of the pieces, is one which was the foundation of most of the formulæ relating to strength of materials originally employed by engineers and given in the older standard treatises on the subject. There can be little doubt that this law is universally true, and indeed it amounts to little more than a definition of similar materials. One case of this law is well known as having been approximately verified by the experiments of Barba and others on stretching of similar specimens of different dimensions. A number of other examples of the application of the law will be found in a small treatise by Professor Kick (*Das Gesetz der Proportionalen Widerstände*, Leipzig, 1885). Any deviations from this law should be due simply to difference in quality of material and mode of manufacture between small pieces and large ones. In framing semi-empirical formulæ for cases in which exact formulæ are inapplicable this law should be borne in mind. Thus in the case of pillars the formulæ proposed by Hodgkinson do not satisfy the law, and for this reason alone should be rejected in favour of the Gordon formula, which does satisfy it.

Factors of Safety.—As remarked in the text (pages 392-394) the principle of a constant factor of safety is in many cases open to question. In a paper contained in the *Transactions of the Institution of Naval Architects* (vol. xxix.) the late engineer-in-chief of the navy (Mr. Sennett) proposes to allow a constant *margin* of safety in the case of marine boilers at high pressure instead of a constant *factor*, and his views were endorsed, in the discussion which followed the reading of the paper, by Mr. A. C. Kirk and Mr. Marshall. The circumstances on which factors of safety depend are explained in the text, and it need only be repeated that though theoretical reasoning and laboratory experiments may furnish valuable indications of the direction in which to move, yet any steps in the direction of lower factors of safety should be very gradual, and only taken by those who possess the widest knowledge and the greatest experience of nearly similar cases. It is impossible *a priori* to foresee all the circumstances which may influence the necessary margin of safety.

References.—For further recent information on strength of materials and various questions in the theory of structures the reader is referred amongst other works to

UNWIN: *The Testing of Materials of Construction.* Longman. 1888.
FIDLER: *A Practical Treatise on Bridge Construction.* Griffin. 1887.

IV.—THEORY OF MACHINES.

(CHS. IV.—XI.)

Wedge Chain.—Among the kinematic chains, involving lower pairing only, should have been included a three-linked chain of three sliding pairs, which may be described as the "wedge chain." Only one mechanism can be obtained from this chain. It is sometimes employed to obtain a sliding motion in a given direction from a different sliding motion at some distant point.

Centrode.—In the first edition of this work this word was spelt "centroid," but this term is now employed by various writers for the centre of mass of a body, and the spelling "centrode" appears preferable.

Inertia of Reciprocating Parts (Page 212).—In considering the effect of a connecting rod on the inertia of the reciprocating parts of an engine, the best method to adopt seems to be to divide the weight of the rod into two parts, one placed at the cross head so as to move with the piston, and the other placed at the crank pin so as to move with the crank; the division being so made that the two weights have the same centre of gravity as the original rod. The first alone is to be considered as one of the reciprocating parts. For it is clear that the energy of translation of the two weights is the same as that of the original rod, and therefore requires the same forces to change it. The energy of rotation is greater, because the radius of gyration of the two weights is necessarily greater than that of the rod, but the error due to this must be very small, and is zero at the ends of the stroke where the correction due to inertia is most important.

Friction of Bearings.—The latest experiments on this subject conclusively show that bearings may be so fully lubricated that their friction becomes independent of the pressure. The theory of fully lubricated bearings has been studied by OSBORNE REYNOLDS in a paper published in the *Philosophical Transactions*, and found to agree well with the experiments. In recent experiments on the friction of steam-engines the friction has been found to be nearly independent of the load.

Dynamometers.—The form of dynamometer shown in Fig. 4, Plate VII., is one which has been much employed, but it has been recently shown that the "compensating levers" used may give rise to serious errors, and it is therefore probable that it will not again be used in accurate work. It has therefore been thought unnecessary to reprint the description given in the Appendix of the first edition.

Integrating Apparatus.—This subject has been exhaustively studied by HELE SHAW in a valuable paper published in the *Proceedings of the Institution of Civil Engineers* (vol. lxxxii.).

Reference.—For further information on subjects connected with the *Theory of Machines* the reader is referred to

KENNEDY: *Mechanics of Machinery.* Macmillan. 1885.

V.—HYDRAULICS.

(CH. XIX., PAGES 405-428. APPENDIX, PAGES 522-524.)

Discharge from Orifices.—Some careful experiments have been made by Mr. Mair (*Proceedings of the Institution of Civil Engineers*, vol. lxxxiv.), from which it appears that very slight variations in sharpness of the edges of an orifice will produce considerable effects on the co-efficient of discharge: the sharper the edge the lower the co-efficient, and a co-efficient as low as ·605 was thus obtained. This is probably due to variations in the co-efficient of contraction. Change of temperature has little or no influence on the discharge from an orifice in a thin plate. In the case of large orifices, the plane of which is vertical, the centre of gravity of the issuing fluid was described as the "centre of flow" in the first edition of this work. This term has now been replaced by "centre of energy," which appears preferable. In his *Civil Engineering* (2nd edition, page 681), RANKINE measures the head to the point where the mean velocity occurs, which is somewhat *above* the centre of the orifice, and might properly be described as the "centre of flow," but this mode of measurement does not appear to the author to be accurate.

Surface Friction.—The formula given in the Appendix (page 524) for the loss of head in a pipe has been very thoroughly compared with experiment by UNWIN, who finds that the index of d, the diameter of the pipe, is not exactly $3 - n$, but requires to be independently determined. His results are given in detail in a valuable series of papers published in *Industries* for the year 1886. In other respects the formula appears to be verified. For practical purposes it seems best to retain the ordinary formula given in the text, and employ a value of $4f$ derived from a table for any particular case for which exact results are required. Such a table is given in the paper cited; the value (·03) of $4f$ employed in the text may be considered as a fair average, but it is subject to great variations. In large pipes in good condition it may be much less.

It has now been conclusively proved by the experiments of MAIR, UNWIN, and others, that surface friction is very considerably influenced by temperature, the co-efficient being smaller by about 1 per cent. for each 5° F. that the temperature rises. Some particulars on this point will be found in MAIR's paper already cited. The results given in the table (page 418) must be taken with this qualification, and no exact comparison can be made between various experiments on the discharge of pipes without paying regard to the temperature at which the experiments were made.

According to the reasoning given on page 430, hydraulic resistance in a fluid completely devoid of viscosity should vary as the square of the velocity, and should depend on temperature only so far as temperature influences density. The diminution of the index in the case of surface friction has long been known, and is due probably to the same cause which makes the friction of a long surface different from that of a short one. The effect of temperature in diminishing surface friction must be connected with the indirect effect of viscosity in extinguishing the eddies produced. Now that the laws of surface friction are more exactly known, it may be hoped that a complete theory may not long be wanting.

VI.—RESISTANCE AND PROPULSION.

(CHS. XIX., XX., PAGES 431, 469-472. APPENDIX, PAGES 524, 525.)

The importance of this subject is so great that, though outside the intended limits of this work, some additional information may be useful. What follows, however, must be considered merely as a few brief notes which would require expansion into two or three long chapters if anything like a full statement were attempted.

Froude's Principle of Similitude.—If L be the length of a vessel in feet, V her speed in knots, and we write

$$V=c\sqrt{L},$$

then the principle established by FROUDE may be conveniently expressed by saying that the general character of the resistance to the motion of a vessel through the water depends not on her absolute speed but on the value of the speed-co-efficient c. In the text (page 431) the symbol K' is used instead of c, which is employed here to avoid confusion with the co-efficient K. In an ordinary merchant steamer c ranges from ·5 to ·7. This may be described as a moderate speed, and the resistance is mainly due to surface friction as stated in the text. In fast mail steamers c ranges from ·7 to 1, and the resistance, speaking very roughly, is about one-half due to surface friction and one-half due to other causes, especially the formation of waves. In torpedo boats c may reach 2·2 or even more; this is an excessive speed, and the resistance is here due mainly to the formation of waves.

In the first two cases the resistance may be expressed to a fair degree of approximation by taking, for values of c *not exceeding unity*

$$K=a(1+bc^2),$$

which is equivalent to employing a simplified form of BOURGOIS' formula (page 525). The value of K is thus expressed in terms of two constants a and b. The value of a may be taken as ·4 and the value of b as unity, but it must be understood that these values are only roughly approximate, that the bottom of the vessel must be supposed clean, and that only the resistance of the bare hull is included, without reference to the effect of appendages to the propelling apparatus or other causes of increased resistance. The resistance given by this formula is also too small (1) for a vessel of small size (say less than 100 feet long), (2) for a vessel of small draught of water such as is employed for river traffic, because the effect of surface friction is then relatively greater than is supposed in the formula.

For values of c which approach unity or exceed unity the resistance due to the formation of waves predominates over the rest. It follows a periodic law alternately increasing with extreme rapidity and then for a time more slowly. This was conclusively proved by Froude, in one of the most celebrated of his papers, to be due to the interference of two independent sets of waves, one generated by the bow of the vessel the other at the stern. Hence at excessive speeds no formula of the simple kind just given will hold good. Such speeds always require excessive engine power, and therefore can only be employed for some special purpose. If the vessel have a long middle body the effect of wave interference will be relatively important at smaller values of c.

If the value of c be the same in similar vessels of different dimensions, the speeds of the vessels are said to "correspond." Froude's principle of comparison further asserts that at corresponding speeds the resistances of the two vessels will be in proportion to their displacements. This may be conveniently expressed by saying that the resistance in lbs. per ton of displacement of a vessel depends not on her absolute speed, but on the value of her speed-co-efficient c. Thus adopting the formula given above, the resistance of a vessel in lbs. per ton is about $8(1+c^2)c^2$.

The principle does not apply exactly to surface friction, the resistance due to which is relatively greater for small dimensions of vessel, but in comparing the resistance of a vessel with that of a model, Froude showed how to introduce a correction which would allow for this, and hence the resistance of a vessel is now determined by experiments made on a model of that vessel.

The resistance thus obtained or roughly estimated as above is the *net* resistance of the vessel, and the corresponding horse-power required at a given speed is the Effective Horse Power.

Indicated Power.—In determining the *indicated* power we may make use of the formula

$$P_m=(1+e)P+P,$$

given on page 243, which connects the actual mean effective pressure (P_m) in an engine with the net pressure (P) available for useful work. The constant P_0 depends on the "constant" engine friction, as explained in the text, being generally as there stated about 2 lbs. per sq. inch, but, of course, subject to variation. The constant e may be taken so as to include: (1) the augmentation of resistance due to the propeller; (2) the variable part of the friction of the engines; (3) the friction of the screw in the water. Its value on this understanding ranges from ·5 to 1 when there is no serious defect in design or condition of vessel and propeller. An average value of e is ·7 for well designed vessels under favourable conditions.

The actual mean effective pressure is fixed by the type of engines it is intended to adopt, hence the *net* pressure is found, and from this the necessary dimensions of cylinders for a given number of revolutions. The revolutions of the engines are found from the pitch and slip of the screw, and thus the indicated horse power is determined. The Efficiency of Propulsion is the ratio of effective to indicated power, which at full power rarely exceeds ·55 in the best examples and is often only ·4. When the engines are not working up to their full power, the effect of the "constant" engine friction is so great that the efficiency is greatly diminished.

Friction of Screws.—It was pointed out by Froude that a screw-blade might be regarded as a body moving nearly edgewise through the water, the small angle of obliquity, which we will call ϕ, depending as it does on the slip, being described as the "slip-angle." This small angle does not exceed 10° in practical cases.

If u be the velocity at which a small element representing a portion of a screw-blade moves through the water, the normal pressure on the element is known to be (when ϕ is small) $a \, . \, \phi \, . \, u^2$ where a is a co-efficient. The tangential resistance due to friction is $f \, . \, u^2$ approximately, and the ratio

$$\phi^1 = \frac{f}{a\phi}$$

is a small angle which, as in the friction of surfaces, may be described as the "friction-angle" (page 222). Proceeding now, as on page 227, the efficiency will be found to be

$$\text{Efficiency} = \frac{\tan\,(\alpha - \phi)}{\tan\,(\alpha + \phi^1)},$$

the connection between ϕ and ϕ^1 being such that, as shown above, $\phi\,\phi^1$ is constant. When $\phi = \phi^1$ the efficiency is greatest, the best pitch angle is then approximately 45°, and the value of the maximum efficiency the same as on the page cited. Thus the friction and efficiency of screw propellers, as determined by Froude's calculation are governed by laws closely analogous to those which refer to an ordinary screw and its nut.

References.—For further information on this subject the reader is referred to various papers by Froude, which will be found in the *Transactions of the Institution of Naval Architects*. It is to be hoped that these papers will be published in a collected form.

VII.—HYDRAULIC MACHINES.

(Ch. XX., Pages 458-469.)

Steady Motion in a Rotating Casing.—If a pipe of any form be attached to a wheel rotating at n revolutions per second an equation can be found by a suitable modification of the reasoning on pages 413-462, which gives the steady motion of water through the pipe. The equation in question is

$$\frac{u^2 - v^2}{2g} + \frac{p}{w} + z = \text{Constant},$$

where v is the velocity of the point in the rotating pipe through which the water is flowing, and the rest of the notation is as in the equation on page 413, to which it reduces when v is zero. If r be the distance of the point from the axis of rotation

$$v = 2\pi n r,$$

and the equation may also be written

$$\frac{u^2}{2g} + \frac{p}{w} + z = \text{Constant} + \frac{4\pi^2 n^2 r^2}{2g}.$$

The total head in the pipe relatively to the rotating wheel is equal to a constant plus the quantity $4\pi^2 n^2 r^2/2g$, which is sometimes called the "head due to centrifugal force," a perfectly legitimate expression when properly understood.

This equation applies to any case of steady motion within a rotating wheel or casing on the same manner as the well known equation on page 413 applies when the casing is at rest. It often may be used with great advantage in questions relating to turbines, centrifugal pumps, or other similar cases.

VIII.—PNEUMATIC MACHINES.

(Ch. XXI., Pages 475-491.)

Expansion and Compression of Air (Page 491).—When air is compressed any moisture which it contains in the form of vapour has little influence on the compression curve. To prevent the temperature from rising considerably it must be passed through tubes exposing a large cold surface. When injected into the compressing cylinder water by its evaporation effectually keeps down the temperature. When moist air expands, the fall of temperature speedily reaches the dew point, and after this a further fall of temperature is checked by the heat given out on condensation of the vapour.

Errata in the First Edition.—The subjoined list of errata in the First Edition of this work is limited to corrections of importance not given in the original, or in the Additional Notes of the present edition :—

Page 59, line 8 from bottom—*for* one-sixth, *read* one-third.

Page 217, line 10 from bottom—*for* $\sqrt{L^2+k^2}$, *read* $L+\frac{k^2}{L}$.

Page 317, line 2—*for* one-sixth, *read* one-third.

Page 487, line 9—*for* the ratio of areas, *read* the square of the ratio of areas.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE.

s

va
of
coll

ro
ca
th

wl
fl
re

an

Th
the
for

T

89080439045

b89080439045a